Phys529B: Topics of Quantum Theory

Lecture 3: Basics of interacting fermions

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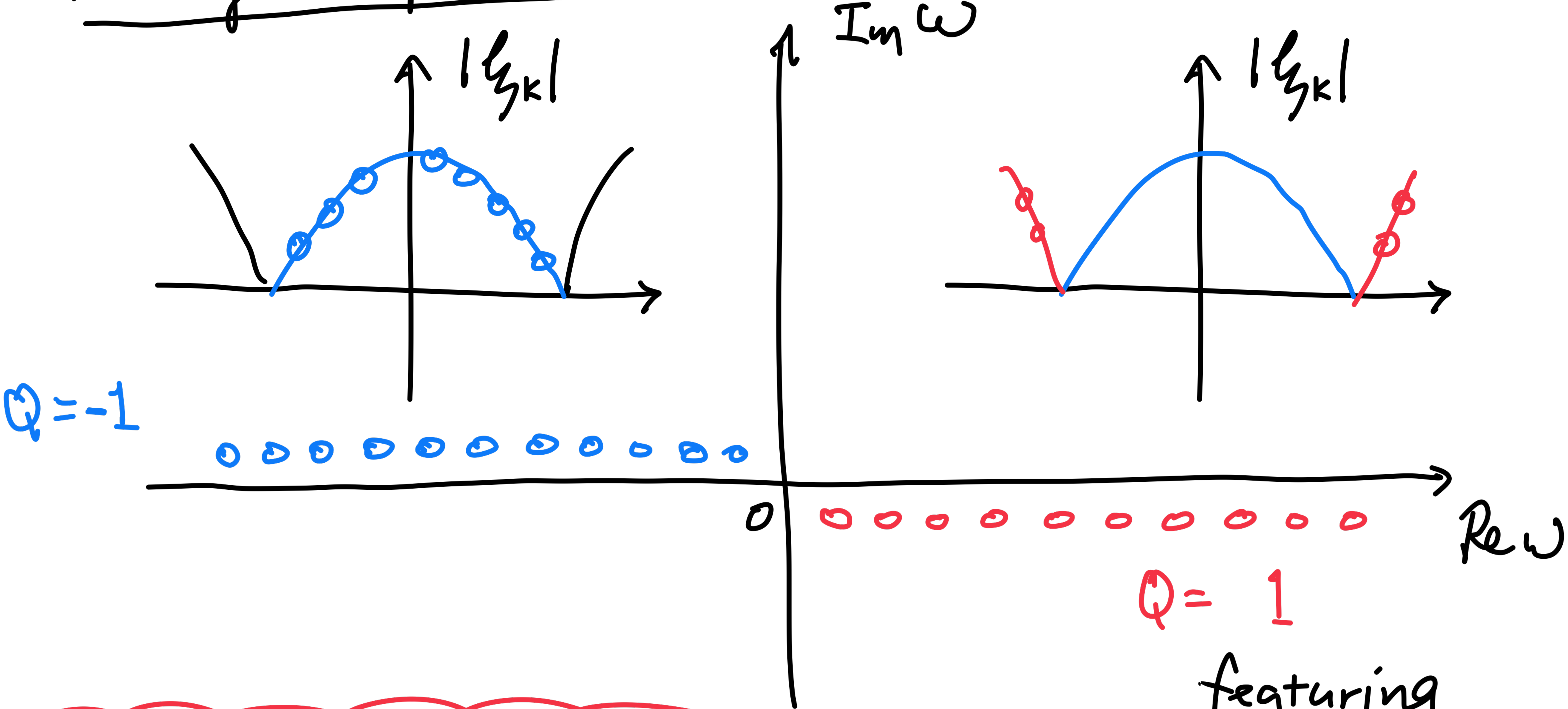
- Summary: non-interacting time ordered green’s function simple structures

- for k> K_F is analytical in the upper half plane (I.e. poles in lower half plane) ; all are Q=+1; For k<K_F, is analytical in the lower half plane (poles in upper half plane). All are Q=-1.

