Phys529B: Topics of Quantum Theory

Lecture 5: basic introduction to interacting fermions

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- Fermi Liquid theory (nice discussions in AGD, chapter 1 and 4)

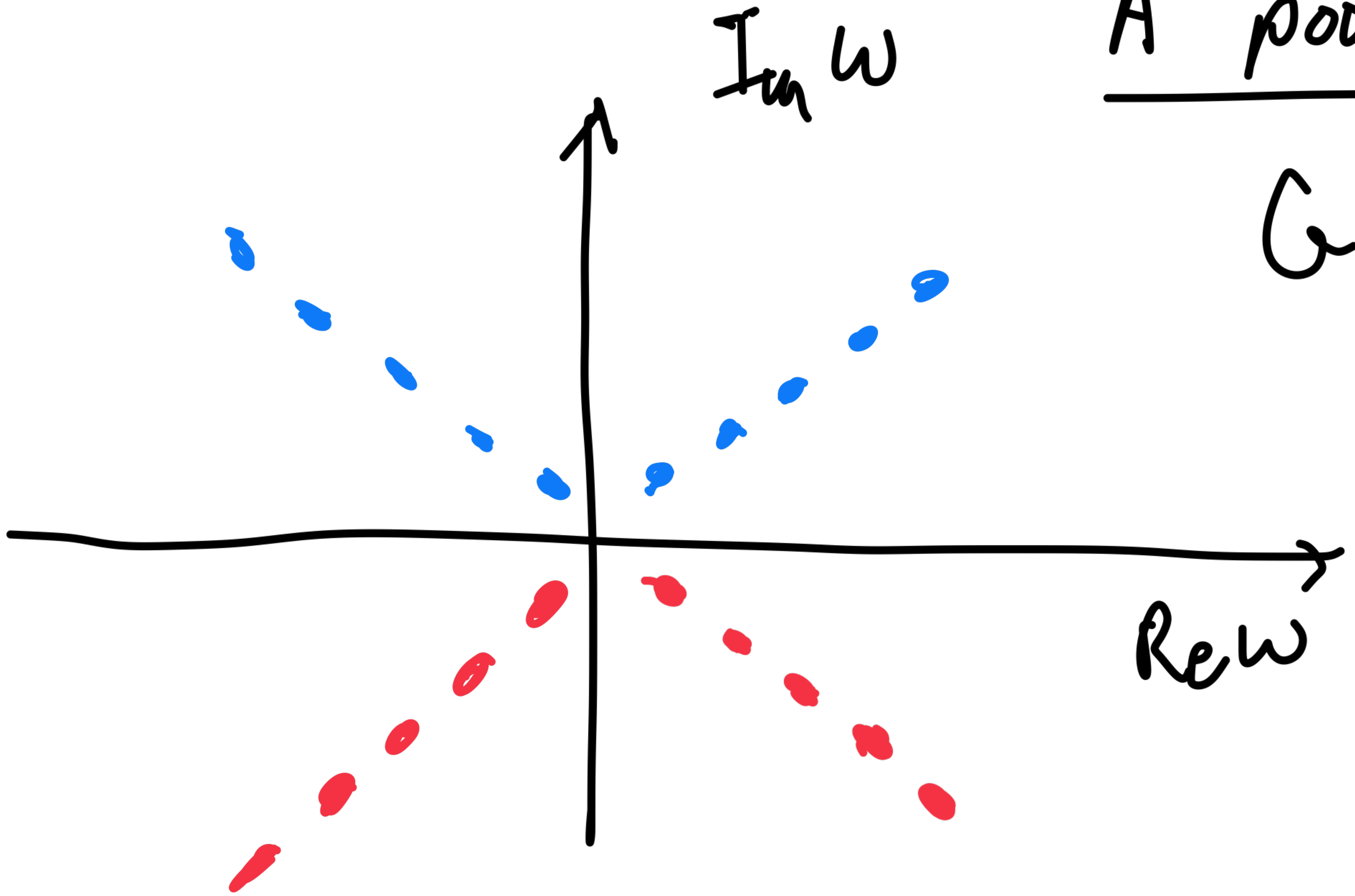
- 1) there is a finite step in the occupation number at exactly K_F. This defines a Fermi surface. $n_{k_F-0} - n_{k_F+0} = Z$

- 2) quasi-particles are of finite life time and become well defined once near Fermi surface,I.e. in the low energy sector. $\frac{1}{\tau_k} = \gamma_k \ll |\xi_k|$

- 3) apart from mass renormalization, wave function renormalization Z occurs at Fermi surface.

- 4) there are low energy emergent bosonic particles.

- 5) for a fixed k, time ordered 'G" is not analytical in either lower or upper frequency planes. However, retarded (advanced) green functions are analytical in lower (upper) plane for any k.(a proof in Lehmann Rep.)



A poor man's approach

$$G(\omega,\vec{k}) = G_R(\omega,\vec{k})\,\theta(\omega) + G_A(\omega,k)\,\theta(-\omega)$$

$$G(\omega,\vec{k}) \simeq \frac{Z}{\omega - \xi_k + i\gamma_k \,\mathrm{sig}\,\omega}$$

$$(|\xi_k| \gg \gamma_k\,,\ Z \leq 1)$$

$$\mathrm{Im}\,G(\omega,\vec{k}) = -\theta(-\omega)\frac{Z\,\gamma_k}{(\omega-\xi_k)^2+\gamma_k^2} + \theta(\omega)\frac{Z\,\gamma_k}{(\omega-\xi_k)^2+\gamma_k^2}$$

Application to $\hat{n}_k = \psi_k^\dagger(0)\psi_k(0) = 1 - \psi_k(0)\psi_k^\dagger(0)$

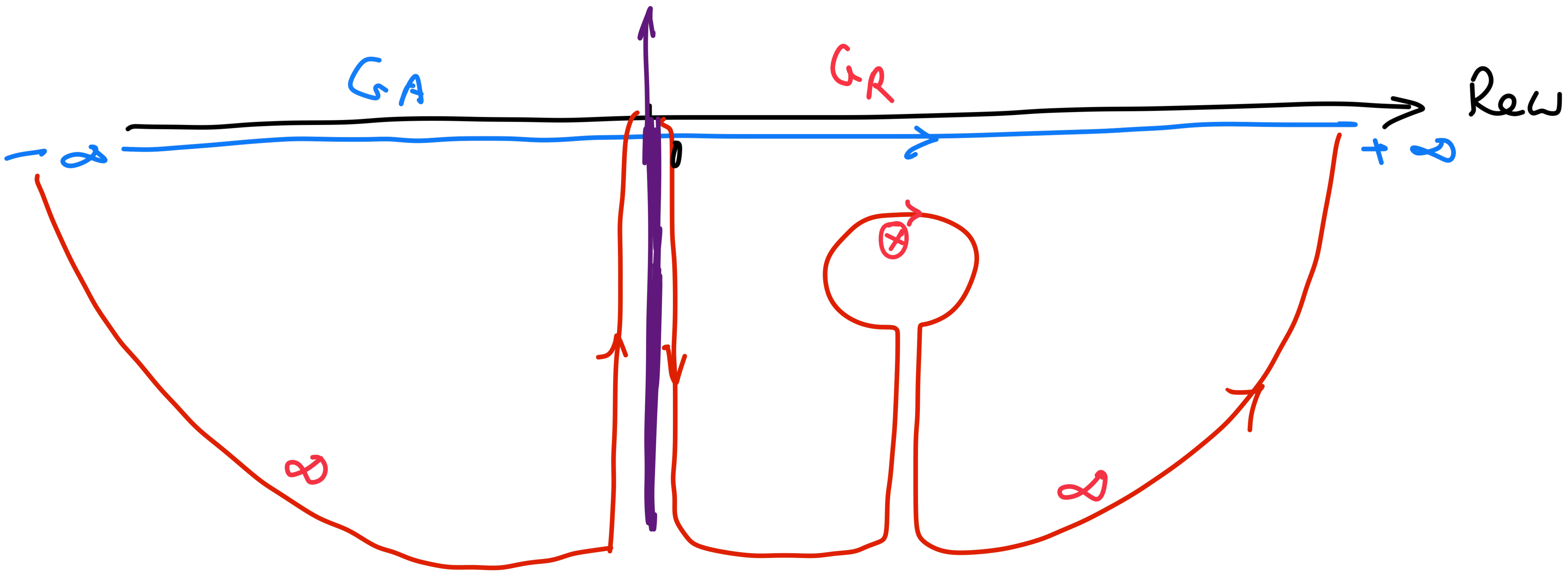
$$-i\,T\,\psi_k(t)\psi_k^\dagger(0) = -i\left[\theta(t)\,\psi_k(t)\psi_k^\dagger(0) - \theta(-t)\,\psi_k^\dagger(0)\psi_k(t)\right]$$

$$1 - n_k = \langle g.s.|\,\psi_k(0)\,\psi_k^\dagger(0)|g.s.\rangle = i\,G(k, t = 0^+)$$

