Phys529B: Topics of Quantum Theory

Lecture 4: basic introduction to interacting fermions

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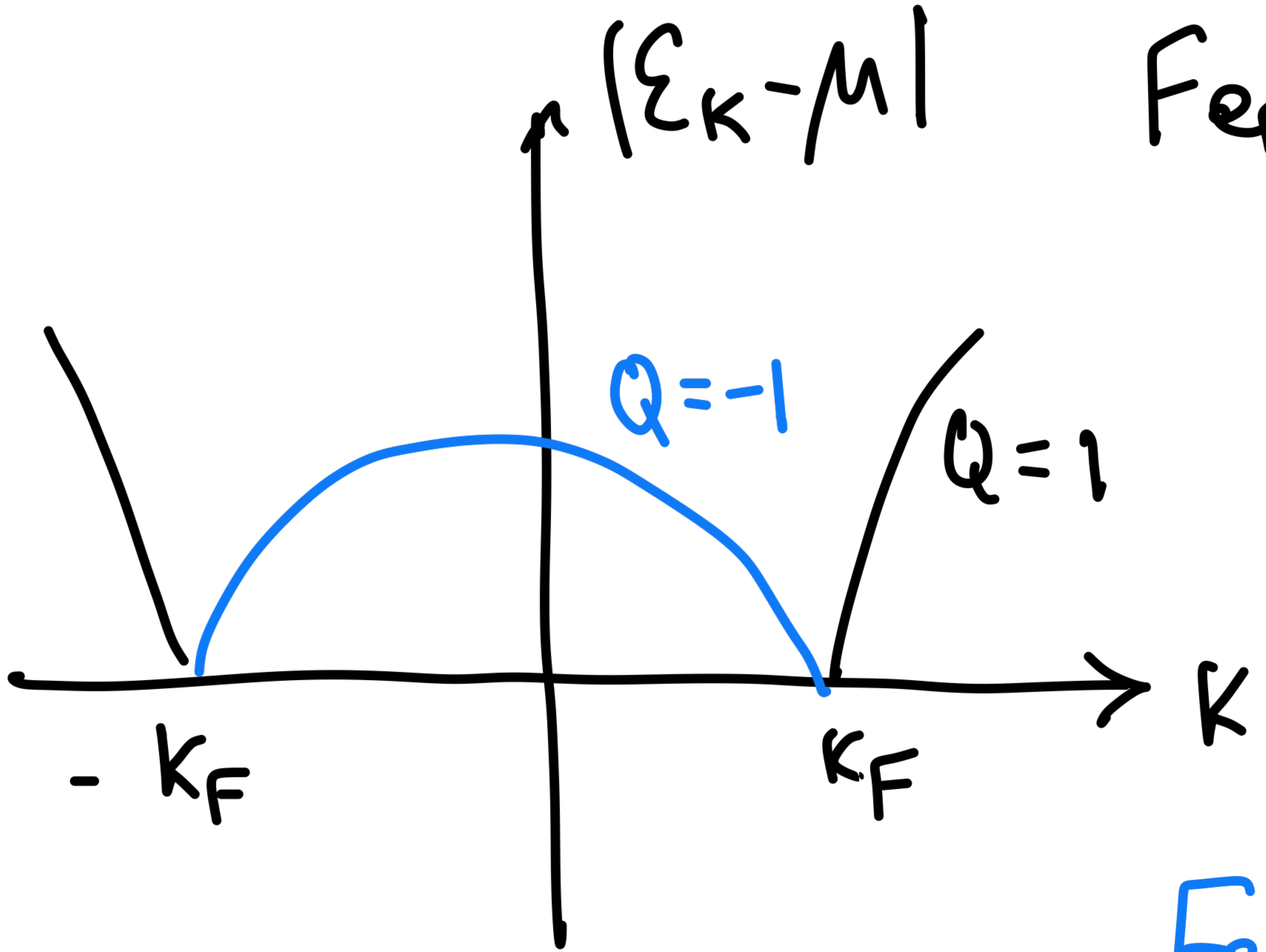