Putting all poles in the same sheet

$G(\vec{k}, \omega)$

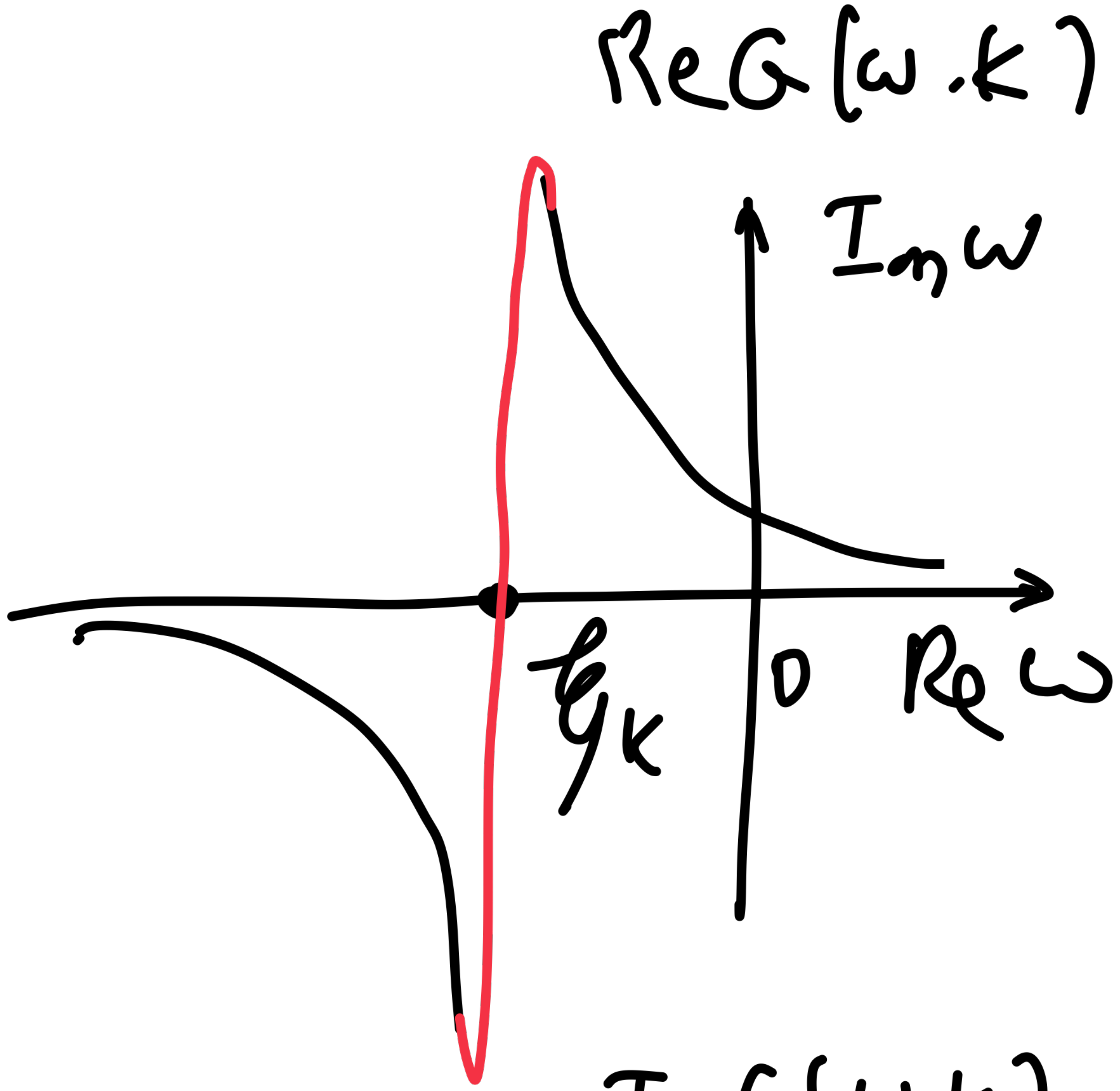


$$\boxed{Z_{\vec{k}} = \mathrm{Res}\; G(\vec{k}, \omega = \xi_k + i\delta_k) = 1, \; \delta_k \to 0}$$

