PHYS 503	Problem set # 3	Due: March 3, 2022, 2pm

1. (10 points) In some applications it is useful to define the phonon Green's function directly in momentum space, $D_q(t) = -i\langle T[u_q(t)u_{-q}(0)]\rangle$, where the displacements $u_q = (a_q + a_{-q}^\dagger)/\sqrt{2M\Omega_q}$ are given in terms phonon creation/annihilation operators. These satisfy canonical bosonic commutation relations $[a_q, a_{q'}^\dagger] = \delta_{qq'}$. The unperturbed phonon Hamiltonian in this notation reads

$$H_{\rm ph} = \sum_q \Omega_q a_q^\dagger a_q \tag{1}$$

where Ω_q is the phonon dispersion.

a) Show by a direct calculation that the frequency-domain phonon GF associated with Hamiltonian (1) is given by $D_q^0(\omega) = 1/M(\omega^2 - \Omega_q^2 + i\eta)$.

b) Find the corresponding Matsubara propagator $\mathcal{D}_q^0(\omega_n)$.

c) From $\mathcal{D}_q^0(\omega_n)$ deduce the retarded phonon GF and the spectral function $A_q(\omega)$.

2. (20 points) Electron motion in crystals is in general affected by the underlying lattice vibrations. The simplest model capturing this physics is defined by the Holstein Hamiltonian

$$H = H_{\rm el} + H_{\rm ph} + \Lambda \sum_{q,k} (a_q^\dagger + a_{-q}) c_{k-q}^\dagger c_k, \tag{2}$$

where $H_{\rm el} = \sum_k \epsilon_k c_k^\dagger c_k$ is the unperturbed electron Hamiltonian, $H_{\rm ph}$ is defined in Eq. (1) and Λ denotes the electron-phonon coupling constant. Treating the last term in Eq. (2) as a perturbation the leading diagrams in the $T = 0$ Feynman-Dyson expansion of the electron propagator are given in the figure below. Solid and dashed lines represent the unperturbed electron and phonon propagators, respectively.

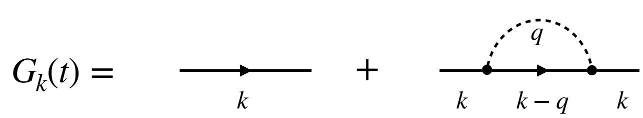
a) Perform the required Feynman-Dyson expansion to second order in Λ and write down the expressions that correspond to the diagrams indicated in the figure.

b) Show that, upon Fourier transforming, the result of part (a) can be written as

$$G_k(\omega) = G_k^0(\omega) + G_k^0(\omega)\Sigma_k(\omega)G_k^0(\omega). \tag{3}$$

Give an expression for the electron self energy $\Sigma_k(\omega)$.

c) Now perform the analogous expansion for the phonon propagator $D_q(t)$. Draw the leading Feynman diagrams and give the expression for phonon self energy $\Pi_q(\omega)$.



3. (10 points) Now consider the electron-phonon problem posed by the Holstein Hamiltonian (2) at a finite temperature T.

a) Perform the Feynman-Dyson expansion of the Matsubara frequency phonon propagator $\mathcal{D}_q(\tau)$ to second order in Λ, find the expression for the Matsubara phonon self energy $\Pi_q(\omega_n)$ in terms of $\mathcal{G}^0_k(\nu_n)$.

b) Using the method of contour integration evaluate the Matsubara sum in the expression for $\Pi_q(\omega_n)$ obtained in part (a).