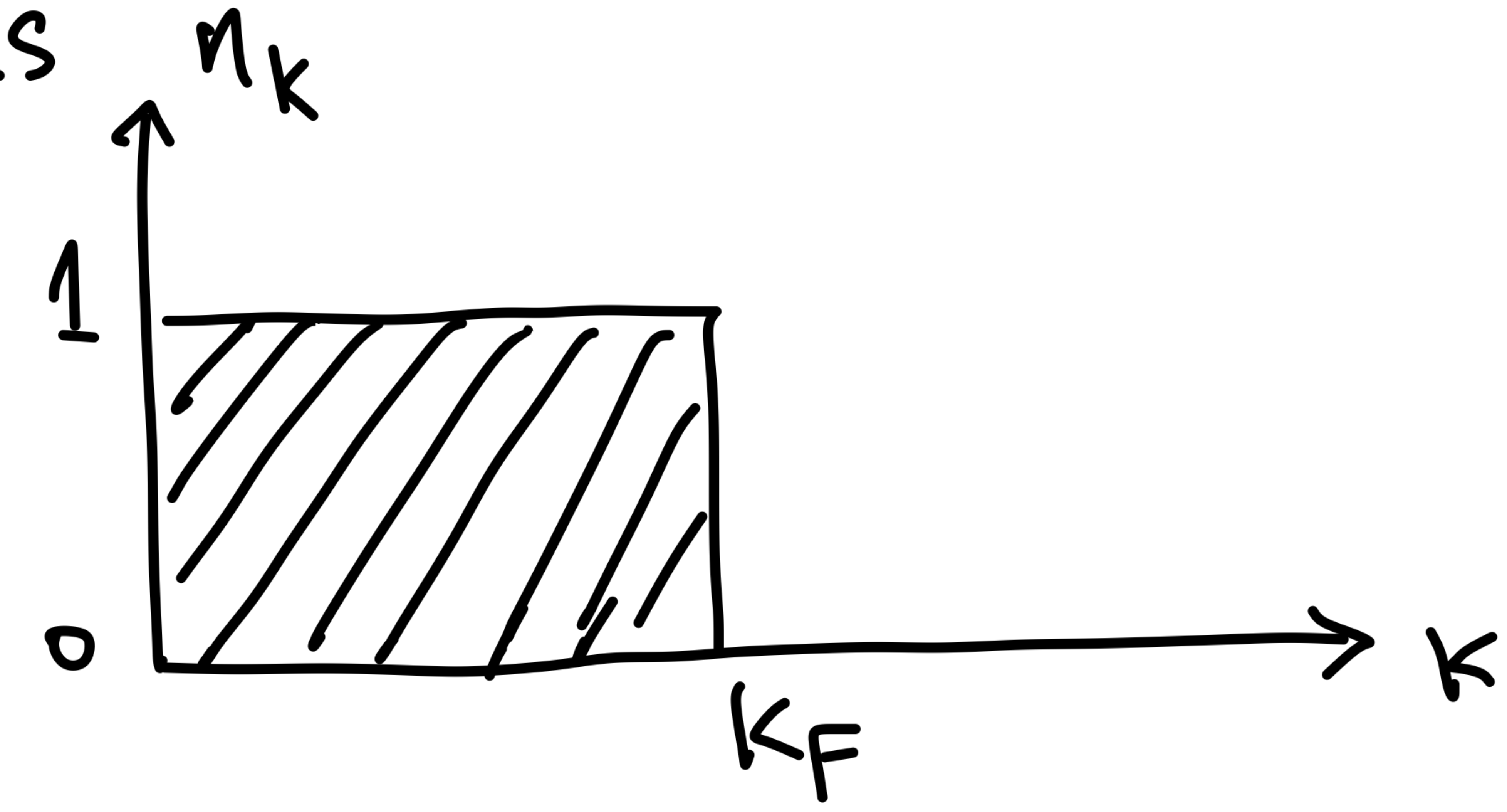
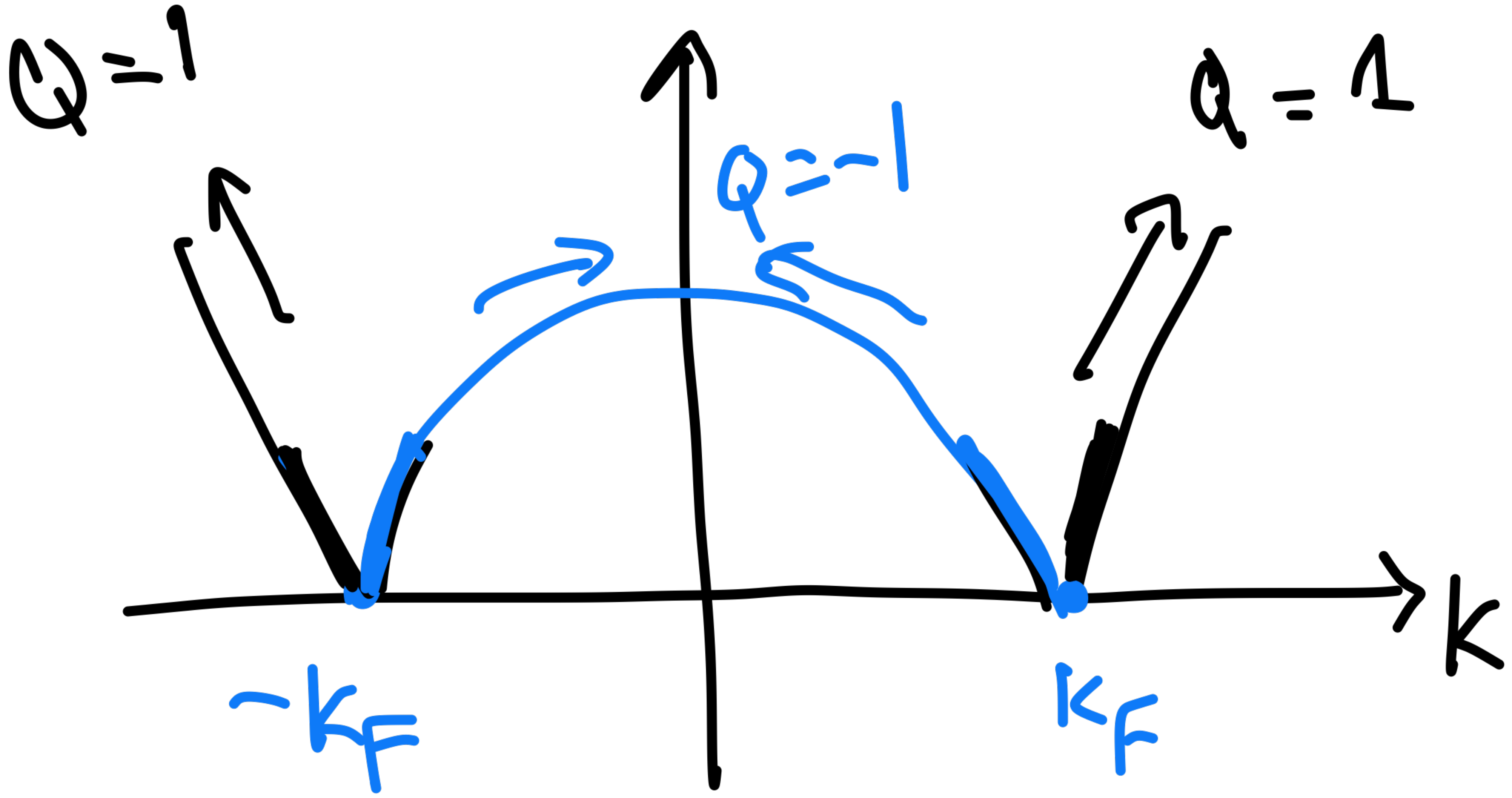
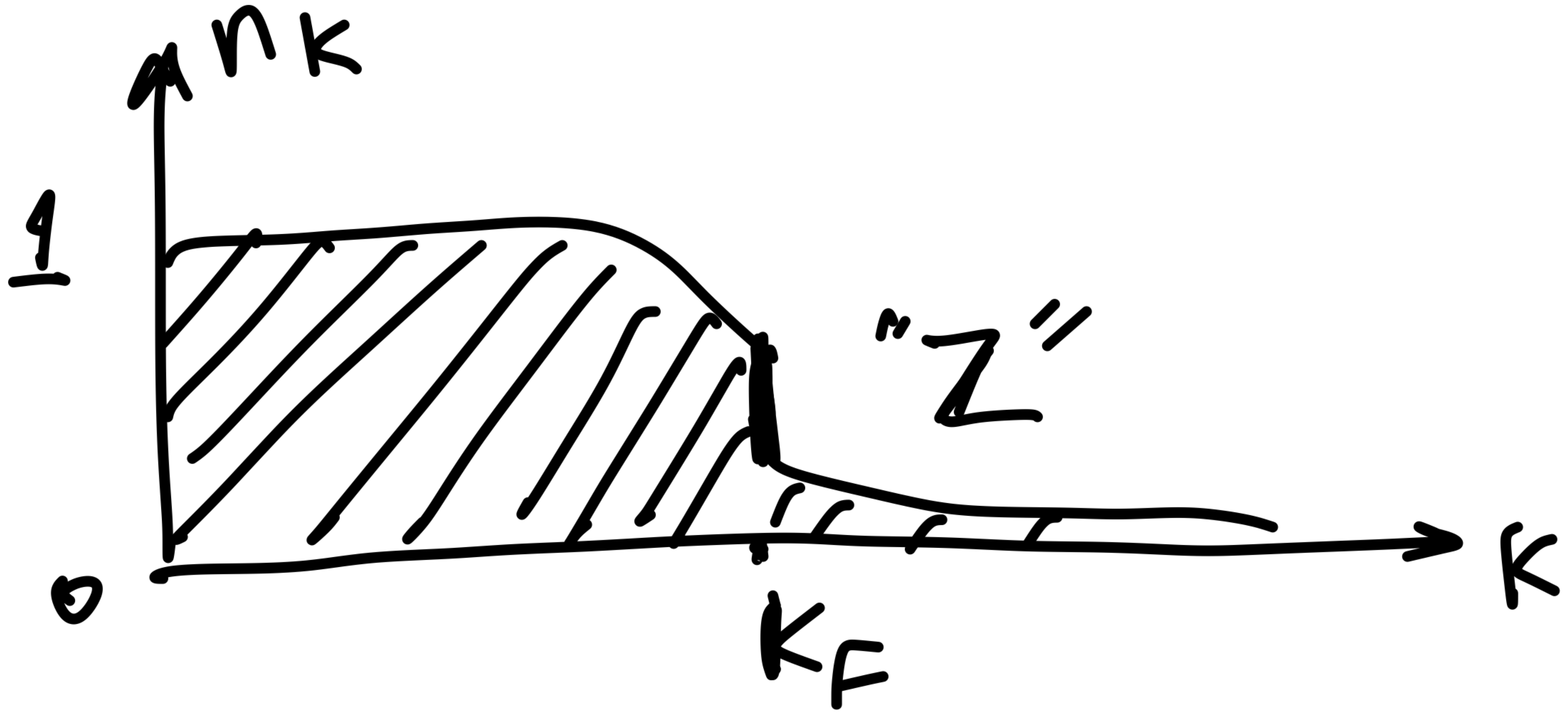
Fermi Gas