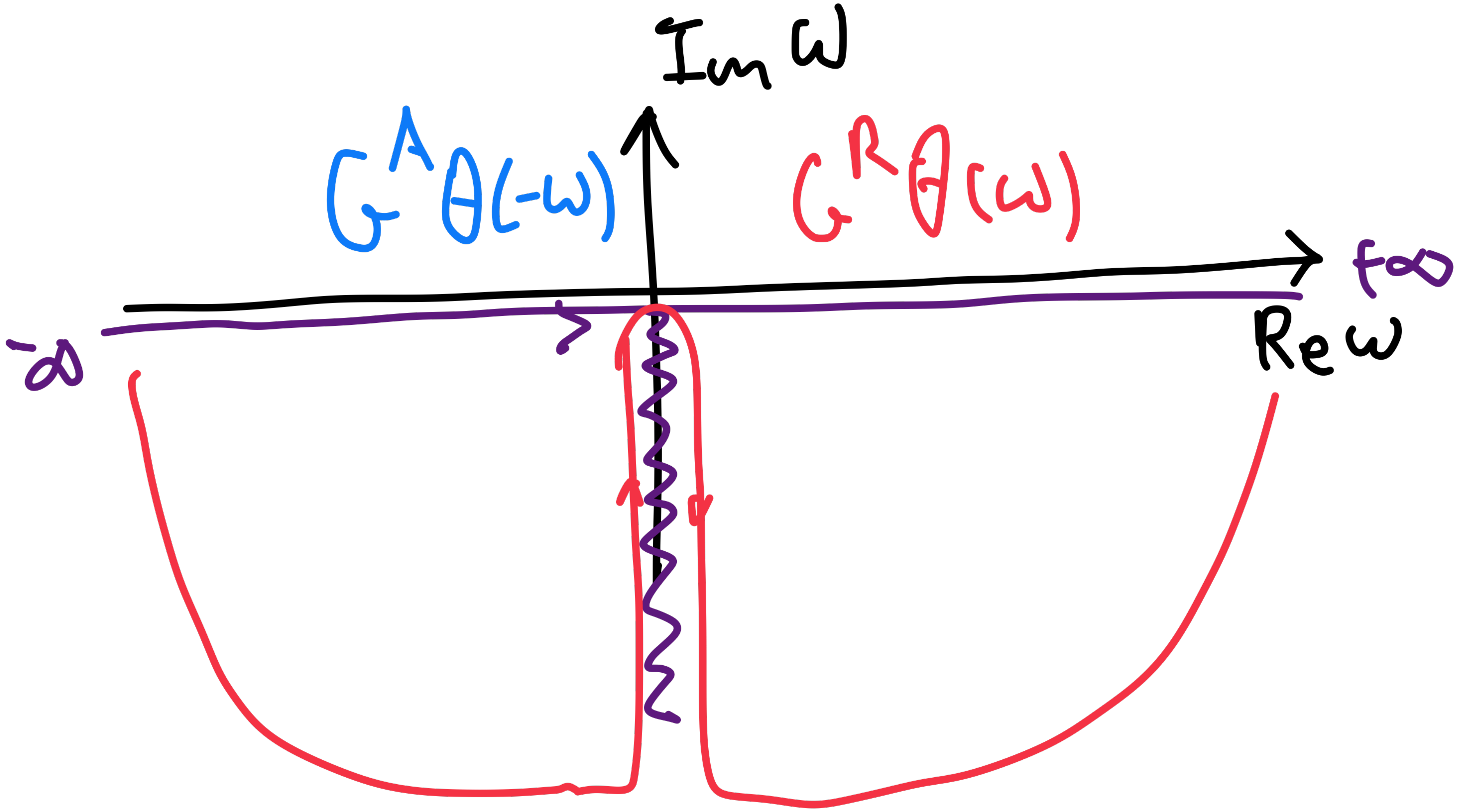
$$1 - n_k = i\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,G(k,\omega)\,e^{-i\omega\cdot 0^+}$$

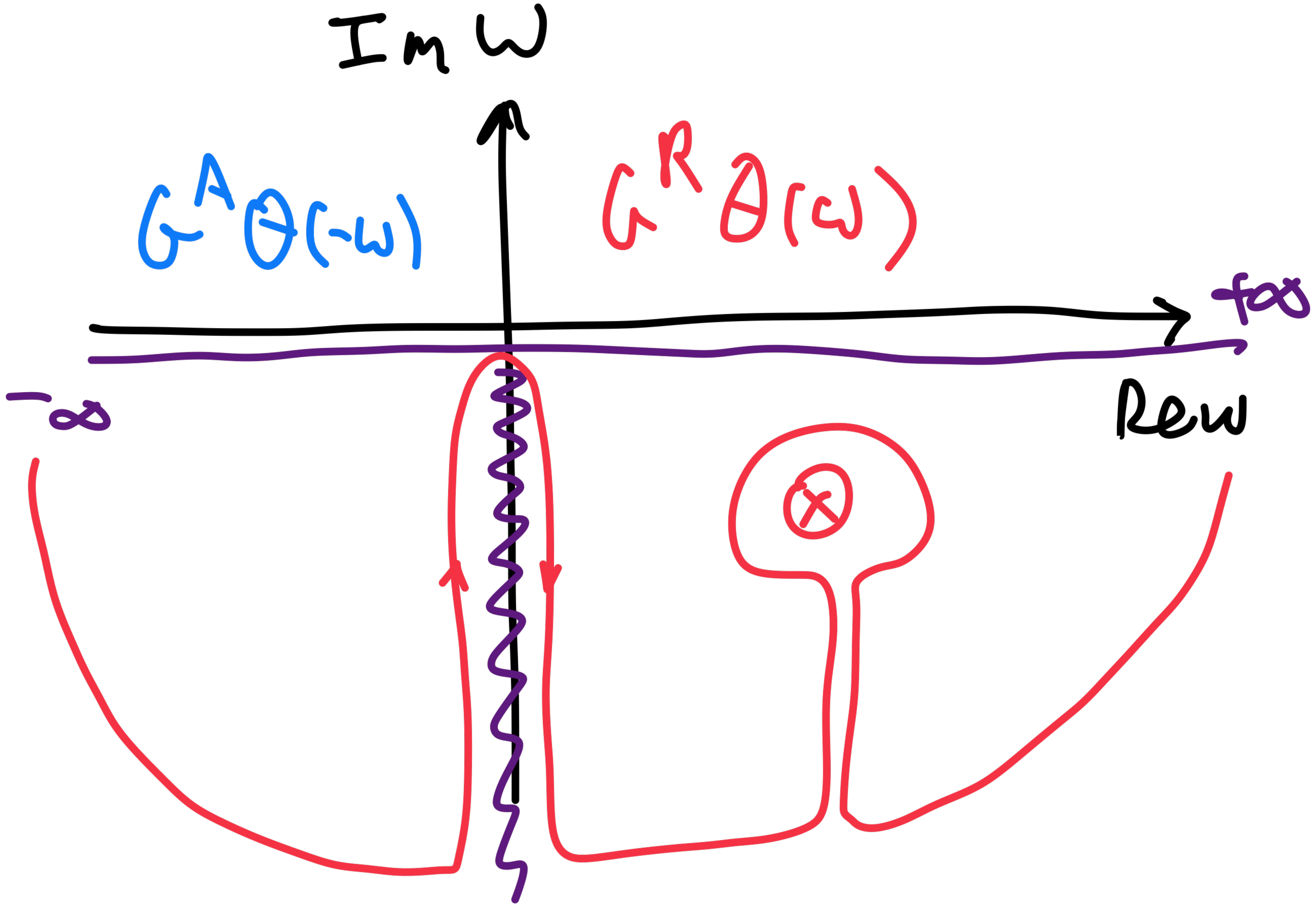
$$1-n_k=\langle g.s.|\,\psi_k(0)\,\psi_k^\dagger(0)\,|g.s.\rangle=-i\,G(\vec{k},t=0^-)=-i\int\frac{d\omega}{2\pi}G(k,\omega)\,e^{-i\omega 0^+}$$

$$G(\vec{k},\omega)=G_R(\vec{k},\omega)\,\Theta(\omega)+G_A(\vec{k},\omega)\,\Theta(-\omega)$$

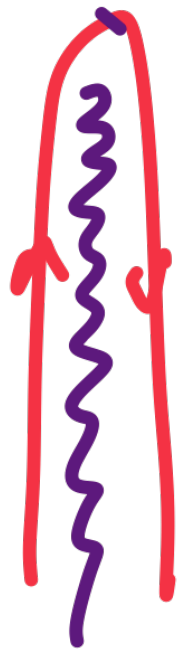


$k = k_F + 0^-$ $\qquad G_R \simeq \frac{Z}{\omega - \xi_k + i\gamma_k} + \dots \qquad$ $k = k_F + 0^+$