featuring fermi Gas Only

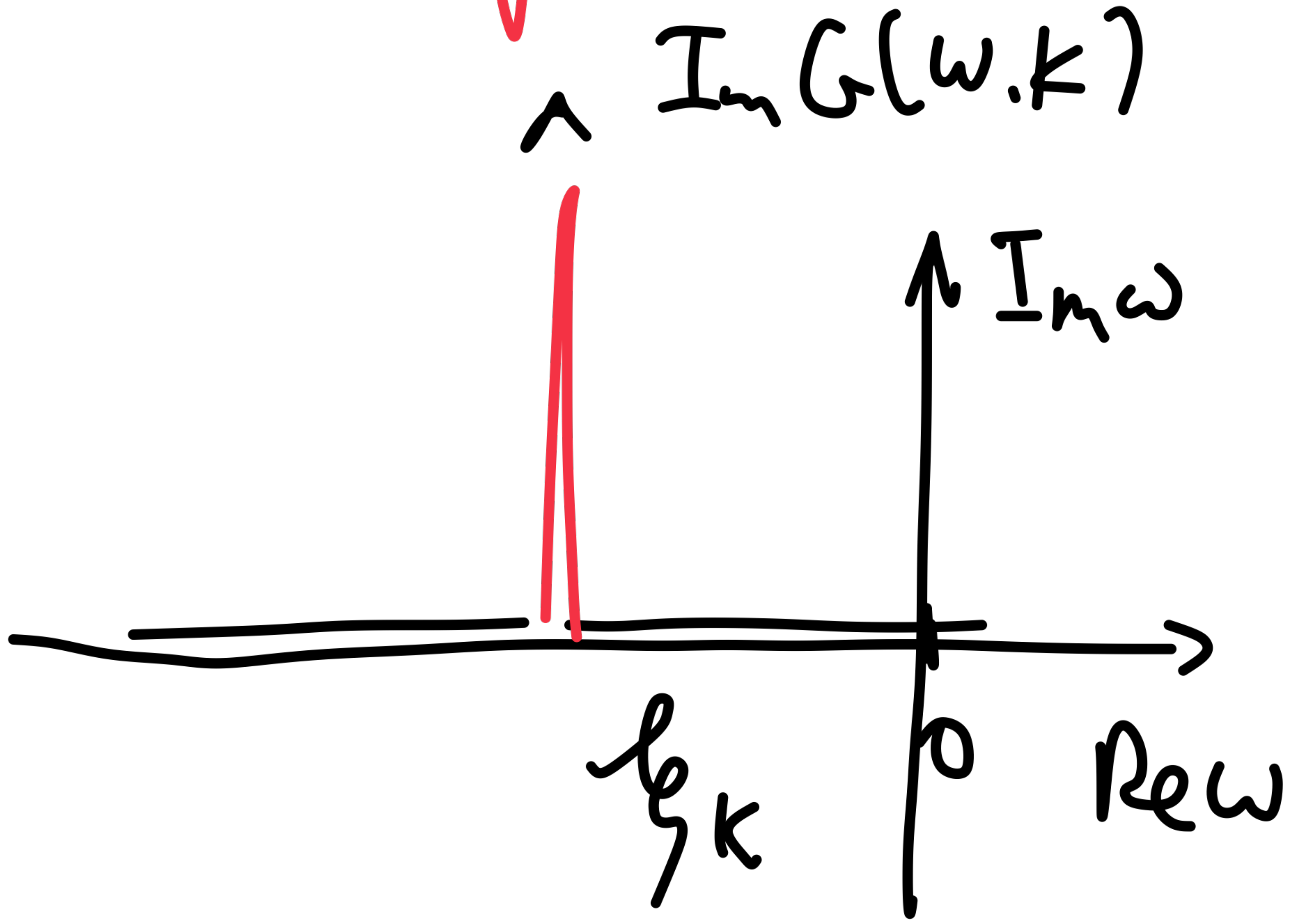
$$G = \frac{1}{\omega - \xi_k + i\delta\, \mathrm{Sig}\,\xi_k}, \quad \text{F. G.} \quad \delta \to 0$$