Fermi liquid





- Fermi Liquid theory (nice discussions in AGD, chapter 1 and 4)

- 1) there is a finite step in the occupation number at exactly K_F. This defines a Fermi surface. $n_{k_F-0} - n_{k_F+0} = Z$

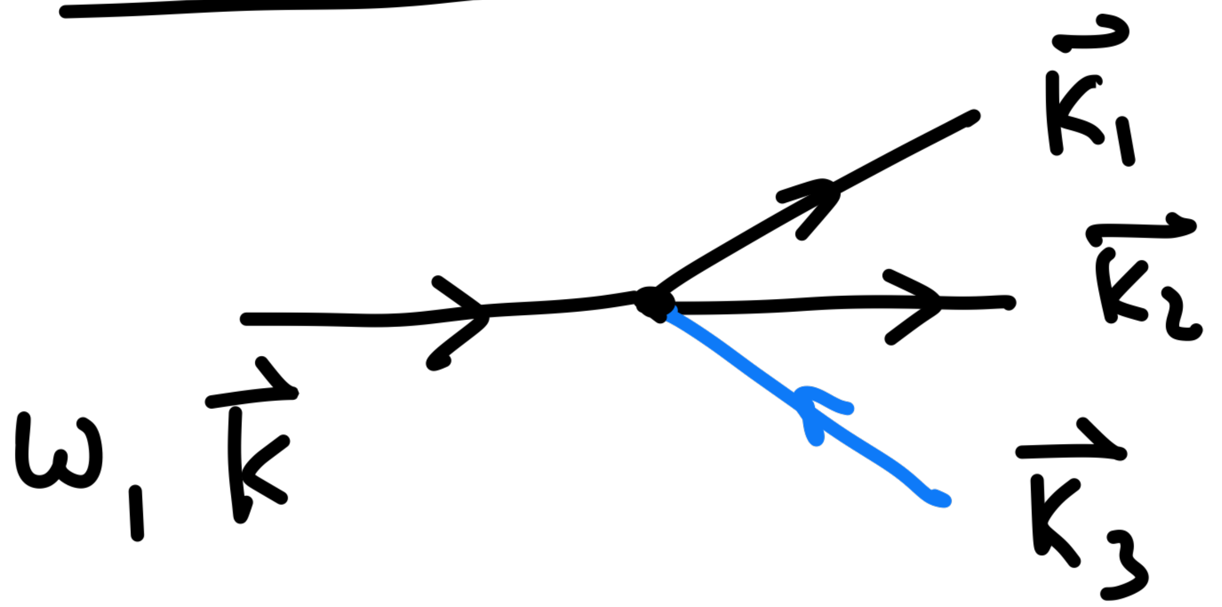
- 2) quasi-particles are of finite life time and become well defined once near Fermi surface,I.e. in the low energy sector. $\frac{1}{\tau_k} = \gamma_k \ll |\xi_k|$

- 3) apart from mass renormalization, wave function renormalization Z occurs at Fermi surface.

- 4) there are low energy emergent bosonic particles.

- 5) for a fixed k, time ordered 'G" is not analytical in either lower or upper frequency planes. However, retarded (advanced) green functions are analytical in lower (upper) plane for any k.(a proof in Lehmann Rep.)

