LECTURE 11

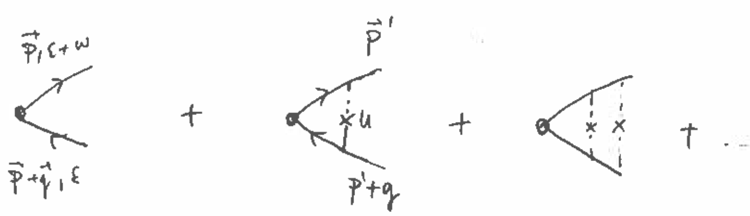
To perform the ladder sum we define

$$\vec{\Pi}(\vec{p};\vec{q};\varepsilon,\omega) = \sum_{p'} \overline{F(\vec{p},\vec{p}'-\vec{q};\varepsilon+\omega)\,F(\vec{p}';\vec{p}+\vec{q};\varepsilon)}\;(\vec{p}'-\tfrac{1}{2}\vec{q})$$

$$\mathcal{R}^{T}_{\alpha\beta}(\vec{q};\omega) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} d\varepsilon \sum_{p} (\vec{p}+\tfrac{1}{2}\vec{q})_{\alpha}\,\Pi_{\beta}(\vec{p};\vec{q};\varepsilon,\omega)$$

$\vec{\Pi}$ can be obtained by iteration of the following Eq:

$$\vec{\Pi}(\vec{p};\vec{q};\varepsilon,\omega) = G(p,\varepsilon+\omega)\,G(p+q,\varepsilon)$$
$$\times\left[(\vec{p}+\tfrac{1}{2}\vec{q}) + \sum_{p'} N u^2(\vec{p}-\vec{p}')\,\vec{\Pi}(\vec{p}';q;\varepsilon,\omega)\right]$$



– see sec. 5.7 for details of the solution

In order to calculate d.c. conductivity we require

$$R_{\alpha\beta}(\vec{q};\omega) = \mathcal{R}_{\alpha\beta}(\vec{q};\omega) - ne\,\delta_{\alpha\beta}$$

in the limit $\omega\to 0$, $q\to 0$. The $\omega\to 0$ limit is tricky.

At frq ω the conductivity is defined by Ohm's law

$$J_\alpha(\omega) = \sigma_{\alpha\beta}(\omega) E_\beta(\omega) = i\omega\, \sigma_{\alpha\beta}(\omega) A_\beta(\omega)$$

A constant electric field corresponds to a divergent vector potential $\vec{A}(\omega) = \vec{E}(\omega)/i\omega$ as $\omega \to 0$. and the d.c. conductivity is obtained from the resp. function as

$$\sigma_{\alpha\beta}(\omega) = \lim_{\omega\to 0} \frac{R_{\alpha\beta}(\omega)}{i\omega} = \lim_{\omega\to 0} \frac{\mathcal{R}_{\alpha\beta}(\omega) - ne^2\delta_{\alpha\beta}/m}{i\omega}$$

(CGS units)

So, we must have $\mathcal{R}_{\alpha\beta}(\omega) \longrightarrow ne\,\delta_{\alpha\beta}$ as $\omega \to 0$ in order to get finite $\sigma_{\alpha\beta}(0)$. It turns out that proper inclusion of vertex corrections is essential to ensure this result.

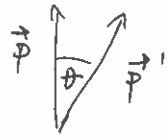
- After a lengthy calculation (see sec. 5.7) one obtains, for a simple free-electron metal with short-range impurities,

$$\overline{\mathcal{R}^T_{\alpha\beta}}(\omega) = \delta_{\alpha\beta}\, \frac{\rho p_F^2}{3}\, \frac{1}{1 - i/\omega\tau_{TR}}$$

where $\rho = mp_F/2\pi^2$ is the electron density of states at the Fermi level, $n = p_F^3/6\pi^2$ is the electron density, and

$$\frac{1}{\tau_{TR}} = 2\pi \rho\, n_{imp} \frac{1}{4\pi} \int d\Omega_{p'} \underline{(1-\cos\theta)}\, U^2(\vec{p}-\vec{p}\,')$$

is the transport lifetime. Here $U(\theta)$ is $U(\vec{p})$ evaluated at the Fermi surface, i.e. for $|\vec{p}| = p_F$ and θ is the polar angle