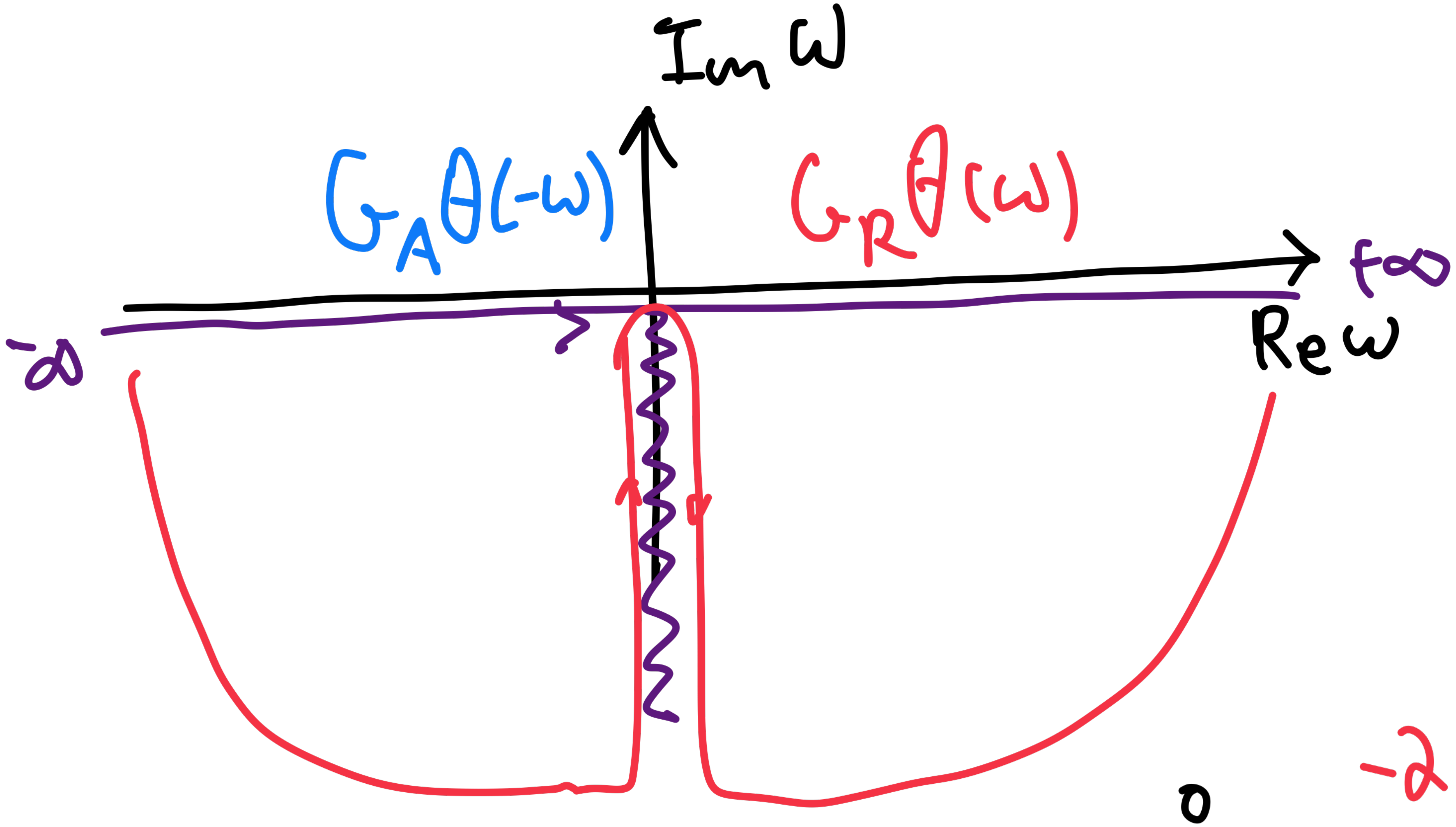
$$n(p_F + 0^-) - n(p_F + 0^-) = \underbrace{\mathrm{Res}\, G_R(\vec{k}, \omega = \xi_k - i\gamma_k)}_{= Z} + $$



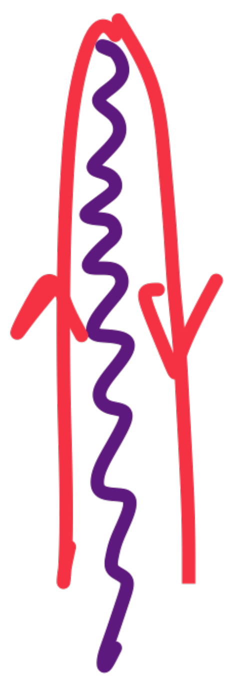
$$\frac{\gamma_k}{|\xi_k|} \xrightarrow{k \to k_F} 0$$

$k = k_F + 0^-$

$$G_R = \frac{Z}{\omega - \xi_k + i\gamma_k}, \quad \gamma_k > 0; \quad G_A = G_R^*$$

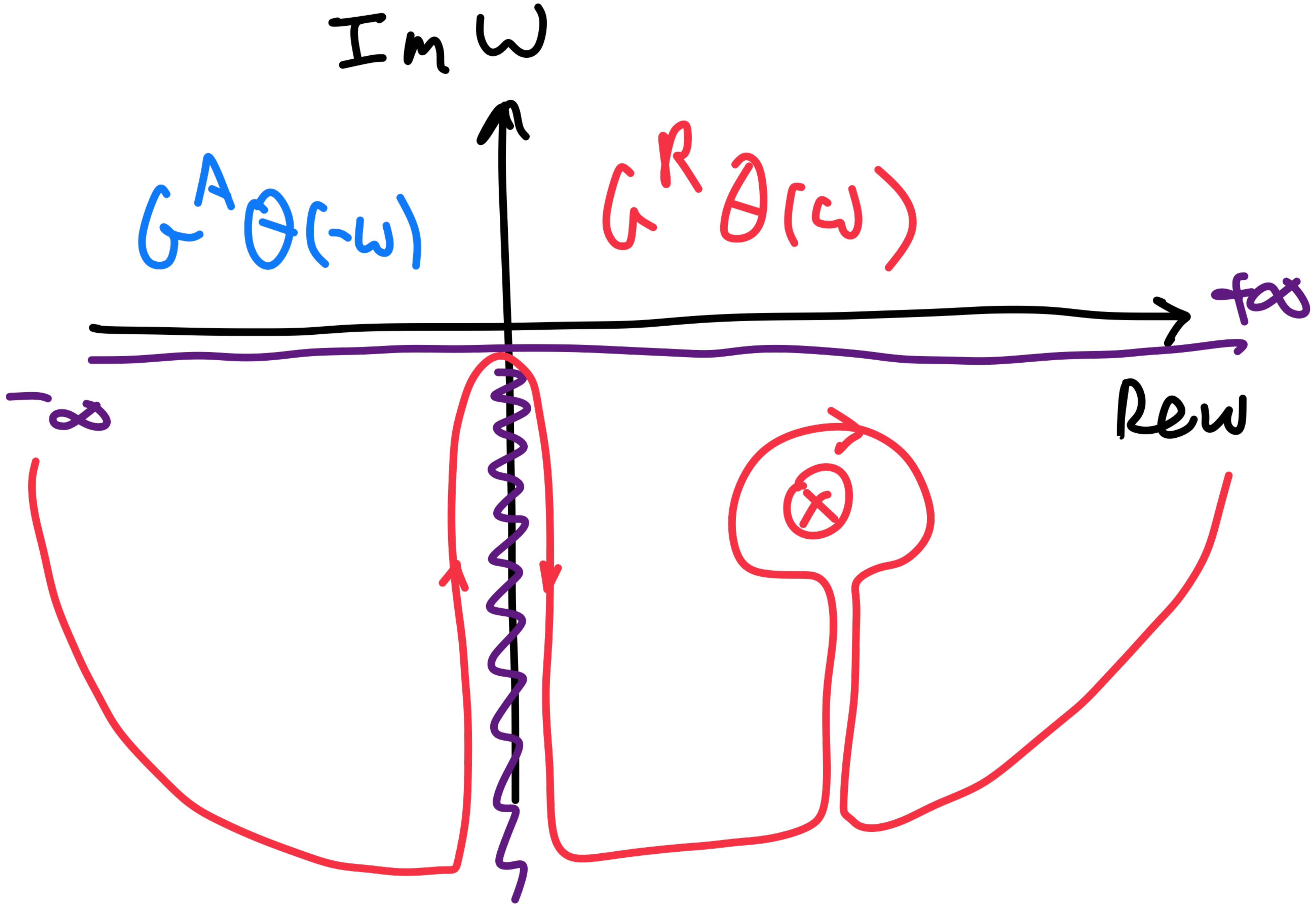


$G_R(\omega, k = k_F + 0^-)$ is analytical when $\mathrm{Re}\,\omega \geq 0$

$n(p_F + 0^-) =$  $= \displaystyle\int_{-i\infty}^{0} (G_A \overbrace{-\,G_R}^{-2i\,\mathrm{Im}\,G_R})\, d\omega \sim 2\frac{Z\gamma_k}{\xi_k} \to 0$