Life time $\tau_k \to \infty$ when $k \to k_F$

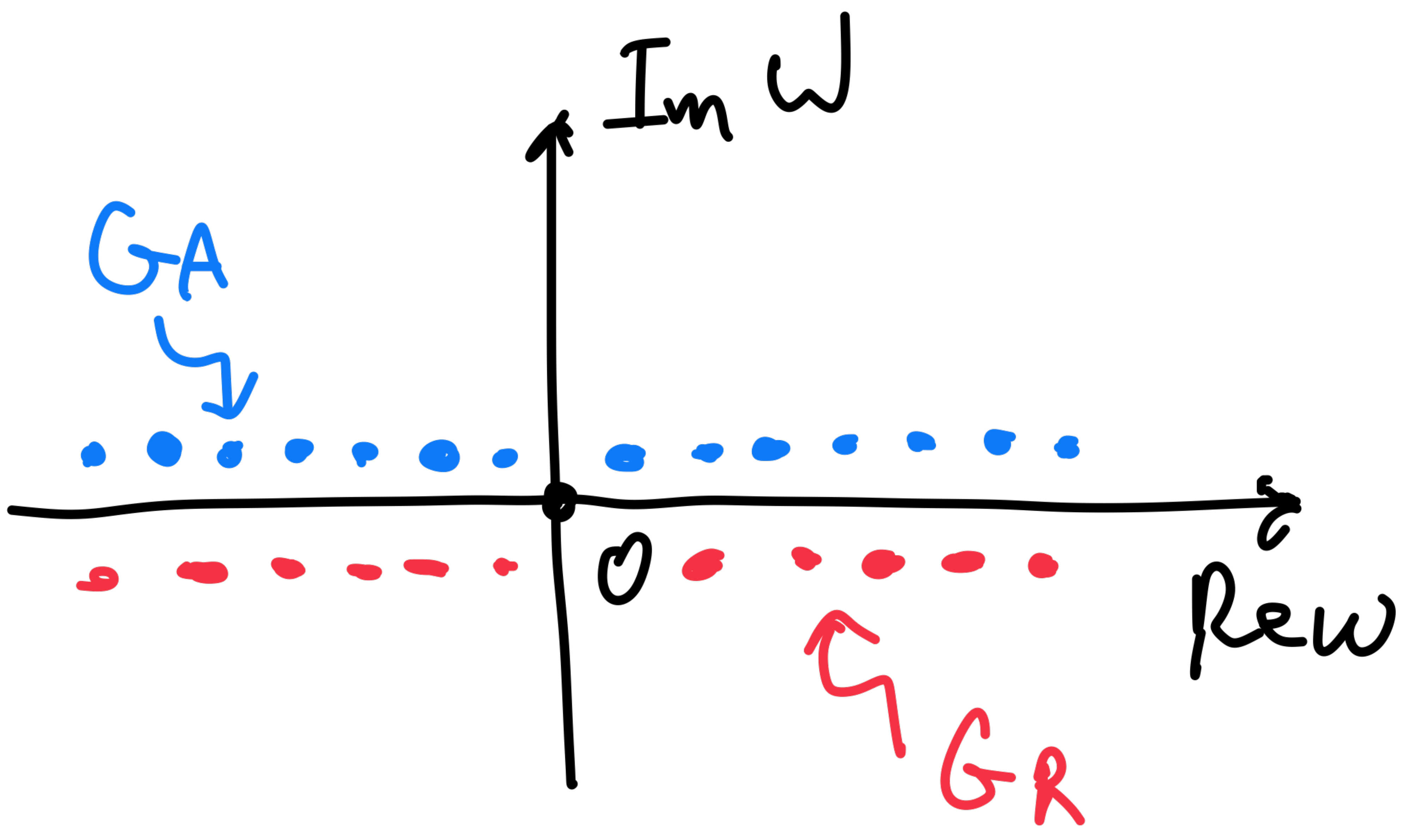


$|\vec{k}|, |\vec{k}_1|, |\vec{k}_2| > k_F, \quad |\vec{k}_3| < k_F$

$$\gamma_k \sim \sum_{\vec{k}_1, \vec{k}_2, \vec{k}_3} \delta\left(\vec{k} - \vec{k}_1 - \vec{k}_2 - (-\vec{k}_3)\right)\delta\left(\xi_{\vec{k}} - \xi_{\vec{k}_1} - \xi_{\vec{k}_2} - (-\xi_{\vec{k}_3})\right) \to \frac{\xi_{\vec{k}}^2}{E_F} \xrightarrow{k \to k_F} 0$$

and $\gamma_k \ll |\xi_{\vec{k}}|$

Very Useful phenomenology F. G



$$G(\omega,\vec{k}) = G_R(\omega,\vec{k})\,\Theta(\omega) + G_A(\omega,\vec{k})\,\Theta(-\omega)$$

G_R: analytical in upper half

G_A: analytical in Lower half

$$G(\omega,\vec{k}) = \frac{1}{\omega - \xi_k + i\delta\,\mathrm{sig}\,\omega} = \overbrace{\frac{1}{\omega - \xi_k + i\delta}}^{G_R}\,\Theta(\omega) + \overbrace{\frac{1}{\omega - \xi_k - i\delta}}^{G_A}\,\Theta(-\omega)$$

($\delta \to 0$ for F. G.)