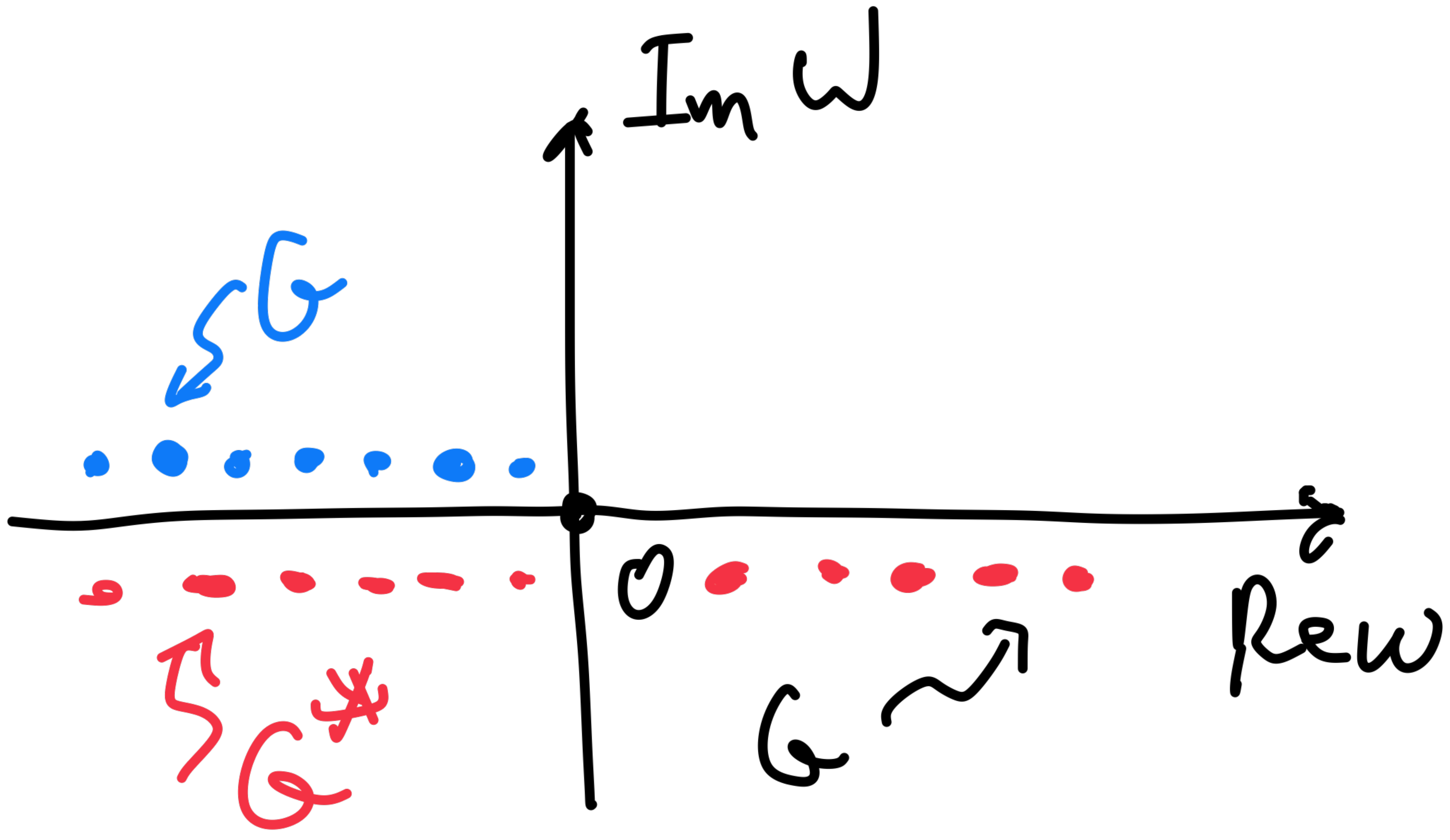
$$\mathrm{Re}\, G(\omega, k) = \frac{\omega - \xi_k}{(\omega - \xi_k)^2 + \delta^2}, \quad \delta \to 0$$



$$\mathrm{Im}\, G(\omega, k) = \frac{-\delta\, \mathrm{Sig}\,\xi_k}{(\omega - \xi_k)^2 + \delta^2}$$

$$\xrightarrow{\delta \to 0} -\pi\, \delta(\omega - \xi_k)\, \mathrm{Sig}\,\omega$$