$$G_R \simeq \frac{Z}{\omega - \xi_k + i\gamma_k}$$

$$k = k_F + 0^+$$



G_R has a pole when Re ω > 0

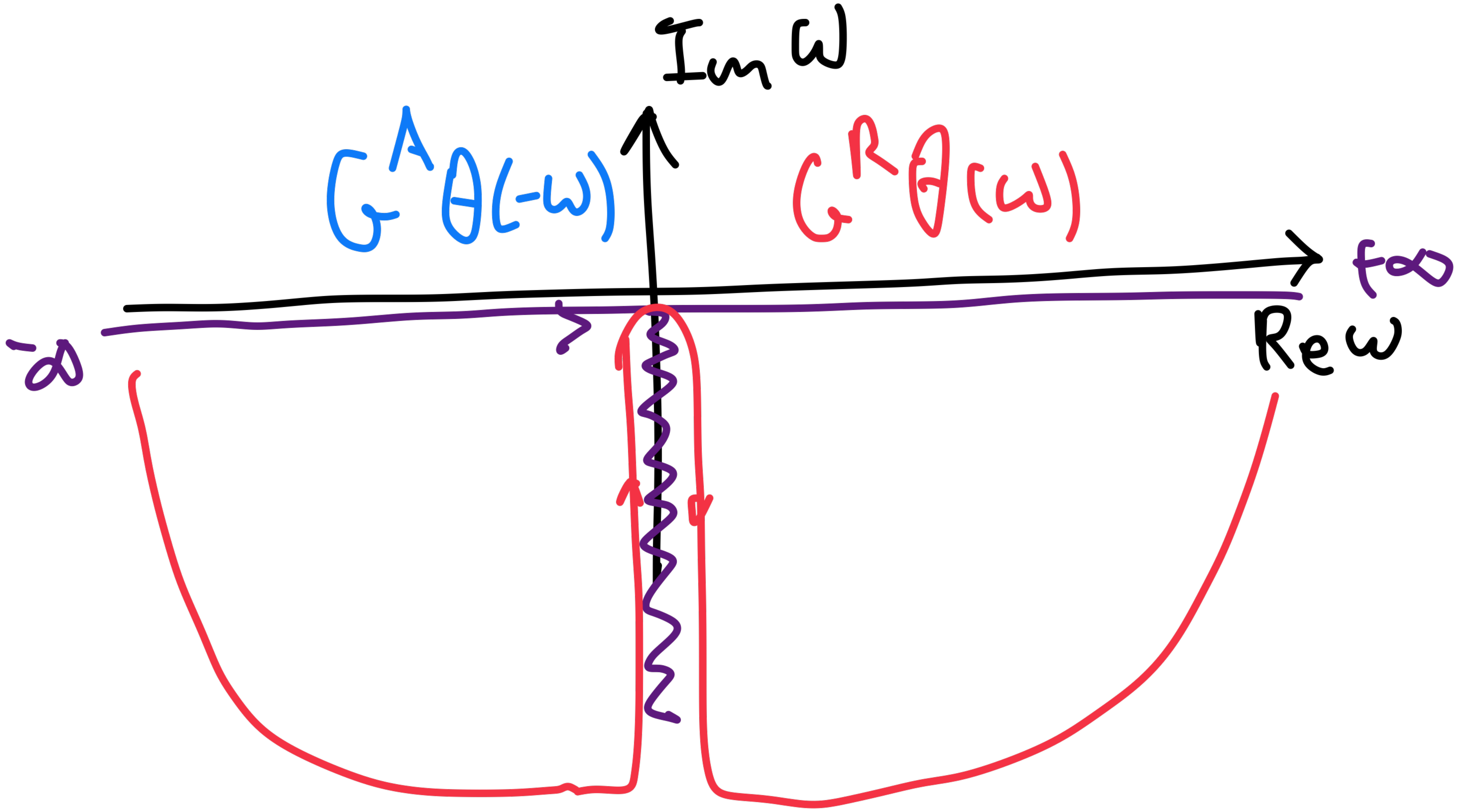
$$n(p_F + 0^+) = -\operatorname{Res} G_R(\vec{k}, \omega = \xi_k - i\gamma_k) + \frac{\gamma_k}{|\xi_k|} \xrightarrow{k \to k_F} 0$$

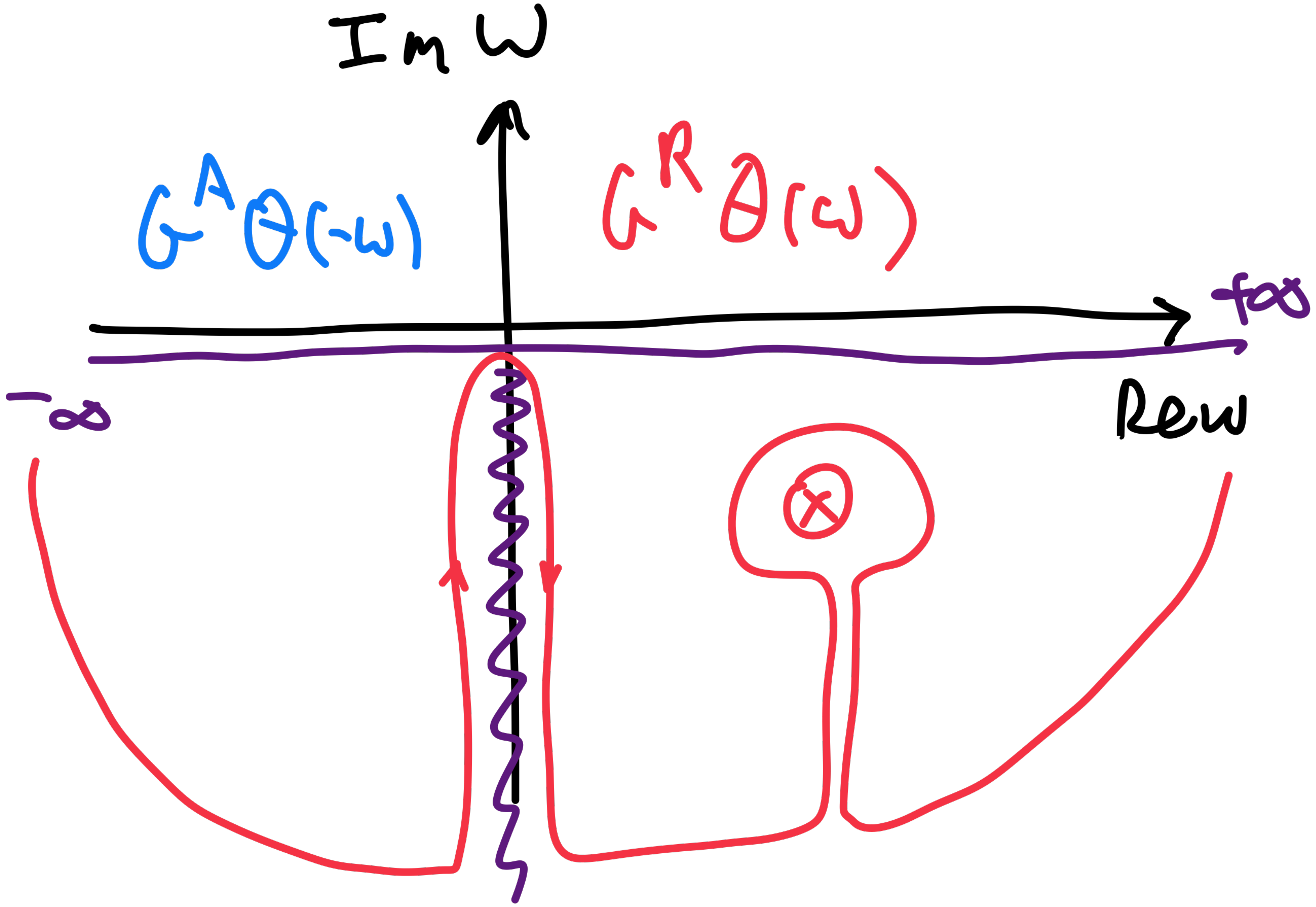
$$\simeq Z$$

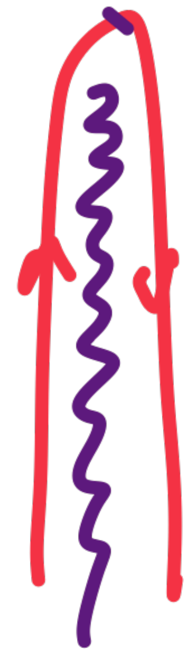
$$G_R \simeq \frac{Z}{\omega - \xi_k + i\gamma_k}$$

$k = k_F + 0^-$

$k = k_F + 0^+$





$$n(p_F+0^-) - n(p_F+0^-) = \underbrace{\operatorname{Res} G_R(k, \omega = \xi_k - i\gamma_k)}_{= Z} +$$  $$\frac{\gamma_k}{|\xi_k|} \xrightarrow{k \to k_F} 0$$

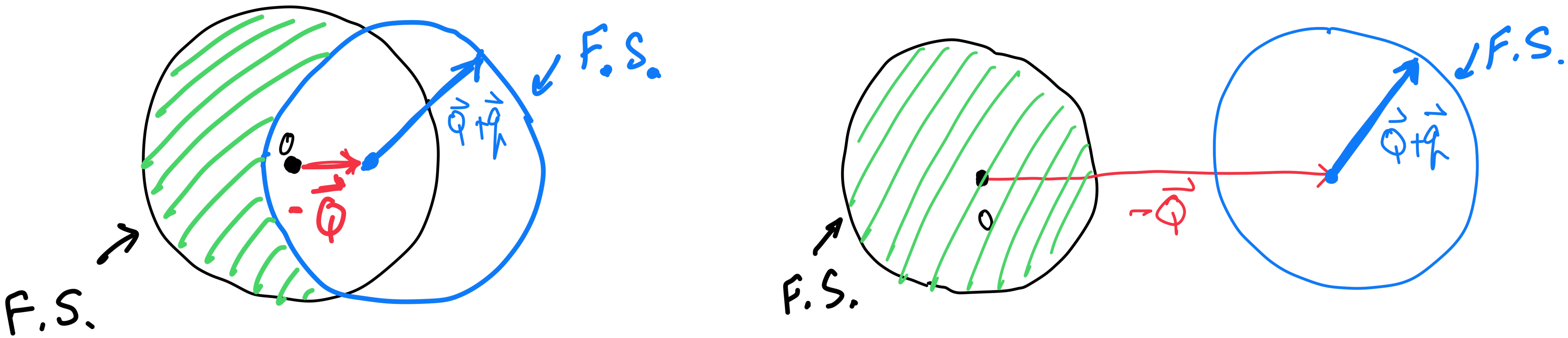
Composite bosonic states as a continuum

$$\phi^{+}_{Q} = \sum_{\vec{q}} A_{Q,\vec{q}}\, \psi^{+}_{\vec{Q}+\vec{q}}\, \psi_{\vec{q}} \qquad (\text{“}Q=0 \text{ Sector”})$$