"Fermi Liquid": pedagogical discussion ↙ "following Lehmann Rep"

$$G(\omega, K) = G_A(\omega, K)\,\Theta(-\omega) + G_R(\omega, K)\,\Theta(\omega)$$

A.L. in Lower half A.L. in Upper half

A "Quasi-particles (holes)" → "poles in Lower (Upper) half-plane for G_R (G_A) away from the Real axis"

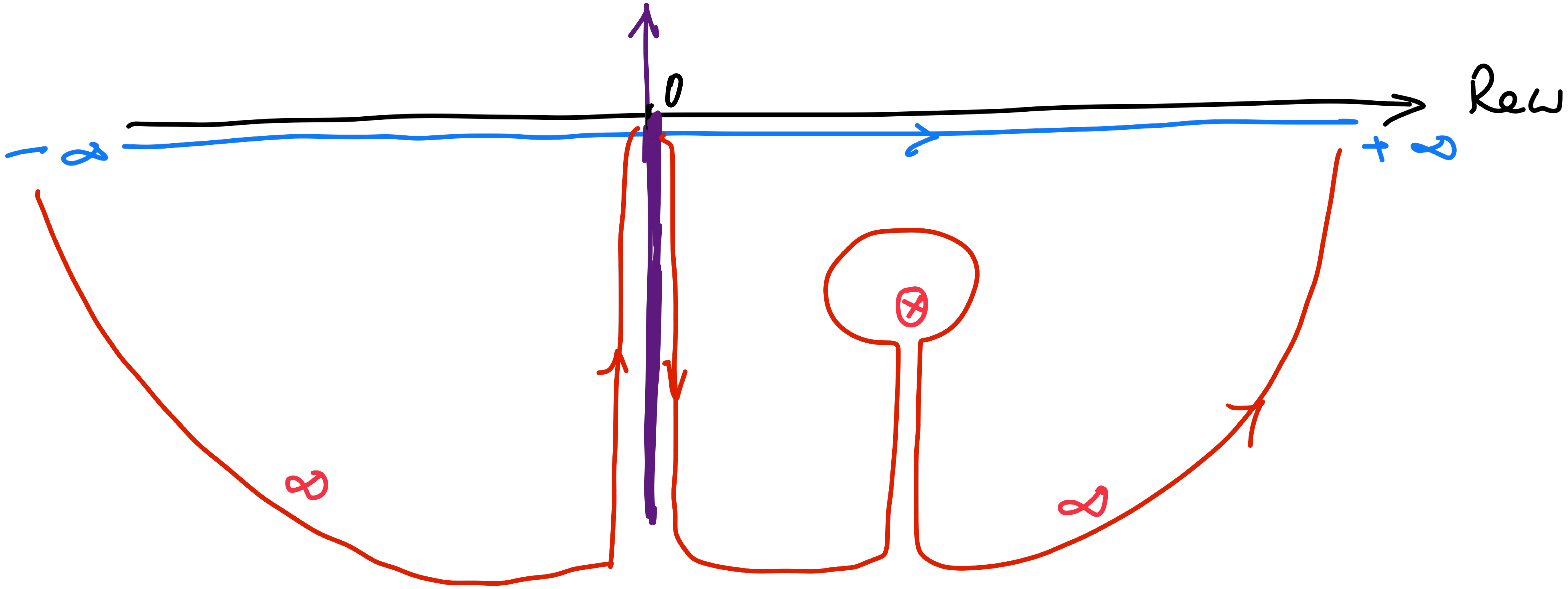
B "$\gamma_K \ll \xi_K$ or $\frac{\gamma_K}{\xi_K} \xrightarrow{K \to K_F} 0$" → "$\delta n_{K=K_F} = Z$"

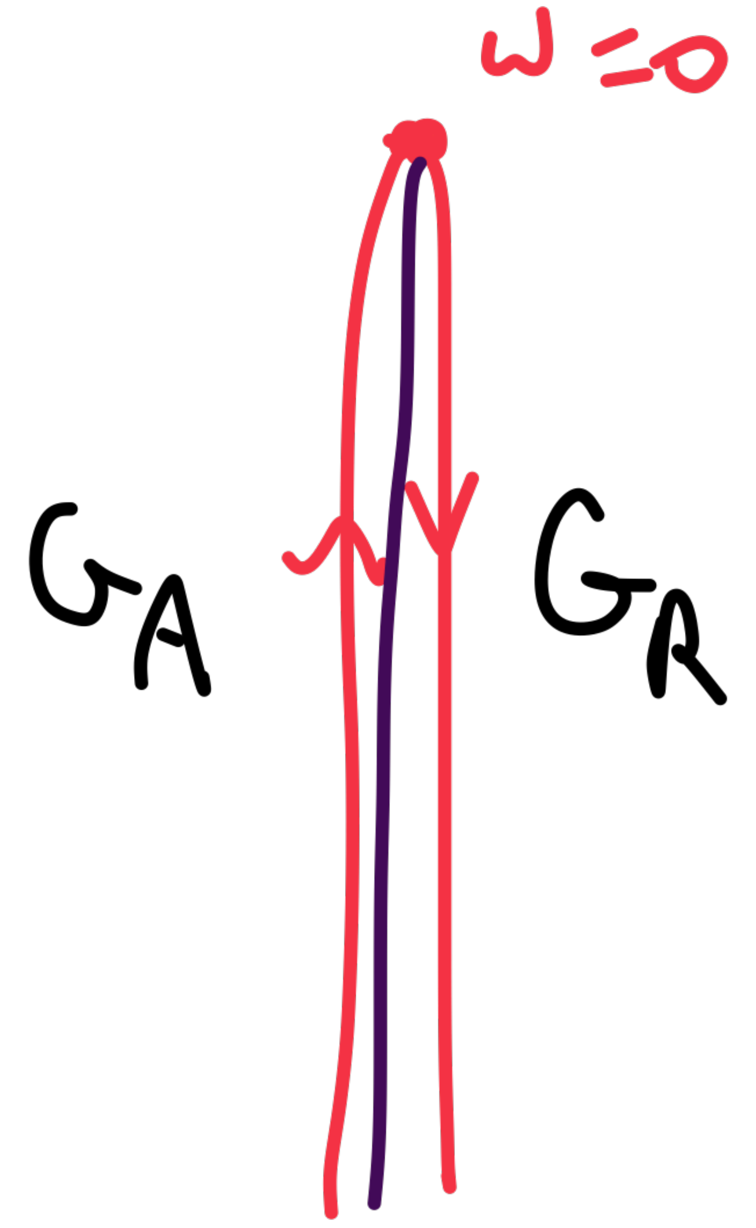
"Residual of poles"

$$G(k, t>0) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \, G(k,\omega) \, e^{-i\omega t} \, , \qquad t>0$$

