PHYS455 Homework set V, Wednesday, Feb 23, 2005
(Due at 11am, March 2, 2005)

1)(0pt) Read the textbook, page 290-303. Understand black body radiation. You will not receive points for this part of the assignment but you will be rewarded in the future!

2)(30pt) To characterize classical EM fields (in a cubic box $V = L^3$), we introduce $\mathbf{A_q}$ vector potential for each wave vector

$$\mathbf{q} = \frac{2\pi}{L}(n_x, n_y, n_z). \tag{1}$$

$n_{x,y,z}$ are integers. Given $\mathbf{A_q}$ for each $\mathbf{q}$, one then completely determines $\mathbf{A}(\mathbf{x})$ via the fourier transformation

$$\mathbf{A}(\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{x}). \tag{2}$$

In quantum EM field theory, EM fields are described by wave functions

$$\Psi_{N_{q1}}(\mathbf{A}_{q1}) \times \Psi_{N_{q1}}(\mathbf{A}_{q2})... \times \Psi_{N_{qj}}(\mathbf{A}_{qj})... \tag{3}$$

Here qj is a wave vector. $\Psi_{N_q}(\mathbf{A}_q)$, $(q = q1, q2, q3...)$ is the eigenstate of the following Harmonic oscillator Hamiltonian for the q-component,

$$H_q = \frac{c^2}{2}\Pi_q^2 + \frac{q^2}{2}A_q^2. \tag{4}$$

Π_q and A_q are a pair of conjugate variables, $[\Pi_q, A_q] = i\hbar$ (one can draw an analogy to a Harmonic oscillator in quantum mechanics as we did before; see more in P455L4.pdf). And the Hamiltonian of EM fields is

$$H = \sum_q H_q. \tag{5}$$

So in this context, we characterize a state $\mathbf{s}$ of EM fields in terms of eigen values for individual q-component harmonic oscillators

$$(N_{q1}, N_{q2}, ...N_{qj}, ...). \tag{6}$$

Each N_q varies from 0 to infinity. And the energy of this state is

$$E_s = \sum_q (N_q + \frac{1}{2})\hbar\omega_q, \hbar\omega_q = \hbar c q \tag{7}$$

where the sum is over all wave vectors.

a) Show that the partition function $Z = \sum_s \exp(-\beta E_s)$ is

$$Z = \prod_q \exp(-\frac{\beta}{2}\hbar\omega_q)\frac{1}{1 - \exp(-\beta\hbar\omega_q)}. \tag{8}$$

Hint: $\sum_s = \sum_{N_{q1}=0}^{\infty}\sum_{N_{q2}=0}^{\infty}\sum_{N_{q3}=0}^{\infty}....$

b) The total energy of EM fields is $U = -\partial \ln Z/\partial\beta$. Show that the energy has the following form

$$U = U_0 + U_T, U_T = \sum_q \hbar\omega_q f_q, f_q = \frac{1}{\exp(\beta\hbar\omega_q) - 1}. \tag{9}$$

Express U_0 explicitly in terms of $\hbar\omega_q$; this is the zero point energy, an energy carried by vacuum. This term was neglected in our discussions about photons for a good reason.

c) The second term in the expression for U, U_T vanishes at $T = 0$. Carry out the sum and express the result in terms of kT, c speed of light etc. This is the total energy of photons discussed in one of lectures.

d) Calculate the total number of photons at given temperature T

$$N(T) = \sum_q f_q. \tag{10}$$

Express the result in terms of temperature kT, c etc.

e) The pressure of photons is $P = -(\partial U/\partial V)_N$. Show that

$$P = \frac{1}{3}\frac{U_T}{V} \tag{11}$$

which vanishes at $T = 0$.

Compare this with the degeneracy pressure (See Eq.(7.44). on page 275 in the textbook.)

3) (30pt) Derive the Stefan's Law for photons in a two dimensional space. The momenta of photons can only have x and y components.

a) Calculate U_T (defined in Eq.9). Hint: in this case

$$\sum_{\mathbf{q}} \to L^2 \int \frac{dq_x}{2\pi}\frac{dq_x}{2\pi}. \tag{12}$$

L^2 is the area of the 2d space. Express the result in terms of kT, c etc. (An integral might appear in the result; you are required to argue that the integral is convergent and is order of unity but not required to evaluate the integral.)

b) Following steps used to derive Eq. (7.98) on page 302, the textbook, study Stefan's law in the two dimensional space. Calculate power per unit *length*.