LECTURE 5

2. Adiabatic hypothesis and the S matrix

Consider hamiltonian $H = H_0 + e^{-\varepsilon|t|} H_1$ (ε small)

$H = H_0$ at $t \to \pm\infty$

$H = H_0 + H_1$ at $t = 0$

Assume at $t \to -\infty$ the system is in the ground state $|\phi_0\rangle$ of H_0. We then turn on H_1 slowly and evolve the system to an eigenstate of $H = H_0 + H_1$ at $t = 0$:

$$|\Psi_0\rangle = \frac{U(0,-\infty)|\phi_0\rangle}{\langle\phi_0|U(0,-\infty)|\phi_0\rangle}$$

If there is no eigenstate crossing then $|\Psi_0\rangle$ is the ground state of $H = H_0 + H_1$ by adiabatic theorem.

- We now want to express the GF in a form that refers to unperturbed ground state $|\phi_0\rangle$

$$G_{ij}(t) = -i\langle\Psi_0|T[u_i(t)\, u_j(0)]\,|\Psi_0\rangle$$

$$U(0,-\infty)|\phi_0\rangle = U(0,\infty)\underbrace{U(\infty,-\infty)}_{S}|\phi_0\rangle = U(0,\infty)S|\phi_0\rangle$$

$S = U(\infty,-\infty)$ "S-matrix"

- Since interaction is zero at $t\to\pm\infty$ the state $|\phi_0\rangle$ can differ from $S|\phi_0\rangle$ only by a phase factor (provided that it is non-degenerate).

$$S|\phi_0\rangle = e^{i\alpha}|\phi_0\rangle$$

- Now we can write the g.F: (for $t>0$)

$$G_{ij}(t) = -i\,\underbrace{\langle\phi_0|S^+U^+(0,\infty)}_{\langle\Psi_0|}\underbrace{U^+(t,0)\,u_i(t)\,U(t,0)}_{u_i^H(t)}\underbrace{u_j(0)}_{u_j^H(0)}\underbrace{U(0,-\infty)|\phi_0\rangle}_{|\Psi_0\rangle}$$

$$= -i\,e^{-i\alpha}\langle\phi_0|\,U(\infty,t)\,u_i(t)\,U(t,0)\,u_j(0)\,U(0,-\infty)\,|\phi_0\rangle$$

$$= -i\,e^{-i\alpha}\langle\phi_0|\,T\big[u_i(t)\,u_j(0)\,\underbrace{U(\infty,t)\,U(t,0)\,U(0,-\infty)}_{U(\infty,-\infty)=S}\,|\phi_0\rangle$$

$$= -i\,e^{-i\alpha}\langle\phi_0|\,T\big[u_i(t)\,u_j(0)\,S\big]\,|0\rangle$$

Also notice that

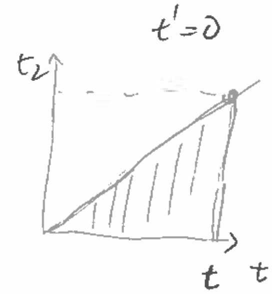
$$\langle\phi_0|S|\phi_0\rangle = e^{i\alpha}\langle\phi_0|\phi_0\rangle = e^{i\alpha}$$

$$\boxed{G_{ij}(t) = -i \frac{\langle \phi_0 | T[u_i(t)\, u_j(0)\, S] | \phi_0 \rangle}{\langle \phi_0 | S | \phi_0 \rangle}}$$

Starting from this expression an expansion can be developed for $G_{ij}(t)$ by systematically expanding S in powers of $H_1(t)$.

Integral eq. for $U(t,t')$:

$$U(t,t') = 1 - i\lambda \int_{t'}^{t} dt_1\, H_1(t_1)\, U(t_1, t')$$



Iterate:

$$U(t,t') = 1 - i\lambda \int_{t'}^{t} dt_1\, H_1(t_1) + (-i\lambda)^2 \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, H_1(t_1)\, H_1(t_2) + \dots$$

This can be manipulated to the following form:

$$U(t,t') = 1 - i\lambda \int_{t'}^{t} dt_1\, H_1(t_1) + \frac{(-i\lambda)^2}{2!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2\, T[H_1(t_1)\, H_1(t_2)] + \dots$$

$$= T \exp\left\{-i\lambda \int_{t'}^{t} dt_1\, H_1(t_1)\right\}$$

S-matrix expansion: $S = U(\infty, -\infty)$

$$S = T \exp\left\{ -i\lambda \int_{-\infty}^{\infty} dt_1\, H_1(t_1) \right\}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-i\lambda)^n}{n!} \int_{-\infty}^{\infty} dt_1 \dots dt_n\, T\left[H_1(t_1) \dots H_1(t_n) \right]$$

Feynman-Dyson expansion

We can now use this to expand the GF:

$$\boxed{G_{ij}(t) = G^0(t)\delta_{ij} - i \sum_{n=1}^{\infty} \frac{(-i\lambda)^n}{n!} \int_{-\infty}^{\infty} dt_1 \dots dt_n \langle \phi_0 | T\left[u_i(t) u_j(0) H_1(t_1) \dots H_1(t_n) \right] | \phi_0 \rangle}$$

(we ignored the denominator $\langle \phi_0 | S | \phi_0 \rangle$ for now)

- Why is this expansion useful?

We have transformed the task of calculating $G_{ij}(t)$ to evaluating expectation values of operators whose time dependence is simple with respect to the ground state $|\phi_0\rangle$ of the unperturbed system H_0.

Interaction picture:

$$u_i = \frac{1}{\sqrt{2M\Omega_0}} (B_i + B_i^+) \qquad H_0 = \sum_i \Omega_0 \left(B_i^+ B_i + \tfrac{1}{2} \right)$$

$$B_i(t) = e^{iH_0 t} B_i e^{-iH_0 t} = e^{-i\Omega_0 t} B_i \qquad B_i |\phi_0\rangle = \langle \phi_0 | B_i^+ = 0$$

→ the task reduces to evaluating expectation values like

$$X = \langle T[B(t_1)\, B^+(t_2)\, B(t_3)\, B^+(t_4)] \rangle_0 \qquad (*)$$

This can be evaluated either directly (LABORIOUS!) or with the help of Wick's theorem.

Wick's theorem

Define a "contraction" of two operators

$$\underbrace{A(t)\, B(t')} \equiv \langle \phi_0 | T[A(t)\, B(t')] | \phi_0 \rangle$$

[Then, exp. value of a T-ordered product of operators like (*) can be expressed as a sum of all possible contractions:]

$$X = \langle T[B(t_1)\, B^+(t_2)\, B(t_3)\, B^+(t_4)] \rangle_0$$

$$= \langle T[B(t_1)\, B^+(t_2)] \rangle_0 \langle T[B(t_3)\, B^+(t_4)] \rangle_0$$

$$+ \langle T[B(t_1)\, B^+(t_4)] \rangle_0 \langle T[B^+(t_2)\, B(t_3)] \rangle_0$$

$$[B, B^+] = 1$$

$$BB^+ - B^+B = 1$$

$$\langle BB^+ \rangle_0 = \langle 1 - B^+B \rangle_0 = 1$$

Suppose $t_1 > t_2 > t_3 = t_4$

$$X = e^{-i\Omega_0(t_1 - t_2)} \langle BB^+ \rangle_0 \langle BB^+ \rangle_0 = e^{-i\Omega_0(t_1 - t_2)}$$