$$\langle g.s. | -i \, \psi_k(t) \, \psi_k^{+}(0) \, | g.s. \rangle$$

$$G(k,\omega) = G_A \, \theta(-\omega) + G_R \, \theta(\omega)$$

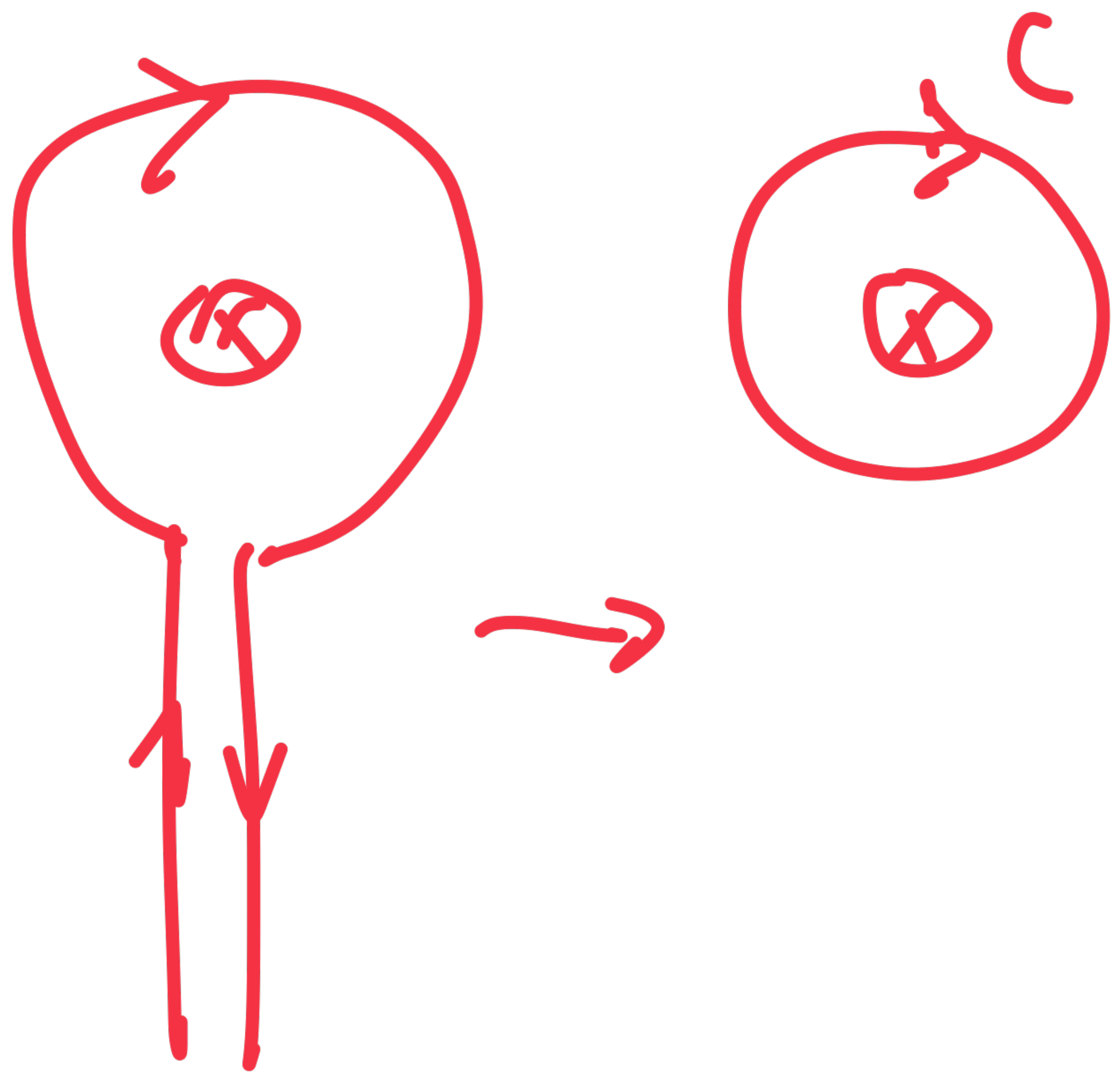




$$\int_{-i\infty}^{0} -2\,\mathrm{Im}\,G_R(\omega,k)\, e^{-i\omega t}\, d\omega \approx \frac{Z\gamma_k}{|\xi_k|^2}\,\frac{1}{t}$$

$$(t \gg |\xi_k|^{-1})$$

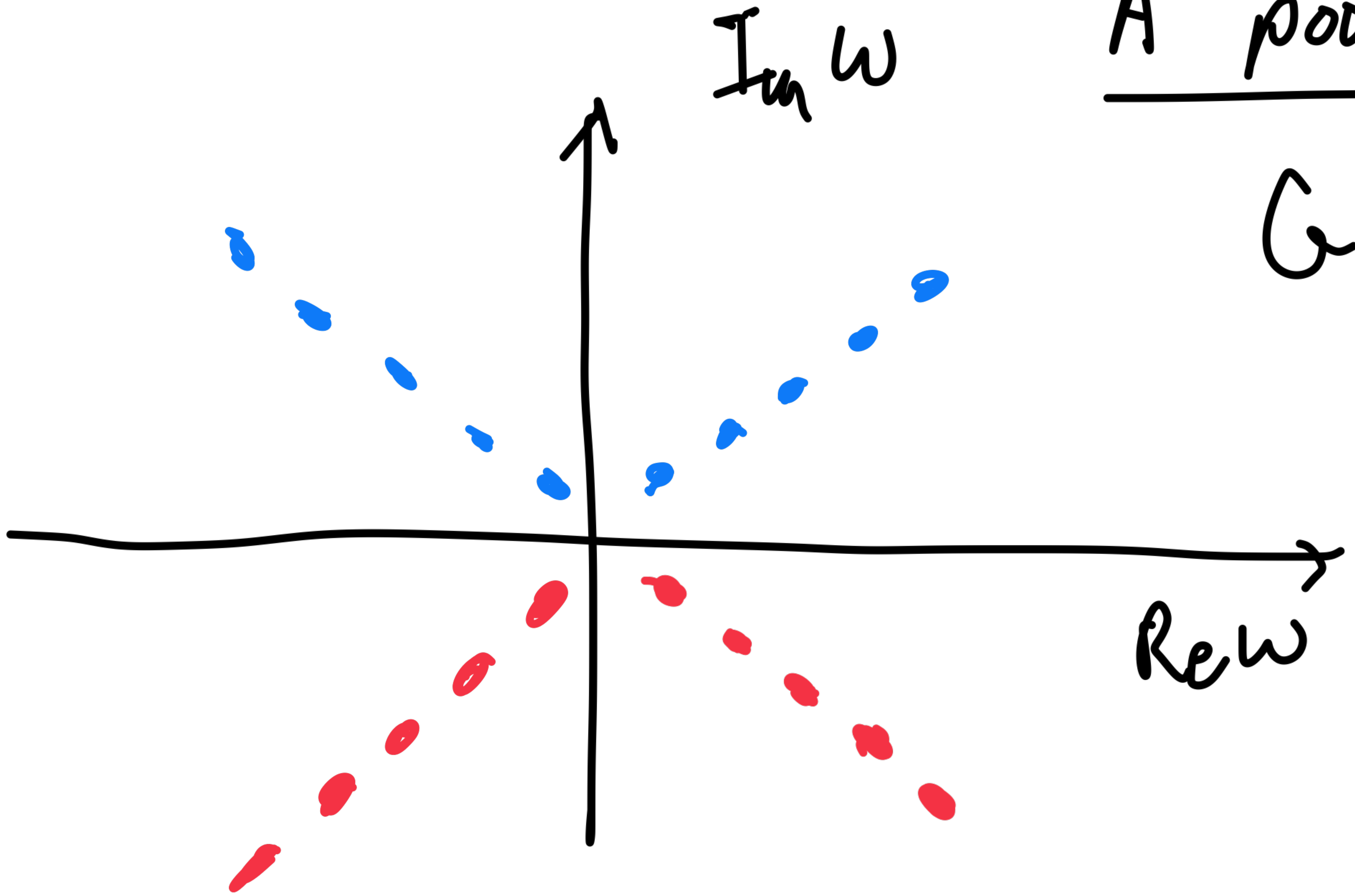
$$G_R(\omega,k) \sim \frac{Z}{\omega-\xi_k+i\gamma_k}, \quad \xi_k>0, \quad \gamma_k>0$$



