Phys 455 Statistical Mechanics Session II
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Fermi-Dirac statistics

1 Introduction

"Happy families are all alike; every unhappy family is unhappy in its own way".
—Leo Tolstoy, *Anna Karenina*

Each fermion just like a unhappy family of Tolstoy's intends to live its own distinct life which we have called a quantum state. Bosons on the other hand can live identical lives. When that happens, especially when the entropy doesn't play an overwhelming role in the free energy, bosons can occupy a single quantum state. That is called Bose-Einstein condensation. Indistinguishable particles are more *emotional* than distinguishable particles. What we have learned in the previous session can be summarized in terms of happy and unhappy families. This is what you have to bring with you even after your summer vacation!!

Let me say it in a slightly different way. Fermions which obey Fermi-Dirac statistics obviously *dislike* each other; a fermion does not tolerate a second fermion in one quantum state and the Pauli exclusion principle exactly expresses this harsh reality of FD statistics. In this sense, Bosons are more tolerating: they are more willing to share a quantum state. They achieve inner happiness because they have done something *platonically-morally right*.

However, dislikeness between fermions is not necessary bad for a system of fermions. If you have been in one of popular Robson St restaurants (say the little Japanese Tapas place at the corner of Robson and Bidwell) , you must have noticed that there are always people waiting outside Friday evening 7 or 8pm. Only when seats are available, people can enter the restaurant to start their dinners. This simple reality in Vancouver westend shows how order is established in a restaurant.

The Pauli exclusion principle as well *enforces* certain unique "order" of fermions. For instance, at T=0 only states within a so-called Fermi surface are occupied but states outside the Fermi surfaces are all empty. In this session I will fully exploit the consequences of the Pauli exclusion principle, or the **dis- likeness** of Fermions **mathematically**.

2 Fermi seas and fermi surfaces

Consider Fermions in a box of size $V = L_x \times L_y \times L_z$. The eigenstates of particles in a box are

$$\begin{aligned}\Psi_{n_x,n_y,n_z} &= \frac{1}{\sqrt{V}} \exp(i\frac{n_x 2\pi}{L_x}x) \times \exp(i\frac{n_y 2\pi}{L_y}y) \times \exp(i\frac{n_z 2\pi}{L_z}z) \\ n_\mu &= 1, 2, 3, ...; \mu = x, y, z.\end{aligned} \quad (1)$$

We have chosen periodical boundary conditions along x, y, z directions.

These are eigenstates for fermions to occupy under the ruling of the Pauli exclusion principle. We label each state as a vector in a three dimensional **momentum** space

$$(n_x, n_y, n_z) \quad (2)$$

where n_μ are integers only. All eigen states then form a lattice in the 3d momentum space. Perfect! So every site of this lattice actually represents an eigenstate with an eigenvalue

$$\epsilon(n_x, n_y, n_z) = \frac{h^2 n_x^2}{2mL_x^2} + \frac{h^2 n_y^2}{2mL_y^2} + \frac{h^2 n_z^2}{2mL_z^2}. \tag{3}$$

Note this is the energy of a particle at given level (n_x, n_y, n_z). For simplicity, we set $L_x = L_y = L_z = L$; $h = 2\pi\hbar$.

FD statistics dictates that all states below μ are filled at $T = 0$. This is obvious from the point of view of the total energy E: particles occupy as many low energy levels as possible before moving to higher energy levels.

So all states satisfy the following condition are occupied

$$\epsilon(n_x, n_y, n_z) \leq \epsilon_F = \mu. \tag{4}$$

The surface defined by the equality above is called the fermi surface (FS) of the fermi gas and the bulk is called fermi sea.

Exercise 1 Calculate the number of quantum states beneath the FS.

For large systems, μ is much larger than the level spacing in the problem. We neglect the discreteness of eigen values in the momentum space. A very useful formula is

$$\sum_{n_x}\sum_{n_y}\sum_{n_z} \to V\int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3}. \tag{5}$$

$V = L^3$.

Hint:

$$\delta n_x = \delta k_x \frac{L_x}{2\pi\hbar}, \sum_x \to \int L_x \frac{dk_x}{2\pi\hbar} \tag{6}$$

Using this formula one finds that

$$\frac{N}{V} = \frac{4\pi k_F^3}{3}\frac{1}{(2\pi\hbar)^3}, \frac{k_F^2}{2m} = \epsilon_F = \mu. \tag{7}$$

k_F is the fermi momentum. So the fermi momentum determines the density of fermions!!

3 Degeneracy pressure

Because fermions carry finite momenta at any finite density, even at $T = 0$, fermions have nonzero kinetic energy which leads to degeneracy pressure of fermi gases.

To appreciate this important aspect of fermi gases, let us recall what happens to "classical" gases describe by the Boltzmann distribution. Let us calculate the partition function of a particle

$$Z_1 = V\int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3}\exp(-\beta\epsilon(\mathbf{k})) = \frac{4\pi V\lambda_T^3}{(2\pi\hbar)^3}\int dx x^2 \exp(-x^2) \tag{8}$$

λ_T is the thermal wave length.

And the energy and pressure of the gas are

$$E = \frac{3}{2}NkT, P = \frac{N}{V}kT \tag{9}$$

a well known result. Note that the pressure goes to zero as the temperature approaches zero because particles lose their momenta.

