P503

LECTURE 2

Iteration solution for G

$$\left(-\frac{d^2}{dt^2} - \Omega_0^2\right) G_{ij}(t) = \frac{1}{M}\delta_{ij}\,\delta(t) + \lambda \sum_k D_{ik}\, G_{kj}(t) \qquad (*)$$

$\lambda = 0$:

$$G^0_{ij}(t) = \delta_{ij}\, G^0(t) \qquad \text{"unperturbed" GF}$$

$$\left(\frac{d^2}{dt^2} + \Omega_0^2\right) G^0(t) = -\frac{1}{M}\,\delta(t) \qquad (**)$$

Solution derived in Eqs. (1.5.2 - 1.5.11), see also HWK 1

$$\boxed{G^0(t) = -\frac{i}{2M\Omega_0}\, e^{-i\Omega_0|t|}}$$

We solve full Eq. $(*)$ by converting it into an integral equation and use $G_{ij}(t) = \delta_{ij}\, G^0(t)$ when $\lambda = 0$:

$$\boxed{G_{ij}(t) = G^0(t)\,\delta_{ij} + \lambda M \int_{-\infty}^{\infty} dt'\, G^0(t-t') \sum_k D_{ik}\, G_{kj}(t')}$$

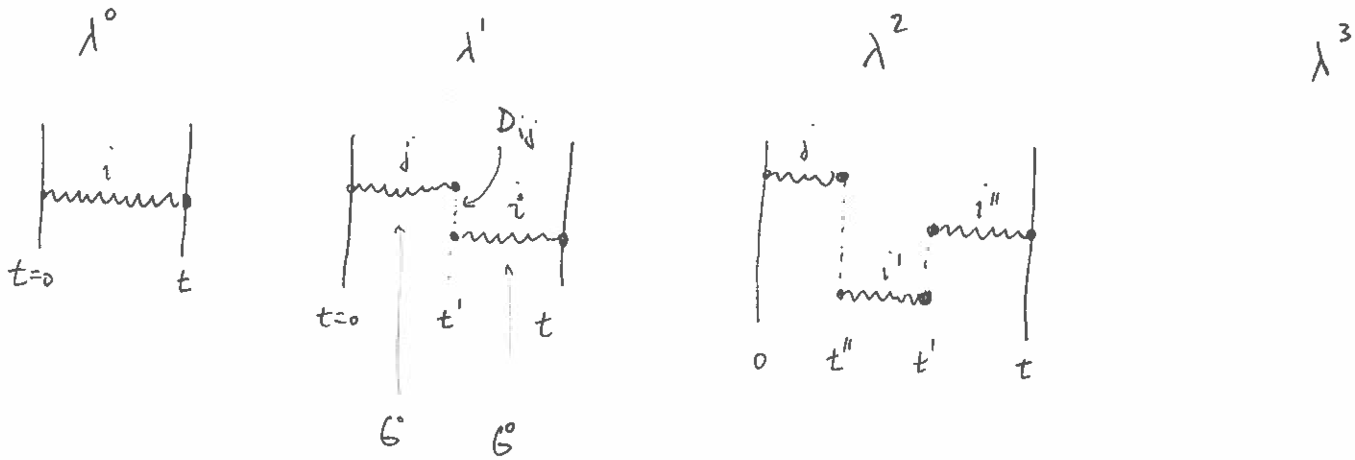
- Verify by applying $\left(\frac{d^2}{dt^2} + \Omega_0^2\right)$ to both sides and using $(**)$.
- This equation for $G_{ij}(t)$ can be solved by iteration:

$$G_{ij}(t) = G^0(t)\,\delta_{ij} + \lambda M \int_{-\infty}^{\infty} dt'\, G^0(t-t')\, D_{ij}\, G^0(t')$$

$$+ (\lambda M)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt'\,dt''\; G^0(t-t')\, D_{ii'}\, G^0(t'-t'')\, D_{i'j}\, G^0(t'')$$

$$+ (\lambda M)^3 \ldots$$

Terms on the RHS can be represented as Feynman diagrams:



Summation of the series

Fourier transform in time:

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, G(\omega) \qquad G(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, G(t)$$

$$G_{ij}(\omega) = G^0(\omega)\,\delta_{ij} + \lambda M\, G^0(\omega) \sum_{\ell} D_{i\ell}\, G_{\ell j}(\omega)$$

Fourier transform in space:

$$G_{ij}(\omega) = \frac{1}{N}\sum_k G_{\vec{k}}(\omega)\, e^{i\vec{k}\cdot(\vec{R_i}-\vec{R_j})}$$

$$G_{\vec{k}}(\omega) = \sum_i G_{ij}(\omega)\, e^{-i\vec{k}\cdot(\vec{R_i}-\vec{R_j})}$$

Note: this is useful if we have translation invariance, namely $G_{ij}(\omega)$ is a function of $(\vec{R_i}-\vec{R_j})$ only

$$\sum_l D_{il} G_{lj}(\omega) = \frac{1}{N^2}\sum_{kk'} D_k G_{k'}(\omega) \sum_l e^{i\vec{k}\cdot(\vec{R_i}-\vec{R_l})} e^{i\vec{k}'(\vec{R_l}-\vec{R_j})}$$

$$= \frac{1}{N^2}\sum_{kk'} D_k G_{k'}(\omega)\, e^{i(\vec{k}\cdot\vec{R_i}-\vec{k}'\cdot\vec{R_j})} \underbrace{\sum_l e^{-i(\vec{k}-\vec{k}')\cdot\vec{R_l}}}_{N\delta_{\vec{k}\vec{k}'}}$$

$$= \frac{1}{N}\sum_k e^{i\vec{k}(\vec{R_i}-\vec{R_j})} D_k G_k(\omega)$$

$$\frac{1}{N}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{R_i}-\vec{R_j})} G_{\vec{k}}(\omega) = G^0(\omega)\,\frac{1}{N}\sum_k e^{i\vec{k}(\vec{R_i}-\vec{R_j})} + \lambda M\, G^0(\omega)\,\frac{1}{N}\sum_k e^{i\vec{k}(\vec{R_i}-\vec{R_j})} D_k G_k(\omega)$$

$$G_k(\omega) = G^0(\omega) + \lambda M\, G^0(\omega) D_k G_k(\omega)$$

$$= G^0(\omega) + \lambda M\, G^0(\omega) D_k G^0(\omega) + (\lambda M)^2 G^0(\omega) D_k G^0(\omega) D_k G^0 + \dots$$

geometric series

$$\boxed{G_k(\omega) = \frac{G^0(\omega)}{1-\lambda M D_k G^0(\omega)}}$$

• To recover $G(t)$ we must

1) Find $G^0(\omega)$
2) Fourier transform $G(\omega) \xrightarrow{FT} G(t)$

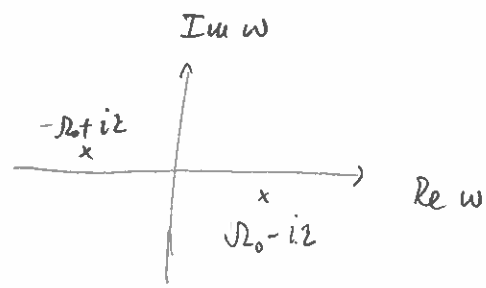
1) $$G^0(t) = -\frac{i}{2M\Omega_0} e^{-i\Omega_0|t|} \qquad FT$$

$$G^0(\omega) = -\frac{i}{2M\Omega_0}\left[\int_{-\infty}^{0} e^{i(\Omega_0+\omega-i\varepsilon)t}\,dt + \int_0^{\infty} e^{i(-\Omega_0+\omega+i\varepsilon)t}\,dt\right]$$

