Phys 455 Statistical Mechanics Session IV
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Various Bosons

1 Origin of Photons

The most nature way to introduce photons is to "quantize" electro-magnetic fields, just like to quantize the motion of a particle in quantum mechanics. This procedure is usually outlined in standard textbooks on quantum field theories (photons are massles spin-one bosons); formally it is therefore beyond the scope of statistical mechanics. (See more discussions in Itzykson and Zuber, Quantum Field Theory, McGraw-Hill Inc.)

I however intend to give some elaborated discussions on the concept of quantum EM fields for it is important to understand the need to introduce objects such as photons. The concept of photons is a natural consequence of quantum EM fields. The relation between

Quantum EM fields and **Classical EM waves**

is equivalent to the relation between

Quantum Mechanics and **Classical Mechanics**.

The only difference is that to describe a particle in classical mechanics or quantum mechanics, we only need to study (x, y, z) of the particle or the wave function $\Psi(x, y, z)$; in the EM theory and Quantum EM field theory, to describe a state of EM fields, we need to introduce a **field**

$$\mathbf{A}(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum A_{\mathbf{q}} \exp(i\mathbf{q} \cdot \mathbf{x}) \tag{1}$$

with infinite degrees of freedom. That is at each point $\mathbf{x}$ (or more conveniently at each wave vector $\mathbf{q}$), we need a vector potential $\mathbf{A}(\mathbf{x})$ (or $\mathbf{A}(\mathbf{q})$) to specify EM fields.

At the classical level, for a given EM field $\mathbf{A}(\mathbf{x})$ is given at each point $\mathbf{x}$ at a given time. This is also what happens in classical mechanics: the coordinate of a particle is always specified at a given moment.

As emphasized above, $\mathbf{A}(\mathbf{x})$ is a "field" introduced to specify the dynamics of EM fields. Just like in QM, we need a wave function $\boldsymbol{\Psi}(x, y, z)$ to characterize a particle, in quantum EM field theory, we need a wave function

$$\Psi(\mathbf{A}(\mathbf{x_1}), \mathbf{A}(\mathbf{x_2}), ..., \mathbf{A}(\mathbf{x_n}), ...) \tag{2}$$

to characterize properties of EM fields. In this equation, I have discretized our space and at each point $\mathbf{x_1}$ there is a vector potential. The wave function yields information about the amplitude of having

$$\mathbf{A}(\mathbf{x}_j) \tag{3}$$

at any point $\mathbf{x}_j$.

More practically, we can use $\mathbf{A}_q$ to specify EM fields. If we know the value of $\mathbf{A}_q$ for each given q, we are able to determine $\mathbf{A}(\mathbf{x})$ at each point. So we can introduce alternatively,

$$\Psi(\{\mathbf{A}_q)\}), \{\mathbf{A}_q\} = (\mathbf{A}_{q_1}, \mathbf{A}_{q_2}, ..., \mathbf{A}_{q_n}, ...) \tag{4}$$

The analogy between EM fields and motion of particles can be drew at a quantitative level if we assume particles are harmonic oscillators. For instance consider the following Hamiltonian,

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2, [P, x] = i\hbar. \quad (5)$$

We have learned in quantum mechanics courses that this defines a harmonic oscillator. The eigen values are

$$\epsilon_n = (n + \frac{1}{2})\hbar\omega. \quad (6)$$

Furthermore, unlike in classical mechanics where a particle should be at

$$x = 0 \quad (7)$$

to minimize the energy, in QM, a ground state is a wave packet where

$$< x^2 >= \frac{\hbar}{m\omega} \quad (8)$$

representing the unique phenomenon of the zero point motion of particles in QM.

It turns out(by this I mean if one goes through the first three chapters of quantum field theory textbooks, one finds the following fact; alternatively one can stretch imagination to agree with me.) that for each Fourier component A_q, the Hamiltonian is identical to a harmonic oscillator

$$H = \sum_q H_q, H_q = \frac{c^2}{2}\Pi_q^2 + \frac{q^2}{2}A_q^2. \quad (9)$$

And the dynamics of EM fields is given by an infinite number of harmonic oscillators with various characteristic frequencies. Let us take it as a fact.

So, it is obvious that for a given q,

$$m\omega^2 \to q^2, \frac{1}{m} \to c^2. \quad (10)$$

And therefore the energy spacing for A_q component is

$$\hbar\omega_q, \omega_q = c|q|. \quad (11)$$

c is the speed of light.

The eigen state of Hamiltonian is

$$\Psi(\{\mathbf{A}_q\}) = \Psi_{N_1}(\mathbf{A}_{q1}) \times \Psi_{N_2}(\mathbf{A}_{q2}) \times \Psi_{N_3}(\mathbf{A}_{q3})... \times \Psi_{N_j}(\mathbf{A}_{q4})... \quad (12)$$

$(N_j + 1/2)\hbar\Omega_{q_j}$ is the eigen value of q_jth component.

Quite amazing! We are actually able to write a wave function for EM fields in our universe. And the vacuum has many activities, much more than anyone could have imagined!

It is easy to show that the partition function is simply

$$Z = \prod_q Z_q, Z_q = \frac{1}{1 - \exp(-\beta\hbar\omega_q)}. \quad (13)$$

Exercise 1: Show this partition function.

Note that in deriving this I have chosen $1/2\hbar\omega_q$ as $\epsilon = 0$ point. One can actually show this would not affect the physical quantity which interests us. If we choose to use $\epsilon_n = (n + 1/2)\hbar\omega_q$, the partition would become

$$Z_q \to Z_q \exp(-\beta \frac{1}{2}\hbar\omega_q). \tag{14}$$

The energy calculated using two different partition functions has the following relation

$$U \to \sum_q \frac{1}{2}\hbar\omega_q + U. \tag{15}$$

The first term is actually the zero point energy of vacuum which usually is not measurable. The heat capacity etc is only determined by the second term and therefore doesn't depend on the choices we made about the energy reference point. In the particular choice made in the lecture

$$\epsilon_n = n\hbar\omega_q \tag{16}$$

I have basically neglected all the zero point energy for a very good reason.

Let us now look at the partition function Z_q. The partition function of Z_q is actually equivalent to that of bosons at $\mu = 0$; this suggests that EM fields should be equivalent to bosons with zero chemical potential. These bosons are precisely so-called "photons".

In terms of harmonic oscillators, one photon state (with q) corresponds to the $n = 1$ level of the qth harmonic oscillator. Two photons at q mode stand for the $n = 2$ level of the qth H.O..

To summarize, we provide two different interpretations about the wave function written above for the EM fields. For each state specified below,

$$(N_{q_1}, N_{q_2}, N_{q_3}, ..., N_{q_j}, ...), \tag{17}$$

the energy is

$$E = \sum_q N_q \hbar\omega_q \tag{18}$$

we either say

a) EM fields are occupying N_qth quantum level of the qth H.O..

or

b) there are N_q photons with momentum q.

In the first representation, I emphasize the wave-like behavior of EM fields, while in the later representation, I emphasize elementary "quanta" in EM fields, or photons. This is the duality in Quantum EM fields. Of course they yield exactly same results for any quantity that interests us.

So we go through this example to demonstrate that photons are elementary particles in EM theories, and E and B are secondary. (I understand and imagine this is counter-intuitive to some of you; unfortunately this is what God designed at the beginning. Submitting oneself to God is always a valid approach!! Alternatively you can visit me during office hours which might be practically more helpful.)

I hope that now the concept of photons appears to be natural to most of you.