Solutions to Phys455 Midterm exam (2005)

a)

$$Z_1 = \frac{L^2}{(2\pi\hbar)^2}\int_0^\infty dp 2\pi p \exp(-\beta\frac{p^2}{2m}). \tag{1}$$

Define $\beta p^2/2m = x$ and take into account $\int_0^\infty dx \exp(-x) = 1$, one obtains

$$Z_1 = \pi\frac{L^2}{\lambda_T^2}, \lambda_T = \frac{2\pi\hbar}{\sqrt{2mkT}}. \tag{2}$$

$N \ll Z_1$ indicates that

$$\frac{N}{L^2} \ll \pi\frac{1}{\lambda_T^2}. \tag{3}$$

This leads to a characteristic temperature $T_c \sim \hbar^2 N 2mL^2$.

b)

$$< N >= \frac{1}{Z}\sum_N NZ(N) = \frac{1}{Z}\sum_N \frac{1}{(N-1)!}Z_1^N \exp(\beta N\mu) = \exp(\beta\mu)Z_1 \tag{4}$$

And one can further show that

$$< N^2 >=< N > + < N >^2 . \tag{5}$$

So

$$< N^2 - < N >^2 >=< N > . \tag{6}$$

c) For 2DEG,

$$\frac{N}{L^2} = \frac{1}{(2\pi\hbar)^2}\int_{p\le p_F} dp 2\pi p = \frac{\pi p_F^2}{(2\pi\hbar)^2} = \frac{1}{2\pi\hbar^2}m\epsilon_F. \tag{7}$$

$$\frac{U}{L^2} = \frac{1}{(2\pi\hbar)^2}\int_{p\le p_F} dp 2\pi p\frac{p^2}{2m} = \frac{1}{2}\frac{p_F^2}{2m}\frac{\pi p_F^2}{(2\pi\hbar)^2}. \tag{8}$$

The last equality leads to

$$\frac{U}{N} = \frac{1}{2}\epsilon_F. \tag{9}$$

d) For relativistic fermions, $\epsilon_F = cp_F$.

$$\frac{N}{L^2} = \frac{\pi p_F^2}{(2\pi\hbar)^2} = \frac{1}{4\pi\hbar^2}(\frac{\epsilon_F}{c})^2. \tag{10}$$

$$\frac{U}{L^2} = \frac{1}{(2\pi\hbar)^2}\int_{p\le p_F} dp 2\pi pc = \frac{1}{3}cp_F\frac{\pi p_F^2}{(2\pi\hbar)^2}. \tag{11}$$

which shows that

$$\frac{U}{N} = \frac{1}{3}\epsilon_F. \tag{12}$$