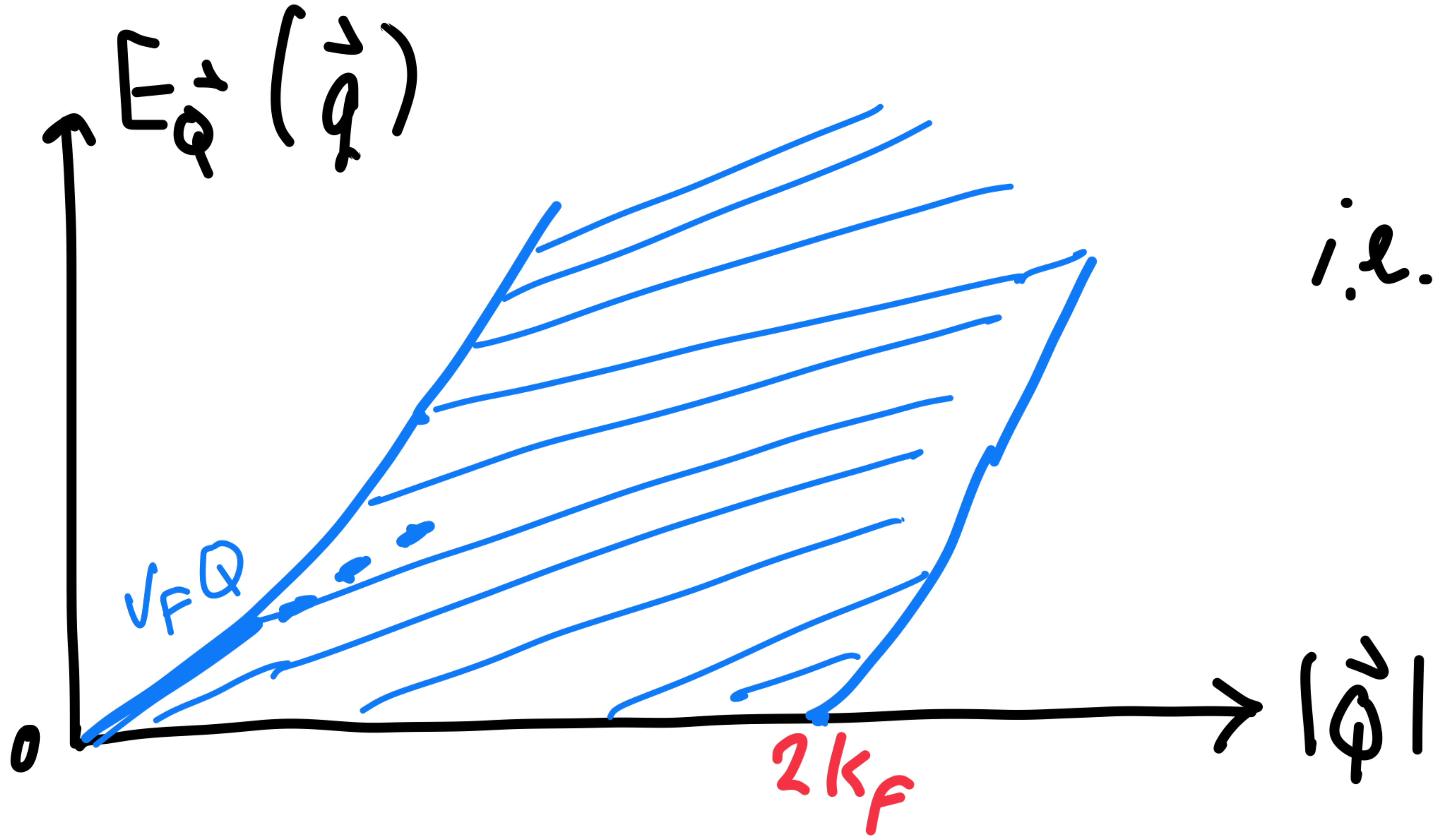
charge neutral sector

$$|\vec{q}| < k_F, \quad |\vec{Q}+\vec{q}| > k_F$$

$|\vec{Q}| < 2k_F$ | $|\vec{Q}| > 2k_F$



"F. G."

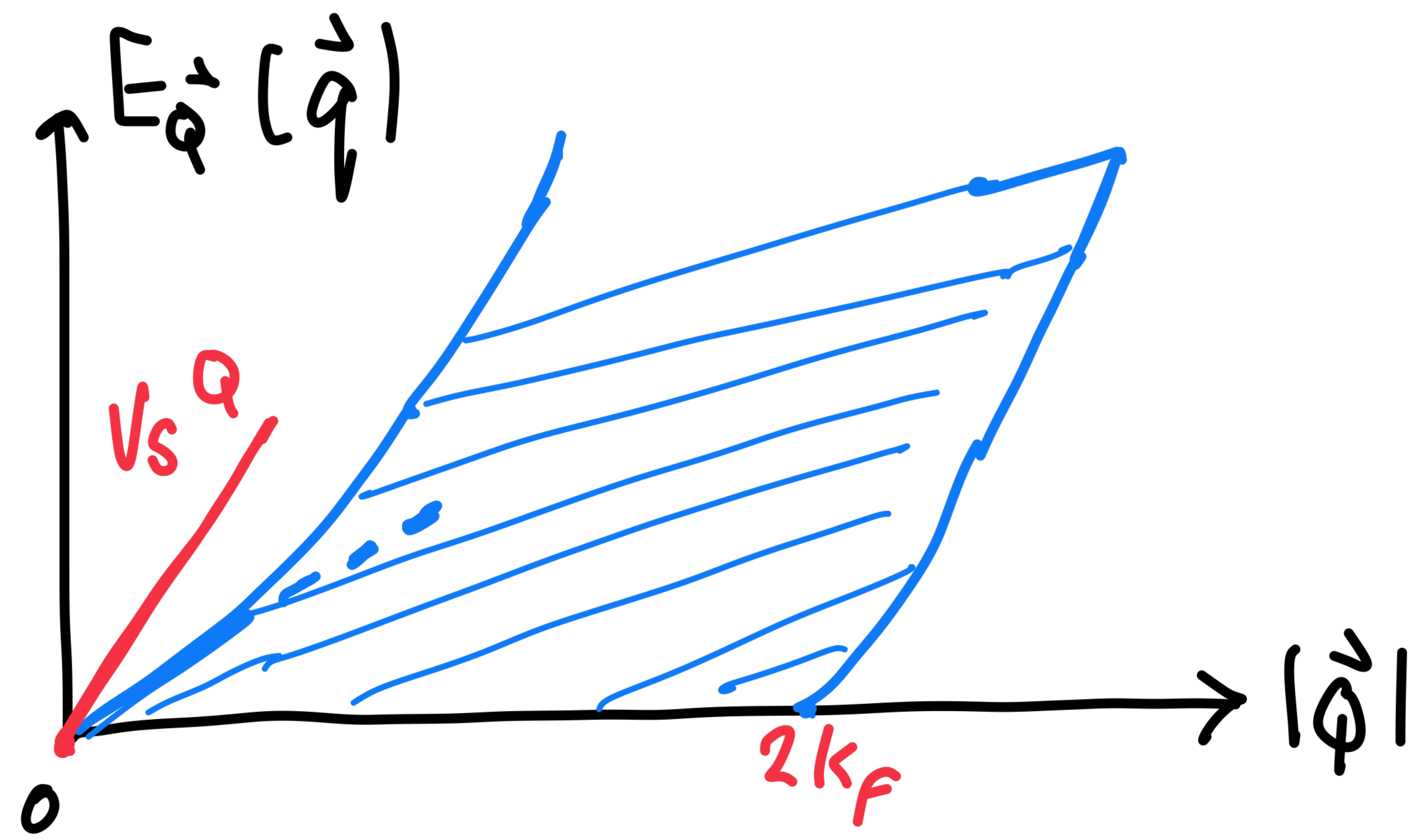


Continuum for bosonic states

i.e. No bosonic "particle"s in F. G.

In F.G., $G_B(\vec{Q},\Omega)$ doesn't have isolated poles.

"F. L"

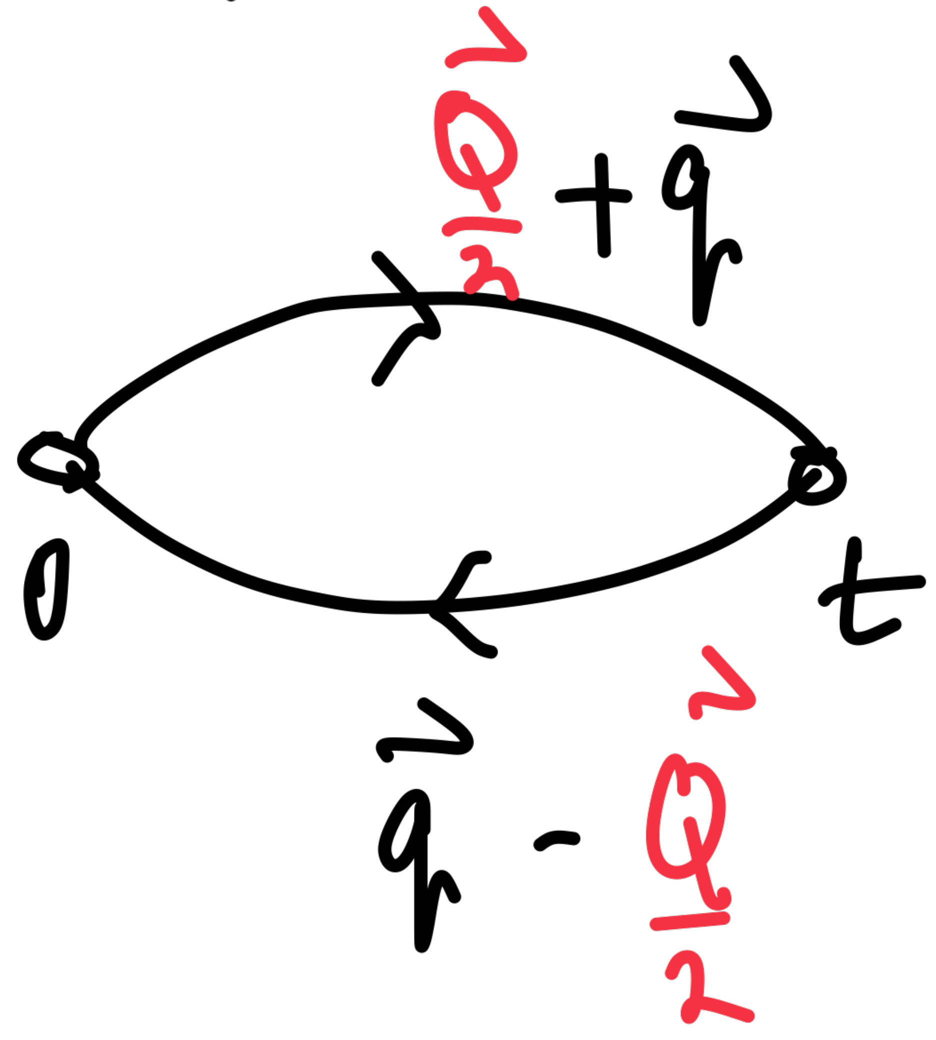


In F.L., $G_B(\vec{Q},\Omega)$ has simple isolated poles, i.e. emergent bosonic fields.