$$\oint_c G_R(\omega,k)\, e^{-i\omega t}\, d\omega = -Z\, \underbrace{e^{-i\xi_k t-\gamma_k t}}_{\text{Quasi-particle}}$$

$$(\xi_k \gg \gamma_k > 0)$$

A poor man's approach

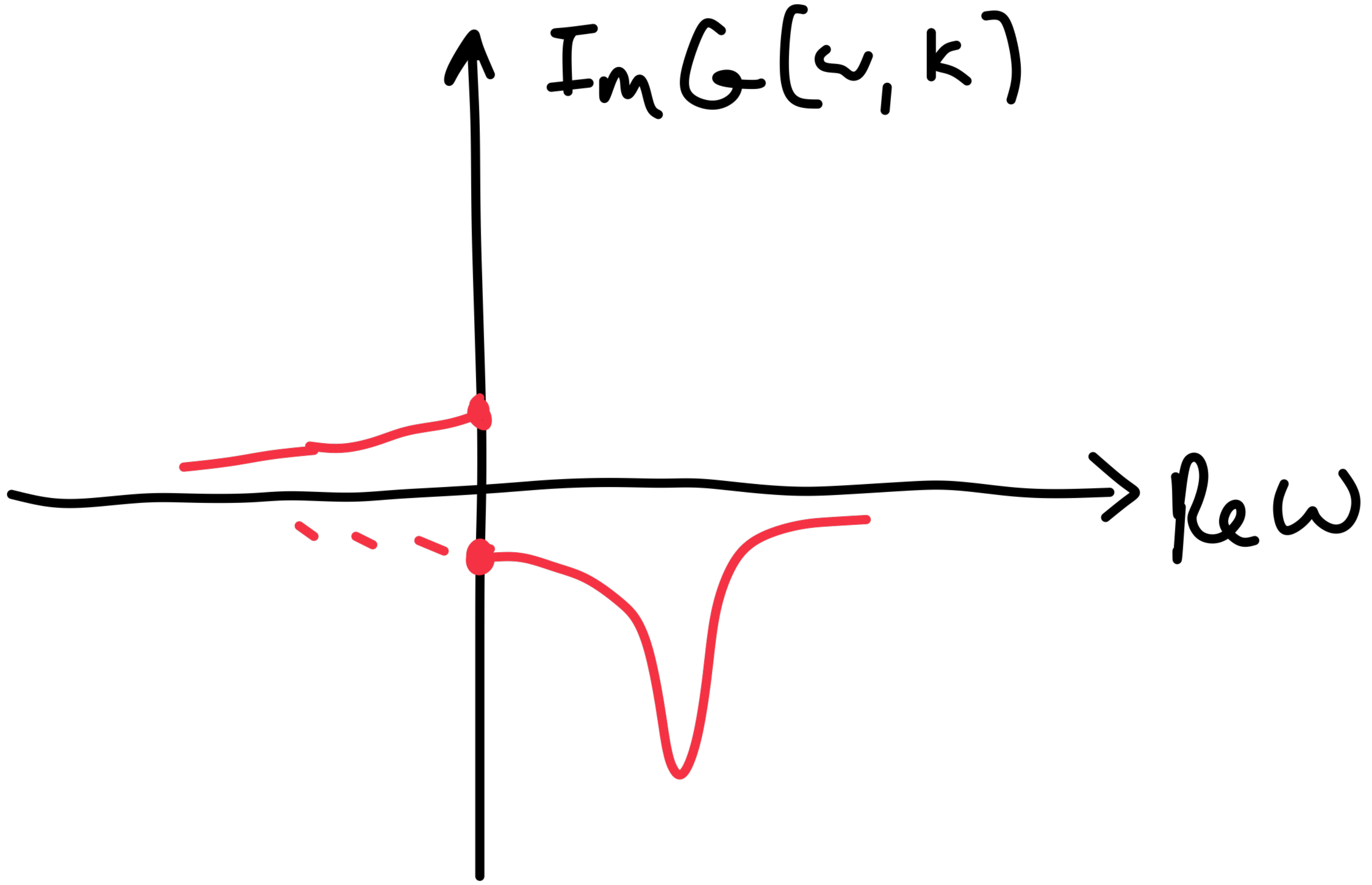


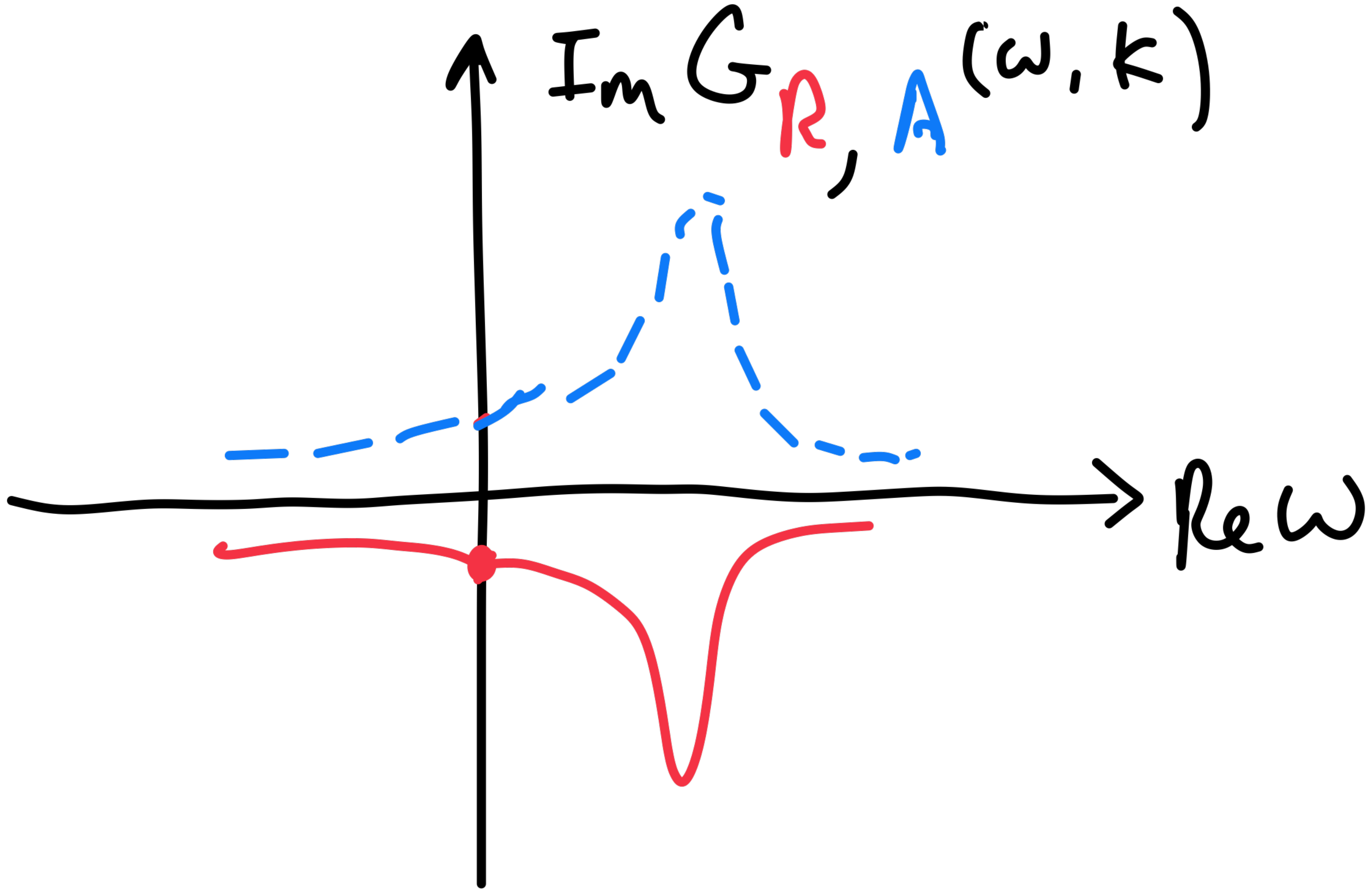
$$G(\omega,\vec{k}) = G_R(\omega,\vec{k})\,\theta(\omega) + G_A(\omega,k)\,\theta(-\omega)$$

$$G(\omega,\vec{k}) \simeq \frac{Z}{\omega - \xi_k + i\gamma_k \,\mathrm{sig}\,\omega}$$

$$\left(|\xi_k| \gg \gamma_k,\ Z \leq 1\right)$$

$$\mathrm{Im}\,G(\omega,\vec{k}) = -\theta(\omega)\frac{Z\,\gamma_k}{(\omega-\xi_k)^2+\gamma_k^{\,2}} + \theta(\omega)\frac{Z\,\gamma_k}{(\omega-\xi_k)^2+\gamma_k^{\,2}}$$





$$\mathrm{Im}\, G(\omega = 0^+) - \mathrm{Im}\, G(\omega = 0^-) \simeq -\frac{2\, Z\, \gamma_k}{\xi_k^2 + \gamma_k^2} \approx -\frac{2 Z \gamma_k}{\xi_k^2}$$