Note: We regularize these integrals by adding a small imag. part to ω: $\omega \to \omega \pm i\varepsilon$
Otherwise integrals strictly do not exist!

$$G^0(\omega) = -\frac{1}{2M\Omega_0}\left[\frac{1}{\omega+\Omega_0-i\varepsilon} - \frac{1}{\omega-\Omega_0+i\varepsilon}\right] = \qquad (\varepsilon>0)$$

$$= \boxed{\frac{1}{M(\omega^2-\Omega_0^2+i\varepsilon)}}$$



2) $$G_k(\omega) = \frac{1}{(G^0)^{-1}-\lambda M D_k} = \frac{1}{M(\omega^2-\Omega_0^2-\lambda M D_k+i\varepsilon)}$$

$$= \boxed{\frac{1}{M(\omega^2-\Omega_k^2+i\varepsilon)}}$$

$$\Omega_k = \Omega_0^2 + \lambda M D_k$$

(i) Now the poles occur for $\underline{\omega = \pm(\Omega_k - i\epsilon)}$

i.e. generally different for different values of $\vec{k}$.

"dispersion relation" or "excitation spectrum"

- interactions shift poles away from unperturbed values $\pm\Omega_0$.

(ii) The t-dep GF is given by inverse FT:

$$G_k(t) = -\frac{i}{2M\Omega_k} e^{-i\Omega_k|t|}$$

(iii) More generally the poles may be situated away from the real axis, e.g.

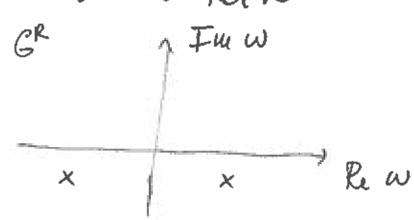
$$G_k(\omega) = \frac{1}{M(\omega^2 - \Omega_k^2 + i\Gamma_k)} \qquad \Gamma_k > 0$$

Then $$G_k(t) \sim e^{-i\Omega_k|t|} \underbrace{e^{-\Gamma_k t}}_{\text{"damping factor"}}$$

$\rightarrow$ the excitation at freq. Ω_k now has a finite lifetime $\tau_k \simeq \Gamma_k^{-1}$

(iv) For retarded/advanced GF we would obtain

$$G_k^{R/A}(\omega) = \frac{1}{M(\omega^2 - \Omega_k^2 \pm i\epsilon\omega)}$$



- Calculation of the ground state energy from $G_k(t)$

$$\Delta E_G = \lim_{t\to 0^+} \tfrac{1}{2} iM \sum_{i\neq j} D_{ij} \int_0^1 d\lambda \, \frac{1}{N}\sum_k G_k(t)\, e^{i\vec{k}\cdot(\vec{R}_i-\vec{R}_j)}$$

$$\sum_{i\neq j} D_{ij}\, e^{i\vec{k}\cdot(\vec{R}_i-\vec{R}_j)} = \frac{1}{N}\sum_j D_k = D_k$$

$$\Delta E_G = \lim_{t\to 0^+} \tfrac{1}{2} iM \int_0^1 d\lambda \sum_k D_k G_k(t)$$

$$G_k(t) = -\frac{i}{2M\Omega_k} e^{-i\Omega_k|t|} \xrightarrow{t\to 0} -\frac{i}{2M\Omega_k}$$

$$= \frac{1}{4}\int_0^1 d\lambda \sum_k \frac{D_k}{\Omega_k}$$

$$= \frac{1}{4}\sum_k \int_0^1 d\lambda \frac{D_k}{\sqrt{\Omega_0^2+\lambda D_k}} = \frac{1}{2}\sum_k \left[\sqrt{\Omega_0^2 + D_k} - \Omega_0\right]$$

- Neutron scattering cross-section can be discussed in a similar manner, see p. 26-28.