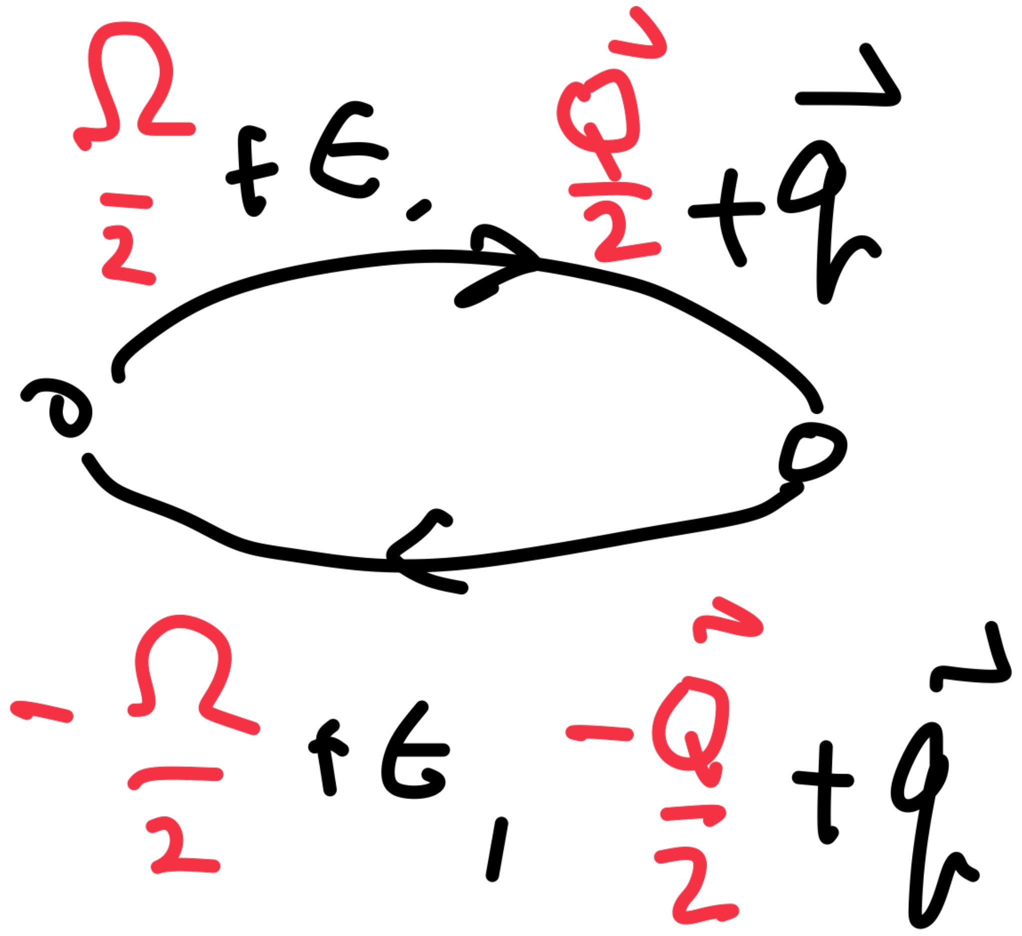
("zero sound")

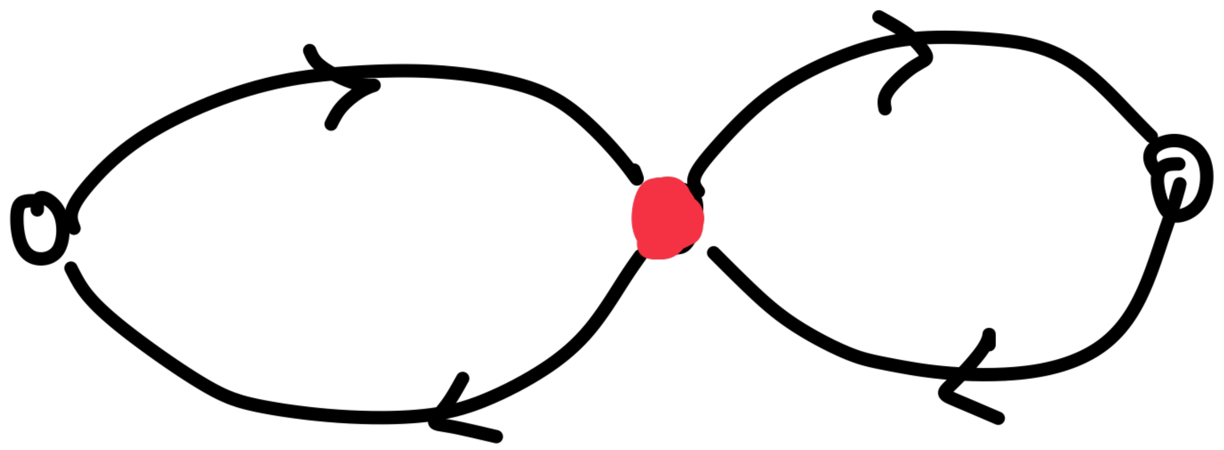
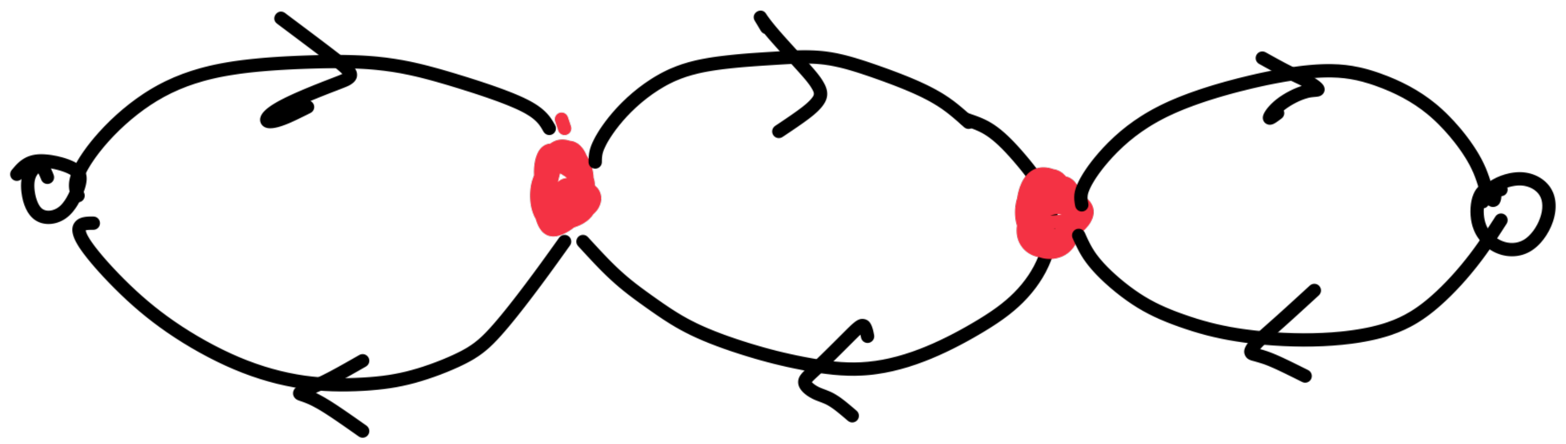
$v_s = v_s(v_F, \tilde{g}) > v_F$

$$G_B(\vec{Q}, t) = \langle 0 | -i \, T \, \phi_{\vec{Q}}(t) \, \phi^{\dagger}_{\vec{Q}}(0) | 0 \rangle, \quad \phi^{\dagger}_{\vec{Q}} = \sum_{\vec{q}} \psi^{\dagger}_{\frac{\vec{Q}}{2}+\vec{q}} \, \psi_{\vec{q}-\frac{\vec{Q}}{2}}$$

