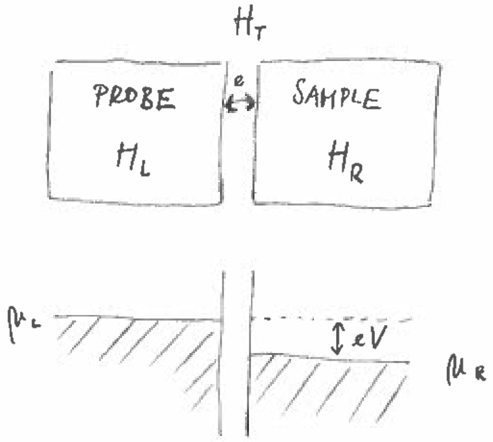
LECTURE 17

Tunneling formalism as a probe of electron spectra in superconductors

[following Mahan sec. 9.3]



- Want to calculate the TUNNELING CURRENT I driven by the voltage difference $eV = \mu_L - \mu_R$

- Use the tunneling Hamiltonian

$$H = \underbrace{H_L + H_R}_{H_0} + H_T \qquad H_T = \sum_{k,p} \left(T_{kp}\, c_k^{+} c_p + h.c. \right) \tag{1}$$

T_{kp}: tunneling matrix element; c_k^{+}: R; c_p: L

- Obtain the current I from the derivative of the number operator $N_L = \sum_p c_p^{+} c_p$

$$I(t) = -e \langle \dot{N}_L(t) \rangle$$

$$\dot{N}_L = i\,[H, N_L] = i\,[H_T, N_L] \tag{2}$$

$$= i \sum_{p,k} \left(T_{pk}\, c_k^{+} c_p - T_{pk}^{*}\, c_p^{+} c_k \right)$$

- To evaluate $\langle \dot{N}_L \rangle$ we go to the interaction picture (with $H_T = H'$) and expand the S matrix to first order:

$$I(t) = -ei \int_{-\infty}^{t} dt' \langle [N_L(t), H_T(t')] \rangle \qquad (3)$$

$$H_T(t') = e^{iH_0 t'} H_T e^{-iH_0 t'} \quad \text{etc.}$$

- Because we have $\mu_L \neq \mu_R$ we must treat chemical potentials carefully. Define

$$\left. \begin{aligned} K_R &= H_R - \mu_R N_R \\ K_L &= H_L - \mu_L N_L \end{aligned} \right\} \quad K_0 = K_R + K_L = H_0 - \mu_R N_R - \mu_L N_L \qquad (4)$$

$$\begin{aligned} H_T(t) &= e^{iK_0 t} \left[e^{it(\mu_L N_L + \mu_R N_R)} H_T e^{-it(\mu_L N_L + \mu_R N_R)} \right] e^{-iK_0 t} \\ &= e^{iK_0 t} \sum_{kp} \left[T_{kp} e^{it(\mu_R - \mu_L)} c_k^{+} c_p + \text{h.c.} \right] e^{-iK_0 t} \end{aligned} \qquad (5)$$

- express $N_L(t)$ in a similar way, substitute to Eq. (3) and simplify. We obtain $I(t) = I_s(t) + I_J(t)$

$$\left[\begin{aligned} I_s(t) &= e \int_{-\infty}^{\infty} dt' \, \theta(t-t') \left\{ e^{ieV(t'-t)} \langle [A(t), A^{+}(t')] \rangle + \text{c.c.} \right\} \\ I_J(t) &= e \int_{-\infty}^{\infty} dt' \, \theta(t-t') \left\{ e^{-ieV(t+t')} \langle [A(t), A(t')] \rangle + \text{c.c.} \right\} \\ A(t) &= \sum_{kp} T_{kp} \, c_k^{+}(t) \, c_p(t) \end{aligned} \right] \qquad (6)$$

- I_s corresponds to the single-electron tunneling which we will consider here

- I_J describes the Cooper pair tunneling associated with the Josephson effect – important when both L and R systems are superconductors, otherwise $I_J = 0$.

- The integrand in $I_s(t)$ has the structure of a RETARDED GF for quantity $A(t)$. Thus we define

$$X_{Ret}(t) = -i\theta(t)\langle[A(t), A^+(0)\rangle$$

$$\text{FT:} \quad X_{ret}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} X_{ret}(t) \tag{7}$$

$$I_s = ie\left[X_{ret}(-eV) - X_{ret}(-eV)^*\right] = -2e\, Im\left[X_{ret}(-eV)\right]$$

- We now evaluate the Matsubara version of X using the Wick's theorem

$$\begin{aligned} X(\omega_n) &= -\int_0^{\beta} d\tau\, e^{i\omega_n\tau}\langle T_\tau A(\tau) A^+(0)\rangle \\ &= -\sum_{\substack{kp\\k'p'}} T_{kp} T^*_{k'p'} \int_0^{\beta} d\tau\, e^{i\omega_n\tau}\langle T_\tau c^+_k(\tau) c_p(\tau) c^+_{p'} c_{k'}\rangle \\ &= \sum_{kp} T_{kp} T^*_{kp} \int_0^{\beta} d\tau\, e^{i\omega_n\tau} \underbrace{\langle T_\tau c_k c^+_k(\tau)\rangle}_{g_R(\vec{k}, -\tau)} \underbrace{\langle T_\tau c_p(\tau) c^+_p\rangle}_{g_L(p, \tau)} \end{aligned} \tag{8}$$

$$\boxed{X(\omega_n) = \sum_{kp} |T_{kp}|^2 \frac{1}{\beta}\sum_m g_L(\vec{p}, \varepsilon_m)\, g_R(\vec{k}, \varepsilon_m - \omega_n)} \tag{9}$$