$$\left(\delta\, \mathrm{Im}\, G(\omega = 0) \to 0 \quad \text{when } \gamma_k = \delta \to 0 \right)$$

Application to $\hat{n}_k = \psi_k^\dagger(0)\psi_k(0) = 1 - \psi_k(0)\psi_k^\dagger(0)$

$$-i\,T\,\psi_k(t)\psi_k^\dagger(0) = -i\left[\theta(t)\,\psi_k(t)\psi_k^\dagger(0) - \theta(-t)\,\psi_k^\dagger(0)\psi_k(t)\right]$$

$$1 - n_k = \langle g.s.|\,\psi_k(0)\,\psi_k^\dagger(0)\,|g.s.\rangle = i\,G(k, t=0^+)$$

$$1 - n_k = i\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,G(k,\omega)\,e^{-i\omega\cdot 0^+}$$

$$1 - n_k = \langle g.s. | \psi_k(0)\, \psi_k^\dagger(0) | g.s. \rangle = -i\, G(\vec{k}, t = 0^-) = -i \int \frac{d\omega}{2\pi} G(k,\omega)\, e^{-i\omega 0^-}$$

$$G(\vec{k},\omega) = G_R(\vec{k},\omega)\,\Theta(\omega) + G_A(\vec{k},\omega)\,\Theta(-\omega)$$