Phys 455 Final Exam (total 130 pts including 30 optional pts)

1 (50pt) **Phonons in 3d crystals**

Phonons represent elementary excitations in a crystal and contribute to the energy of a crystal. A phonon with momentum $\hbar Q$ and frequency $\Omega(Q) = v_s Q$ has energy $\hbar Q v_s$, v_s is the speed of sound.

a)(10) The Debye momentum $\hbar Q_D$ is defined by the following equation

$$\sum_{|Q| \leq Q_D} = N \tag{1}$$

where N is the number of atoms or lattice sites in a 3d crystal. Express Q_D in terms of the lattice constant a of a cubic crystal(the distance between two neighboring sites).

b)(5) The Debye frequency is defined as the frequency of phonons with the Debye momentum $\Omega_D = \Omega(Q_D)$. Using this definition, express the speed of sound in terms of Ω_D, Q_D.

c)(20) The total energy of phonons is

$$U = \sum_Q N_Q \hbar\Omega(Q); N_Q = \frac{1}{\exp(\hbar\Omega(Q)\beta) - 1}. \tag{2}$$

N_Q is the distribution function of phonons. Calculate the energy of phonons and express it in terms of the Debye frequency Ω_D and kT.

d)(15) The heat capacity is defined as $C = \partial U/\partial T$. Find out the temperature dependence of C in two limits: a) low temperatures ($kT \ll \hbar\Omega_D$) and b) high temperatures ($kT \gg \hbar\Omega_D$).

2(50pt) **Condensation of atoms**

a)(5) For ideal gases, the dilute limit is defined as the limit when

$$N \ll Z_1 = \sum_p \exp(-\epsilon_p \beta), \epsilon_p = \frac{p^2}{2m}. \tag{3}$$

Calculate Z_1 explicitly and show that the above equation indicates that

$$kT \gg \frac{1}{2m}(\frac{N}{V})^{2/3}. \tag{4}$$

Here N is the number of particles and V is the volume.

b)(5)The chemical potential in this limit is defined as

$$N = \sum_p \exp(-\beta(\epsilon_P - \mu)). \tag{5}$$

Express μ in terms of temperature, the number density and mass of particles; show that μ is negative in the dilute limit.

c)(20) At low temperatures, the chemical potential of bosonic particles is zero. The total number of particles is

$$N = N_0(T) + \sum_{p \neq 0} N_p, N_p = \frac{1}{\exp(\beta\epsilon_p) - 1}. \tag{6}$$

Evaluate the sum above and show that $N_0(T)$, the number of particles in the ground state is

$$N_0(T) = N(1 - (\frac{T}{T_c})^{3/2}). \tag{7}$$

Express T_c in terms of the number density of particles (up to a factor of order of unity).

d)(10) Calculate the total energy of atoms at low temperatures $U = \sum_p N_p \epsilon_p$. (Again up to a constant; the chemical potential is zero.)

e)(10) Estimate the energy per particle when the temperature is much lower than T_c. Show it is much smaller than kT required by the equal partition theorem.

3 (Optional 30pt) **Vibrations in 2d crystals**

A phonon also represents correlated vibrations of atoms in a crystal. Consider the Fourier component of $\mathbf{U}(\mathbf{R})$, the displacement of an atom at site $\mathbf{R}$.

$$\mathbf{U}(\mathbf{R}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{Q}} \mathbf{U}(\mathbf{Q}) \exp(-i\mathbf{Q} \cdot \mathbf{R}). \tag{8}$$

a)(10) In the presence of a single phonon with momentum $\hbar Q_0$, the quantum average of $U(Q)U(Q)$ is given as

$$< U(Q)U(Q) >= \frac{\hbar}{M\Omega(Q)} \delta_{Q,Q_0}. \tag{9}$$

Demonstrate that

$$< |U(R) - U(0)|^2 > \tag{10}$$

is a periodical function of R. That is a phonon stands for a propagating wave with wave length being $2\pi/Q_0$.

b)(20) At temperature T,

$$<< U(Q)U(Q) >>= \frac{\hbar N_Q}{M\Omega(Q)}. \tag{11}$$

N_Q is the distribution of phonons (see problem 1). $<<>>$ stands for quantum and statistical average. Evaluate fluctuations in 2d and show that in 2d crystals fluctuations of atoms defined below

$$<< U^2(R) >> \tag{12}$$

are divergent at any finite temperatures. Hint: $\sum_R << U^2(R) >>= \sum_Q << U^2(Q) >>$.