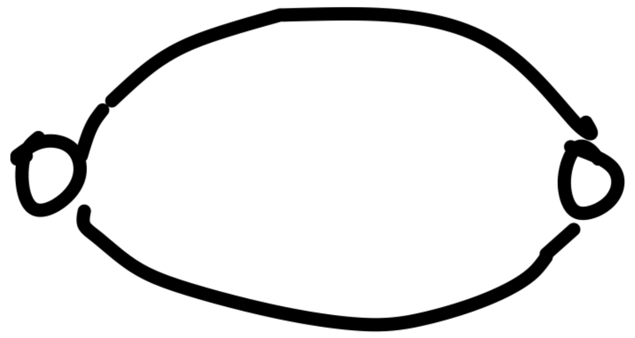
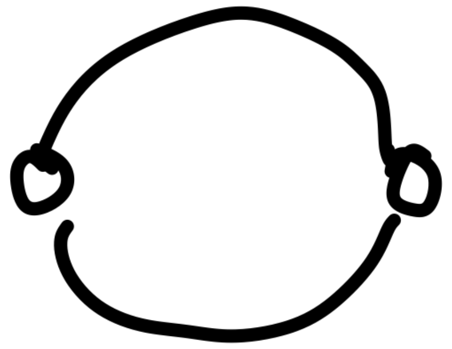
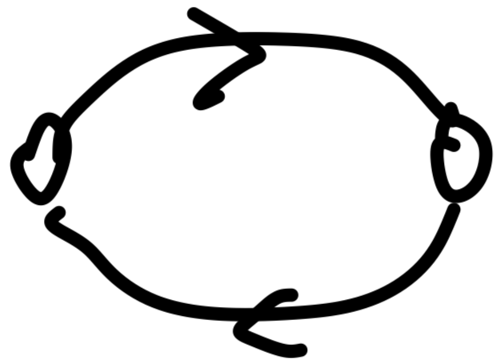
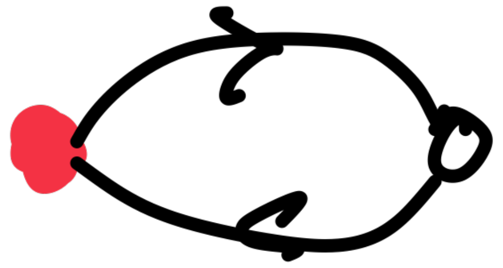


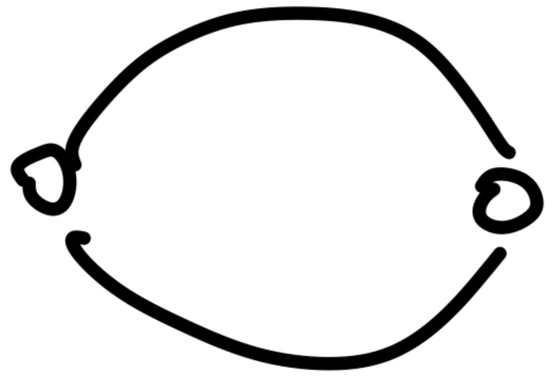
$\longrightarrow \quad G_B(\vec{Q}, \Omega)$



+  +  + ...

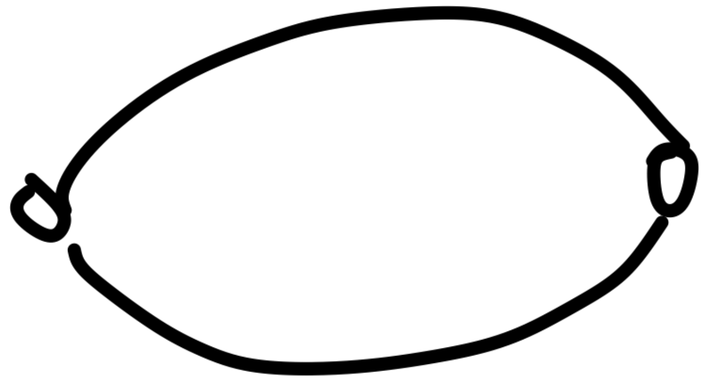
● : interaction "g"

$G_B(\Omega, \vec{Q}) =$  / ($1 - g$ ), g  $=$ 

 $$-i\int \frac{d\epsilon}{2\pi}\frac{d\vec{q}}{(2\pi)^3}\, G\left(\frac{\Omega}{2}+\epsilon, \frac{\vec{Q}}{2}+\vec{q}\right) G\left(-\frac{\Omega}{2}+\epsilon, -\frac{\vec{Q}}{2}+\vec{q}\right)$$

$$= \int \frac{d\vec{q}}{(2\pi)^3}\, \frac{1}{\Omega - v\vec{q}\cdot\vec{Q}}\, \Theta\left(\xi_{\frac{\vec{Q}}{2}+\vec{q}}\right)\Theta\left(\xi_{-\frac{\vec{Q}}{2}+\vec{q}}\right)$$

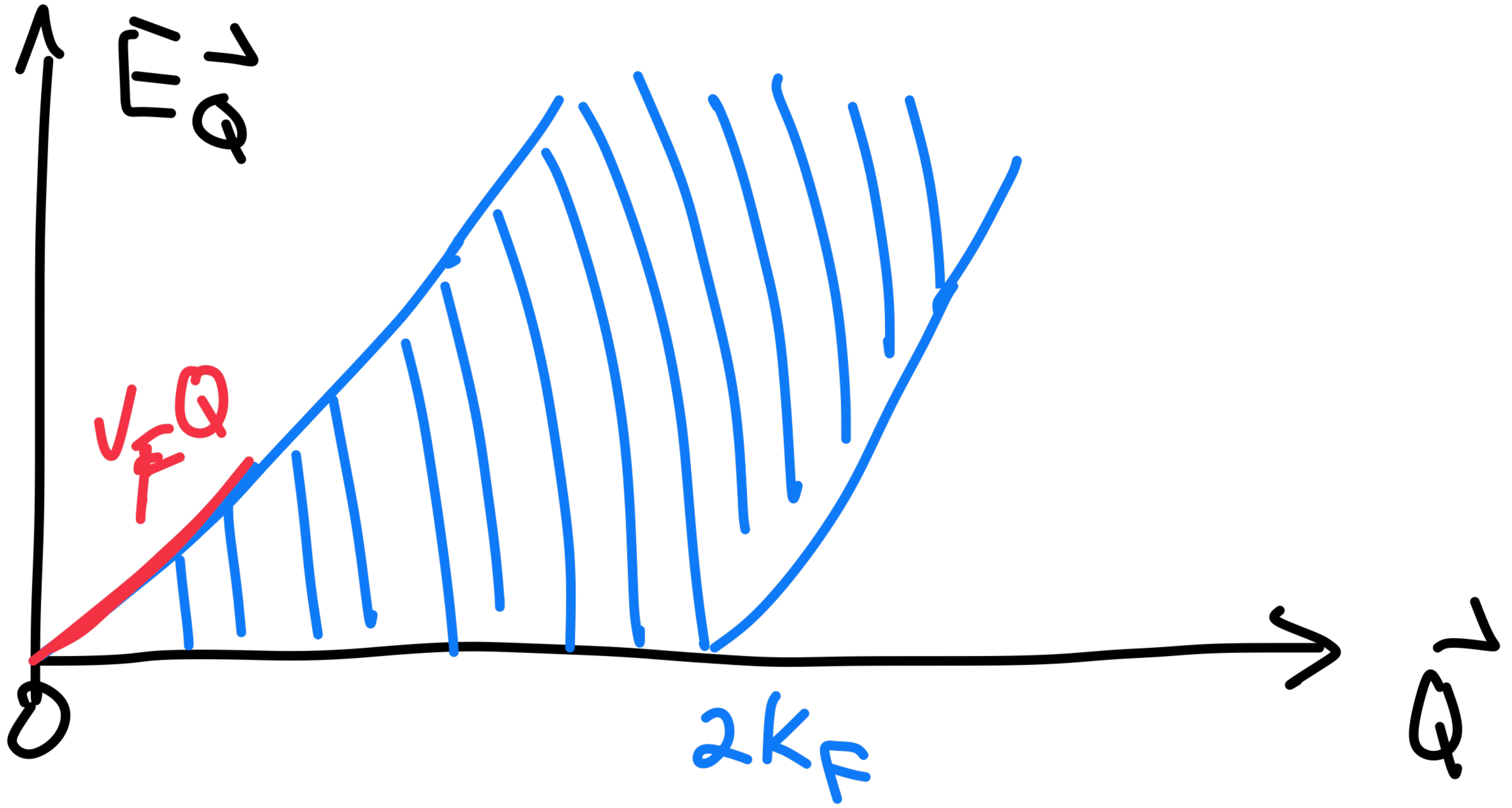
...