• τ_{TR} is unaffected by "forward scattering" i.e. when $\theta \simeq 0$.

$$\overline{R}^{T}_{\alpha\beta}(\omega) = \delta_{\alpha\beta} \frac{ne^2}{m} \frac{1}{1 - i|\omega|\tau_{TR}} \simeq \delta_{\alpha\beta} \frac{ne^2}{m} \Big(\underset{\uparrow}{1} + i|\omega|\tau_{TR} + \dots \Big)$$

exactly cancels the diamagnetic term

$$\boxed{\begin{aligned} R_{\alpha\beta}(\omega) &= \delta_{\alpha\beta}\, i\omega\sigma \\ \sigma &= \frac{ne^2\tau_{TR}}{m} \end{aligned}}$$

$\leftarrow$ Ohm's law, obtained from fully quantum theory.

Interacting electron gas

- Read intro to Ch 6 (p. 123-124)

- we will now include e-e interactions

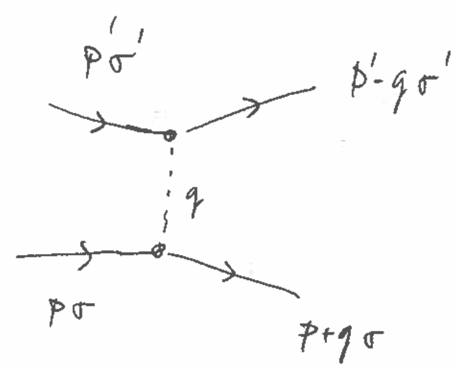
$$H = H_0 + H_1$$

$$H_0 = \sum_i \frac{p_i^2}{2m} \qquad H_1 = \frac{1}{2} \sum_{i \neq j} V(\vec{x}_i - \vec{x}_j)$$

- Second quantized description

$$H_0 = \sum_{p\sigma} \varepsilon_p c^{\dagger}_{p\sigma} c_{p\sigma}$$

$$H_1 = \frac{1}{2} \sum_q V(\vec{q}) \sum_{\substack{pp' \\ \sigma\sigma'}} c^{\dagger}_{p+q\sigma} c^{\dagger}_{p'-q\sigma'} c_{p'\sigma'} c_{p\sigma}$$



$$V(\vec{q}) = \int d^3\vec{x}\, e^{i\vec{q}\cdot\vec{x}} V(\vec{x})$$

The Hartree-Fock approximation - review

- approximates H_1 by assuming that fluctuations are small

$$c^{\dagger}_{p+q\sigma} c^{\dagger}_{p'-q\sigma'} c_{p'\sigma'} c_{p\sigma} \simeq c^{\dagger}_{p+q\sigma} \overbrace{\langle c^{\dagger}_{p'-q\sigma'} c_{p'\sigma'} \rangle}^{\delta_{p'-q,p'} \delta_{\sigma'\sigma'} f_{p'\sigma'}} c_{p\sigma} + \ldots$$

$$- c^{\dagger}_{p+q\sigma} c_{p'\sigma'} \langle c^{\dagger} \ldots c \ldots \rangle \mp$$

$$= \delta_{q,0} f_{p'\sigma'} c^+_{p\sigma} c_{p\sigma} - \delta_{p'-q,p} \delta_{\sigma\sigma'} f_{p\sigma} c^+_{p+q} c_{p'\sigma'}$$

$$H^{HF} = \sum_{p\sigma} \varepsilon_p c^+_{p\sigma} c_{p\sigma} + V(0) \Big(\sum_{p'\sigma'} f_{p'\sigma'} \Big) \sum_{p\sigma} \varepsilon_p c^+_{p\sigma} c_{p\sigma}$$
$$- \sum_{p'} V(\vec{p}\,' - \vec{p}) f_{p'\sigma} \sum_{p} c^+_{p\sigma} c_{p\sigma}$$

$$H^{HF} = \sum_{p\sigma} \varepsilon^{HF}_p c^+_{p\sigma} c_{p\sigma}$$

$$\varepsilon_p^{HF} = \varepsilon_p + V(0) \sum_{p'\sigma'} f_{p'\sigma'} - \sum_{p'} V(p'-p) f_{p'\sigma}$$