useful phenomenology

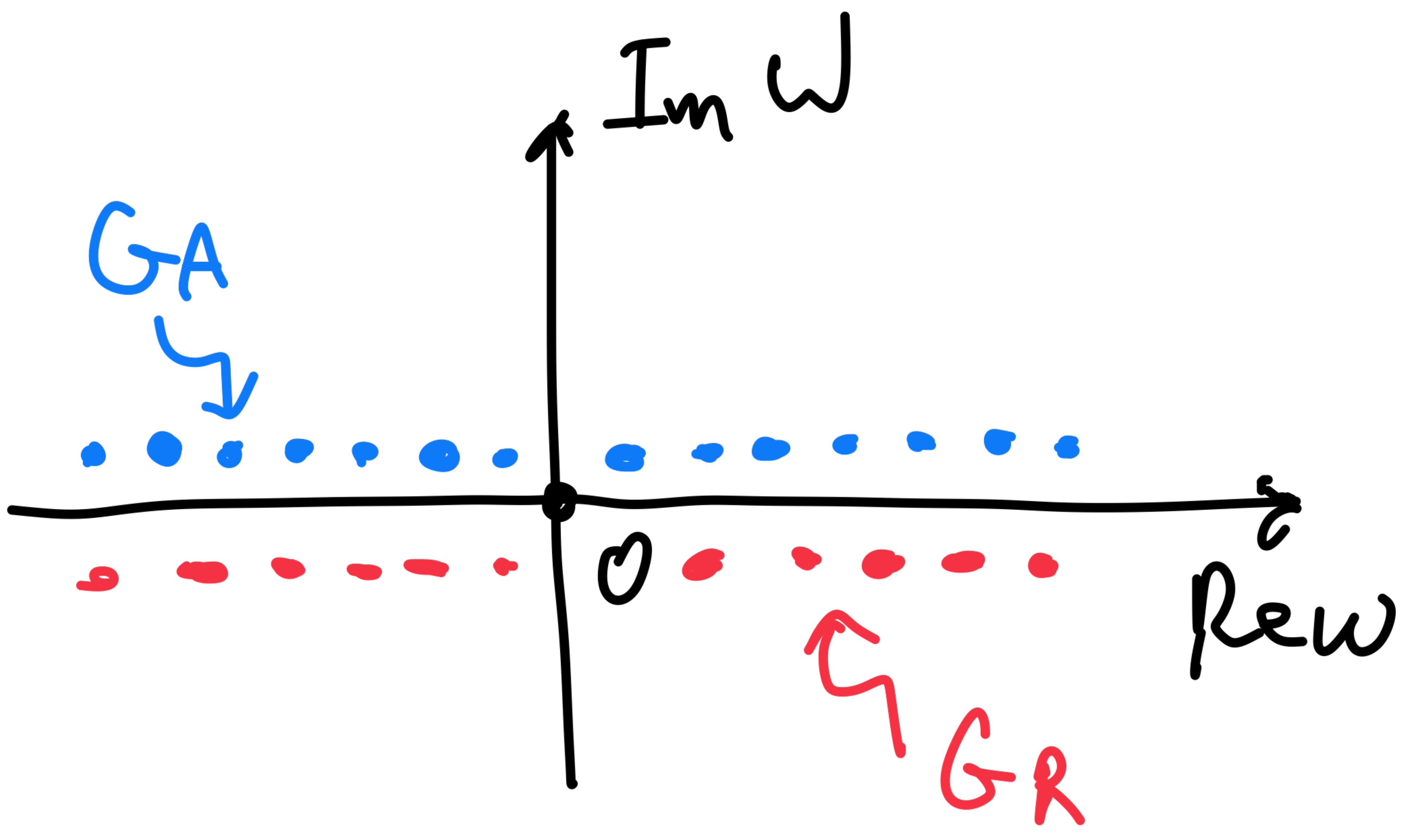
$$G_R(\omega,\vec{k}) = G(\omega,\vec{k})\,\theta(\omega) + G^*(\omega,\vec{k})\,\theta(-\omega)$$

$$G_A(\omega,\vec{k}) = G^*(\omega,\vec{k})\,\theta(\omega) + G(\omega,\vec{k})\,\theta(-\omega)$$

$$G(\omega,\vec{k}) = \frac{1}{\omega-\xi_k+i\delta\,\mathrm{sig}\,\omega} = \frac{1}{\omega-\xi_k+i\delta}\,\theta(\omega) + \frac{1}{\omega-\xi_k-i\delta}\,\theta(-\omega)$$

($\delta \to 0$ for F. G.)