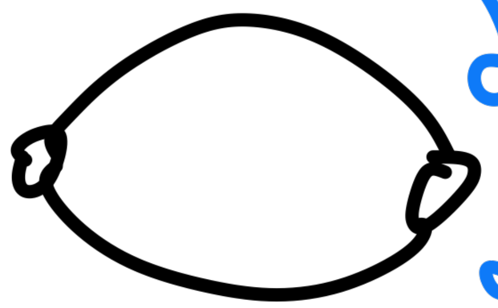


$$= A(k_F)\left[1 + \frac{\Omega}{v_F Q}\ln\left|\frac{\Omega + v_F Q}{\Omega - v_F Q}\right|\right]$$

$$+ B(k_F)\left[i\pi\,\Theta(v_F Q - \Omega)\,\Theta(\Omega) - i\pi\,\Theta(-\Omega)\,\Theta(v_F Q + \Omega)\right]$$

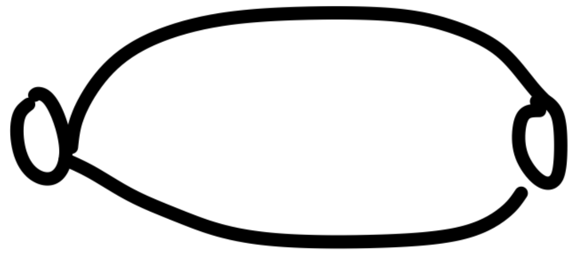
$v_F Q > \Omega > 0$ $\qquad$ $0 > \Omega > -v_F Q$



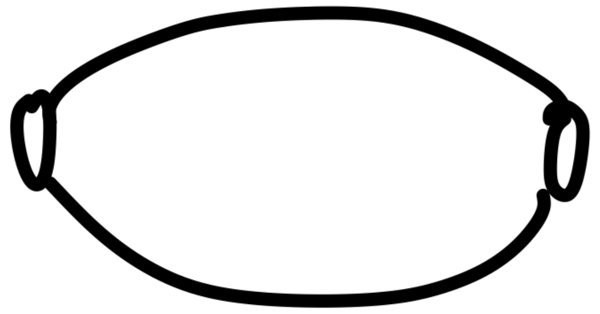
$G_B(\Omega, \vec{Q}) =$ 

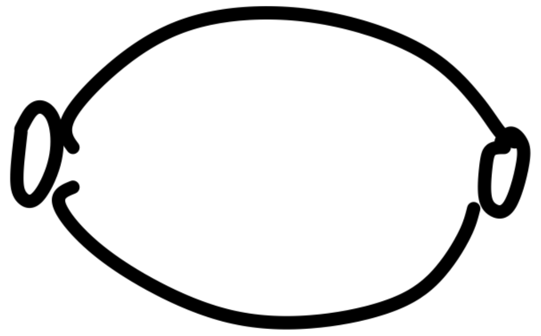
for F. G.

F. L.

$G_B =$ 

divided by

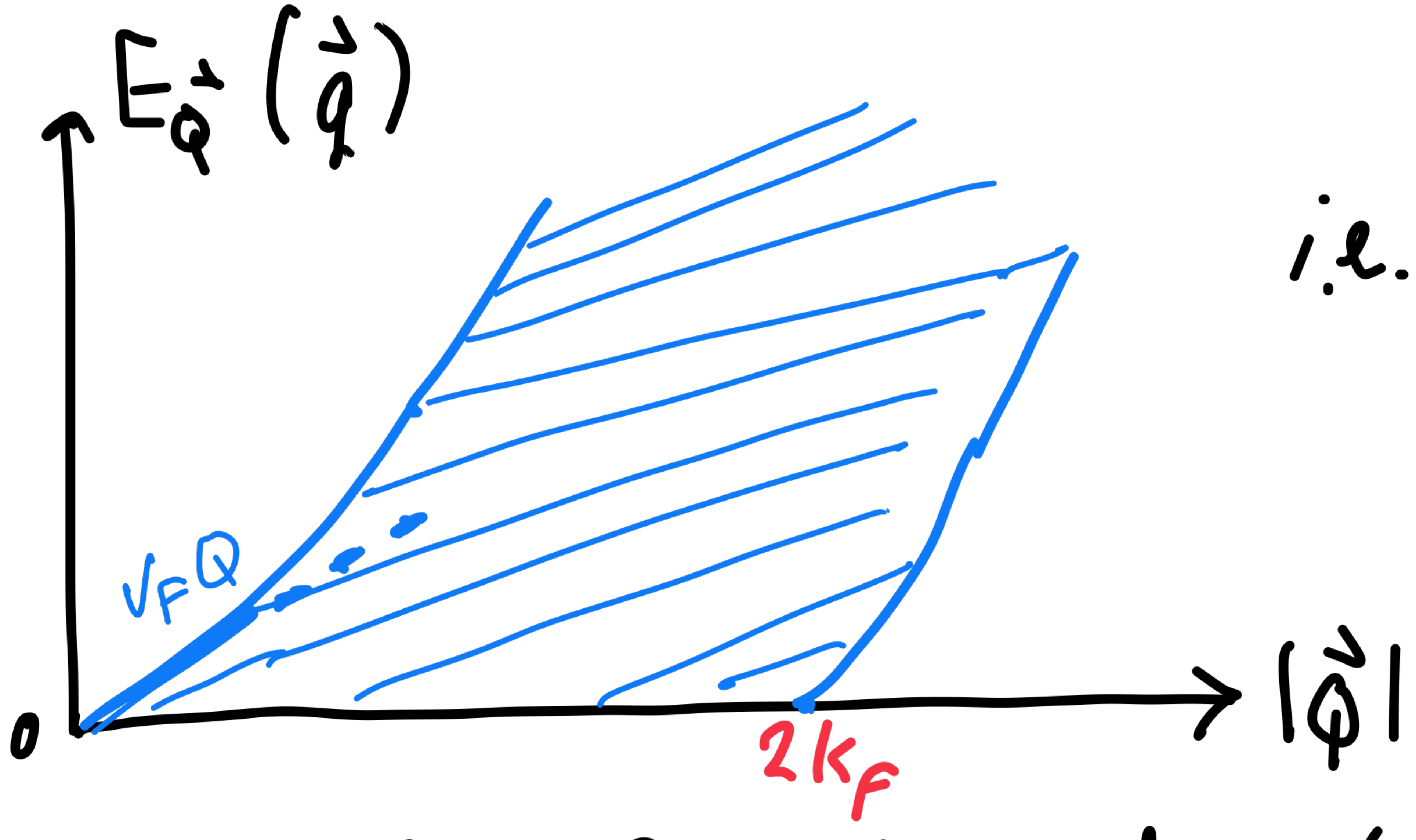
$1 - g$ 

$1 - g$  $= 1 - g\, A(k_F) \left\{ 1 + \frac{\Omega}{v_F Q} \ln \frac{\Omega + v_F Q}{\Omega - v_F Q} \right\} = 0$

Real if $\Omega > v_F Q$

$g \to 0, \quad \Omega - v_F Q \;\hat{=}\; 2 v_F Q\, e^{\frac{1}{g A(k_F)}}$

"F.G."



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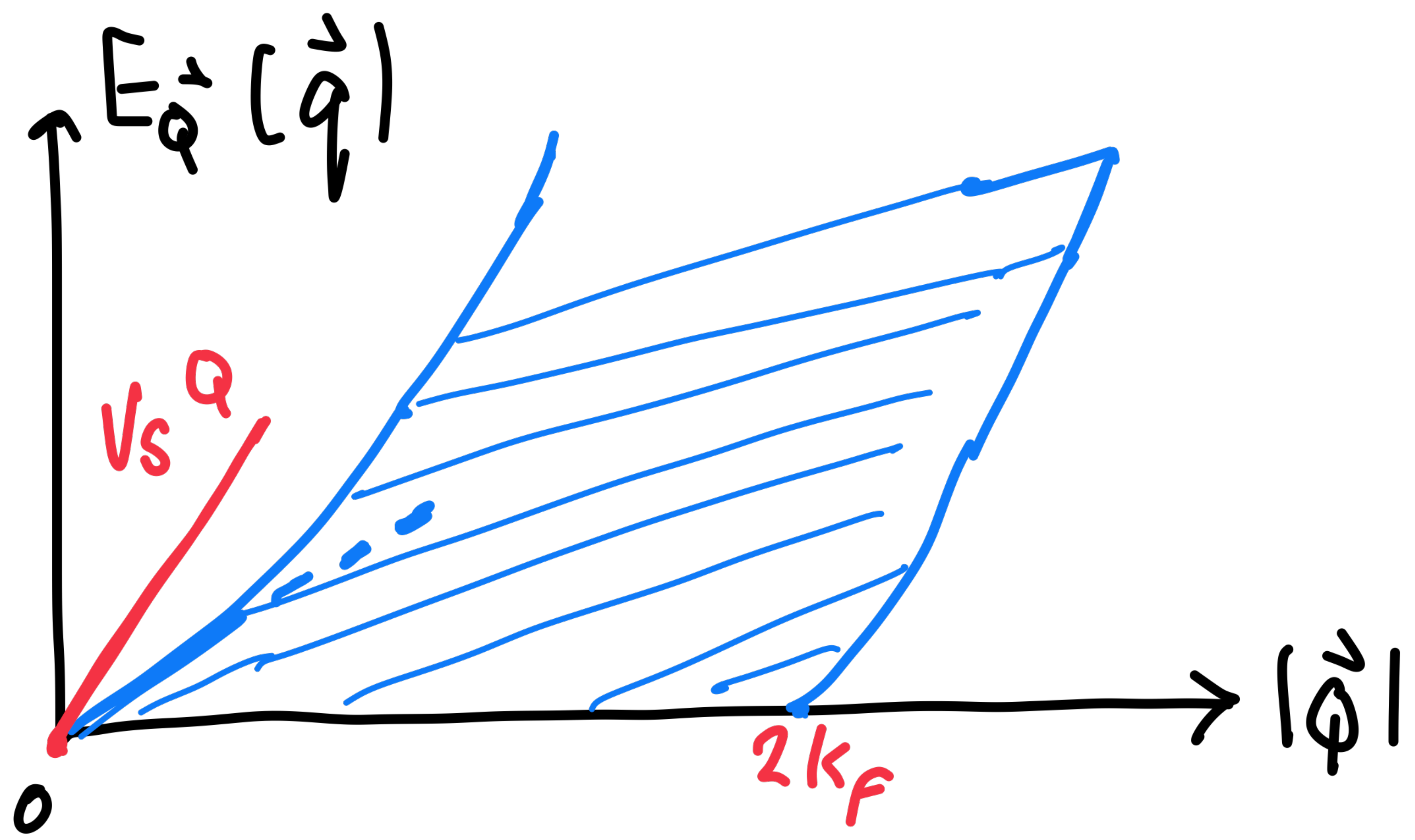

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