Hartree term — Fock (exchange) term

HF from a perturbation expansion of G

Consider one-particle GF

$$G(\vec{p}, t) = -i \langle \Psi_0 | T [\tilde{c}_p(t) \, \tilde{c}^+_p(0)] | \Psi_0 \rangle \quad \leftarrow \text{Heisenberg}$$

$$= -i \frac{\langle \phi_0 | T [c_p(t) \, c^+_p(0) \, S] | \phi_0 \rangle}{\langle \phi_0 | S | \phi_0 \rangle} \quad \leftarrow \text{Interaction}$$

$$c_p(t) = e^{iH_0 t} c_p e^{-iH_0 t} \qquad S = U(\infty, -\infty) = T \exp\Big\{ -i \int_{-\infty}^{\infty} dt_1 H_1(t_1) \Big\}$$

- It is easy to show that

$$c_p(t) = e^{-i\varepsilon_p t} c_p \,, \qquad c_p^{+}(t) = e^{i\varepsilon_p t} c_p^{+}$$

Feynman - Dyson :

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \ldots dt_n \; T\left[H_1(t_1) \ldots H_1(t_n) \right]$$

We will employ Wick's theorem to construct FD expansion of $G(\vec{p}, t)$ in terms of unperturbed $G^{0}(\vec{p}, t)$ obtained previously.

- First order $(n=1)$

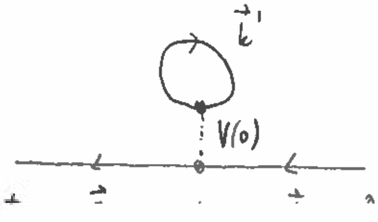
$$G^{(1)}(\vec{p}, t) = (-i)^2 \, \frac{1}{2} \sum_{q k k'} V(q) \int_{-\infty}^{\infty} dt_1$$

(2)

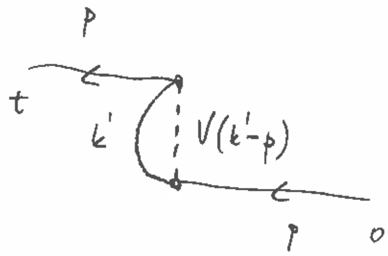
$$\langle \phi_0 | \, T\left[c_p(t) \, c_p^{+}(0) \, c_{k+q}^{+}(t_1) \, c_{k'-q}^{+}(t_1) \, c_{k'}(t_1) \, c_k(t_1) \right] | \phi_0 \rangle$$

(1)

(1) $$i^3 \sum_{k'} \int_{-\infty}^{\infty} dt_1 \, V(0) \, G^{0}(p, t - t_1) \, G^{0}(k', 0^-) \, G^{0}(p, t_1)$$



(2) $$-i^3 \sum_{k'} V(k'-p) \int_{-\infty}^{\infty} dt_1 \; G^0(\vec{p}, t-t_1)\, G^0(k', 0^-)\, G^0(p, t_1)$$



So, we have to first order (after FT in time)

$$G^{(1)}(p,\varepsilon) = -i\, G^0(p,\varepsilon) \left[V(0) \sum_{k'} G^0(k', t=0^-) \right] G^0(p,\varepsilon)$$
$$+ i\, G^0(p,\varepsilon) \left[\sum_{k'} V(k'-p)\, G^0(k', t=0^-) \right] G^0(p,\varepsilon).$$