- $$g(\vec{k}, \varepsilon_m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varepsilon \, \frac{A(\vec{k}, \varepsilon)}{i\varepsilon_m - \varepsilon}$$ ← use the spectral representation

$$\frac{1}{\beta} \sum_m g_L(\vec{p}, \varepsilon_m) \, g_R(\vec{k}, \varepsilon_m - \omega_n) =$$ ↙ evaluate the Matsubara sum

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\varepsilon \, d\varepsilon' \, A_L(\vec{p}, \varepsilon) \, A_R(\vec{k}, \varepsilon') \underbrace{\frac{1}{\beta} \sum_m \frac{1}{i\varepsilon_m - \varepsilon} \, \frac{1}{i\varepsilon_m - i\omega_n - \varepsilon'}}_{\frac{n_F(\varepsilon) - n_F(\varepsilon')}{\varepsilon - \varepsilon' - i\omega_n}} \qquad (10)$$

- Analytic continuation:

$i\omega_n \longrightarrow \omega + i\varepsilon$, Im

$$\to \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\varepsilon \, d\varepsilon' \, A_L(\vec{p}, \varepsilon) \, A_R(\vec{k}, \varepsilon') \underbrace{\mathrm{Im} \, \frac{n_F(\varepsilon) - n_F(\varepsilon')}{\varepsilon - \varepsilon' - \omega - i\varepsilon}}_{[n_F(\varepsilon) - n_F(\varepsilon')] 2\pi \, \delta(\varepsilon - \varepsilon' - \omega)} \qquad (11)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varepsilon \, [n_F(\varepsilon) - n_F(\varepsilon - \omega)] \, A_L(\vec{p}, \varepsilon) \, A_R(\vec{k}, \varepsilon - \omega)$$

- The final result (the "tunneling current formula")

$$\boxed{\begin{aligned} I_s &= -2e \, \mathrm{Im} \, [X_{ret}(-eV)] \\ X_{ret}(\omega) &= i \sum_{k,p} |T_{kp}|^2 \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} [n_F(\varepsilon) - n_F(\varepsilon - \omega)] \, A_L(\vec{p}, \varepsilon) \, A_R(\vec{k}, \varepsilon - \omega) \end{aligned}} \qquad (12)$$

- Assume that the probe (H_L) is made from a simple metal whose spectral function is

$$A_L(\vec{p},\omega) = 2\pi\,\delta(\omega - \xi_p) \qquad \xi_p \text{ - electron dispersion}$$

$$X_{ret}(\omega) = i \sum_{k,p} |T_{kp}|^2 \left[n_F(\xi_p) - n_F(\xi_p - \omega) \right] A_R(\vec{k}, \xi_p - \omega) \tag{13}$$

→ we need the spectral function of the superconductor A_R

$$\hat{g}(\vec{k},\omega_n) = - \frac{i\omega_n + \xi_k \tau^3 + \Delta\tau^1}{\omega_n^2 + \xi_k^2 + \Delta^2} \qquad i\omega_n \to \omega + i\varepsilon$$

$$\hat{G}_{ret}(\vec{k},\omega) = - \frac{\omega + \xi_k \tau^3 + \Delta\tau^1}{-(\omega+i\varepsilon)^2 + \xi_k^2 + \Delta_k^2} \xrightarrow{(1,1)} \frac{\omega + \xi_k}{(\omega+i\varepsilon)^2 - E_k^2} \tag{14}$$

$$\hat{G}_{ret}(\vec{k},\omega)_{11} = \frac{u_k^2}{\omega + i\varepsilon - E_k} + \frac{v_k^2}{\omega + i\varepsilon + E_k} \;) \qquad \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{\xi_k}{E_k}} \tag{15}$$

$$\boxed{A_R(\vec{k},\omega) = 2\pi \left[u_k^2\,\delta(\omega - E_k) + v_k^2\,\delta(\omega + E_k) \right]}$$

← spect. function of a BCS super conductor

- substitute into (13) and evaluate the current.

- commonly used approximation is $|T_{kp}|^2 = T^2$

$$\sum_k \to \int d^3k \to N_R \int d\xi_k$$

$$I_S = + \underbrace{2e\, T^2 N_L N_R}_{\sigma_0} \int d\xi_k\, d\xi_p \left[n_F(\xi_p) - n_F(\xi_p + eV) \right] A_R(\vec{k}, \xi_p + eV)$$

"normal-state conductance"

- The quantity actually measured is "differential conductance"

$$g(V) = \frac{dI}{dV} \simeq e\sigma_0 \int d\xi_k\, d\xi_p \underbrace{\left(- \frac{\partial n_F(\xi_p)}{\partial \xi_p} \right)}_{\xrightarrow{T \to 0}\ \delta(\xi_p)} A_R(\vec{k}, \xi_p + eV)$$

$$\xrightarrow{T=0} e\sigma_0 \int d\xi_k\, A_R(\vec{k}, eV) \qquad (16)$$

- For a BCS superconductor this can be evaluated (see e.g. Mahan's book):

$$\boxed{g(V) = eV\sigma_0\, \mathrm{Re}\left(\frac{1}{\sqrt{(eV)^2 - \Delta^2}} \right)} \qquad (17)$$

"BCS tunneling conductance"

Note $g(V) = 0$ for $|eV| < \Delta$

$g(V) \to \sigma_0 \quad |eV| \gg \Delta$