Exercise 2 Calculate the energy, pressure of a classical gas.

Now let us carry out parallel calculations for fermions.

$$\frac{E}{N} = \frac{V}{N}\int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} f_{FD}(\epsilon_{\mathbf{k}})\epsilon(\mathbf{k}) = \frac{3}{5}\epsilon_F. \tag{10}$$

Very nice discussions about the degenerate gas pressure can be found in textbook, page 272-282.

At last I want to introduce the density of states.

$$\int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \to \int d\epsilon N(\epsilon); N(\epsilon) = \frac{mk(\epsilon)}{4\pi^2\hbar^3}. \tag{11}$$

4 Responses of electrons to Zeeman fields

Spintronic is one of major directions in condensed matter physics. It is interesting to see how electron spins are manipulated these days in laboratories. Here I will only discuss responses of fermi seas to external Zeeman fields.

Most of fermions we are familiar with actually carry spin-1/2. A particle with spin-1/2 can be considered as a magnetic dipole which is coupled to a Zeeman field

$$H = -g\sigma_z H_z \tag{12}$$

g depends on a few internal properties of particles and we simply name it as a g-factor characterizing coupling between spin and external fields. Electron spins are specified by three Pauli matrices in non-relativistic limit but here you can think σ_z as a variable taking either plus (spin-up) or minus one (spin-down).

For an electron with spin-up or down, the total energy is

$$\epsilon(\mathbf{k};\sigma_\mathbf{z}) = \mathbf{\frac{\hbar^2 k^2}{2m} - g\sigma_z H_z}. \tag{13}$$

The Fermi momenta for spin-up(k_{F+}) and spin-down (k_{F-}) particles are different.

$$\frac{\hbar^2 k_{F+}^2}{2m} + \frac{\hbar^2 k_{F-}^2}{2m} = 2\epsilon_F, k_{F+} - k_{F-} = \frac{4mgH_z}{k_{F+} + k_{F-}}. \tag{14}$$

Exercise 4 Calculate the energy as the function of H_z, the magnetization M_z, and the Pauli spin susceptibility χ_s for Fermi gases using the following definitions

$$M_z = \frac{\partial E}{\partial H_z}, \chi_s = \frac{\partial M_z}{\partial H_z}. \tag{15}$$

5 Excitations: Particles and Holes

Before I talk about transport of electrons let me first speak about excitations. And to simplify the situation let us only consider $T = 0$ case.

In the ground state, we have all particles occupying states below the fermi surface. The many-body state of N-particles can be specified as

$$f(\mathbf{k}) = 1, \epsilon_k \leq \epsilon_F; f(\mathbf{k}) = 0, \epsilon_k > \epsilon_F. \tag{16}$$

Consider a state of $N+1$ particles with the extra particle occupying $\mathbf{k}_0(|\mathbf{k}| > k_F)$. $f(\mathbf{k})$ is the same as above except

$$f(\mathbf{k}) = 1, \text{when } \mathbf{k} = \mathbf{k}_0. \tag{17}$$

One can evaluate the momentum, energy, spin and charge of this state. One arrives the conclusion that this state of $N+1$ particles represents a particle with mass m, momentum $\mathbf{k}$, spin 1/2, charge e and velocity $v_F\hat{\mathbf{k}}$. Well, this is a result one should expect!

However, there is another kind of excitations which correspond to removal of a particle from below the fermi surface. The $N-1$ particle state is still defined by Eq.16 except the particle at $\mathbf{k}_0(|\mathbf{k}_0| < k_F)$ is removed, i.e.

$$f(\mathbf{k}) = 0, \text{when } \mathbf{k} = \mathbf{k}_0. \tag{18}$$

One can show that

$$E(N-1) - E(N) = \sum_{|\mathbf{k}| \leq k_F, \mathbf{k} \neq \mathbf{k}_0} (\epsilon(k) - \mu) - \sum_{|\mathbf{k}| \leq \mathbf{k_F}} (\epsilon(k) - \mu) = -(\epsilon_{k_0} - \mu) > 0. \tag{19}$$

Indeed an excitation!!

What I am going to show next is that this hole-like excitation (positively charged) moves in the opposite direction of momentum $\mathbf{k_0}$!! For the given state specified above,

$$Q = \sum_{|\mathbf{k}| \leq k_F, \mathbf{k} \neq \mathbf{k}_0} \mathbf{k} - \sum_{|\mathbf{k}| \leq k_F} \mathbf{k} = -\mathbf{k}_0 \tag{20}$$

So the excitation is of momentum $\mathbf{Q} = -\mathbf{k}_0$ and energy $\epsilon_{\mathbf{Q}} = \mathbf{Q}\mathbf{v}_Q$. Obviously

$$v_{\mathbf{Q}} = v_F\hat{\mathbf{Q}} = -v_F\hat{\mathbf{k}}. \tag{21}$$

The excitation moves along $-\mathbf{k}_0$ direction, very very unusual! This is a distinct feature of a hole-like excitation in fermionic systems.

Exercise 1 Show a hole-like excitation with a $S_z = 1/2$ electron removed from beneath a fermi surface carries $S_z = -1/2$.

6 Collective excitations and Zeroth sounds

TBA

7 2DEG and Quantized Conductance

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8 Charge quantization and Coulomb Blocade

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