"F. G". :



useful phenomenology

$$G(\omega,\vec{k}) = G_R(\omega,\vec{k})\,\Theta(\omega) + G_A(\omega,\vec{k})\,\Theta(-\omega)$$

analytical in upper half

analytical in lower half

$$G(\omega,\vec{k}) = \frac{1}{\omega - \xi_k + i\delta\,\mathrm{sig}\,\omega} = \overbrace{\frac{1}{\omega - \xi_k + i\delta}}^{G_R}\,\Theta(\omega) + \overbrace{\frac{1}{\omega - \xi_k - i\delta}}^{G_A}\,\Theta(-\omega)$$

($\delta \to 0$ for F. G.)

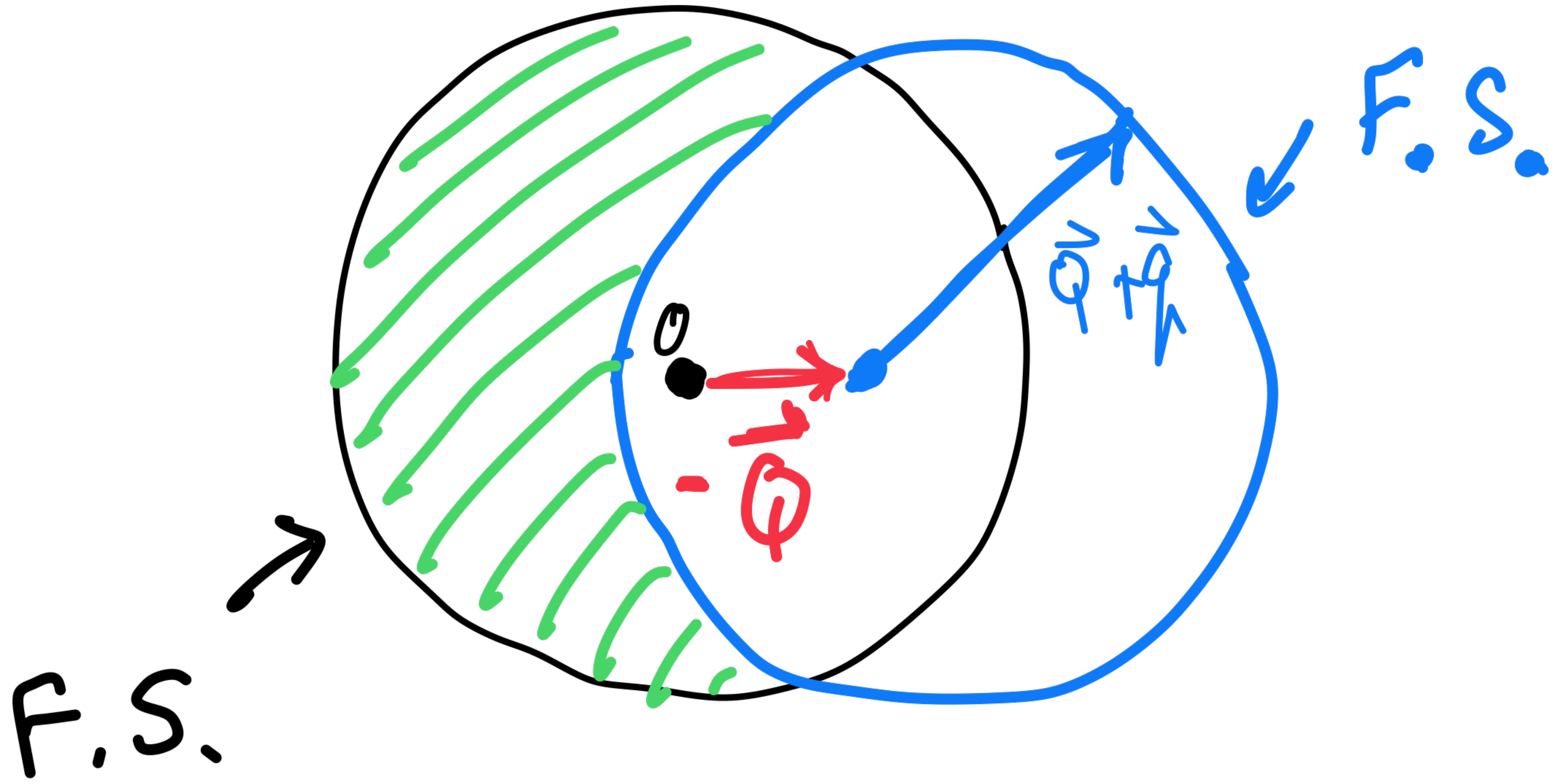
Composite bosonic states as a continuum

$$\phi^{\dagger}_{Q} = \sum_{\vec{q}} A_{Q,\vec{q}}\, \psi^{\dagger}_{\vec{Q}+\vec{q}}\, \psi_{\vec{q}} \quad (\text{"}Q=0 \text{ Sector"})$$

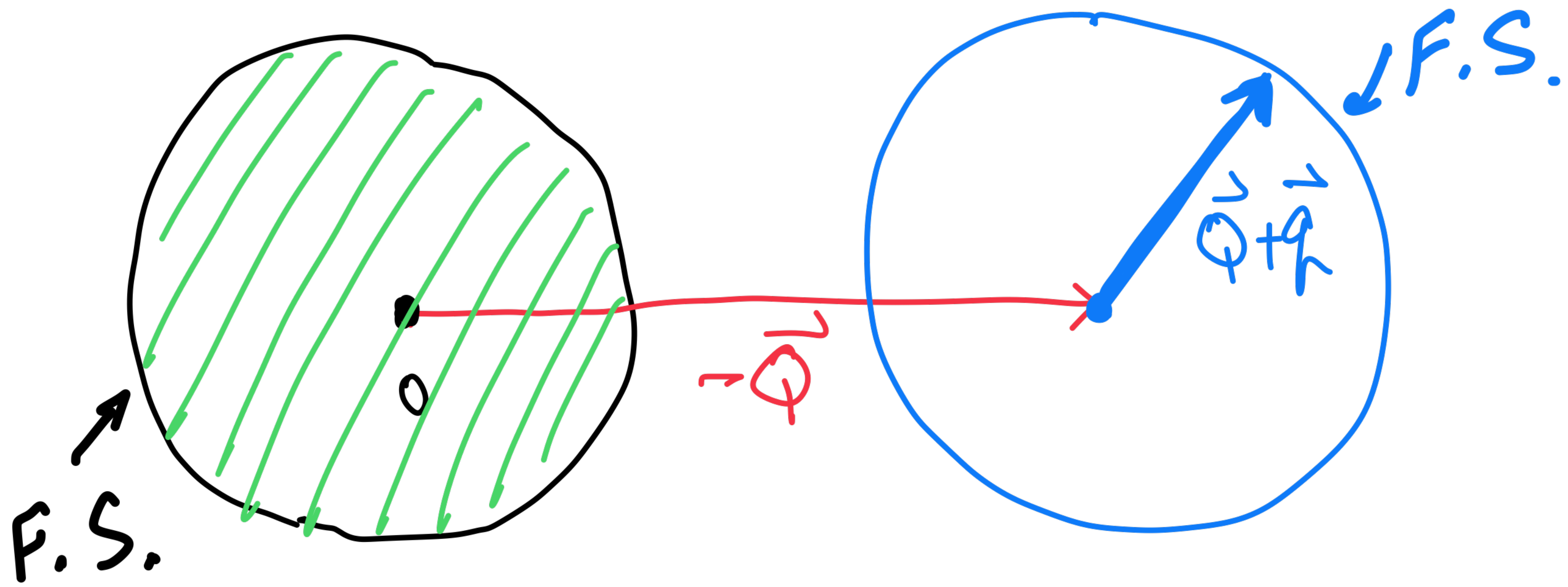
charge neutral sector

$$|\vec{q}| < k_F, \quad |\vec{Q}+\vec{q}| > k_F$$

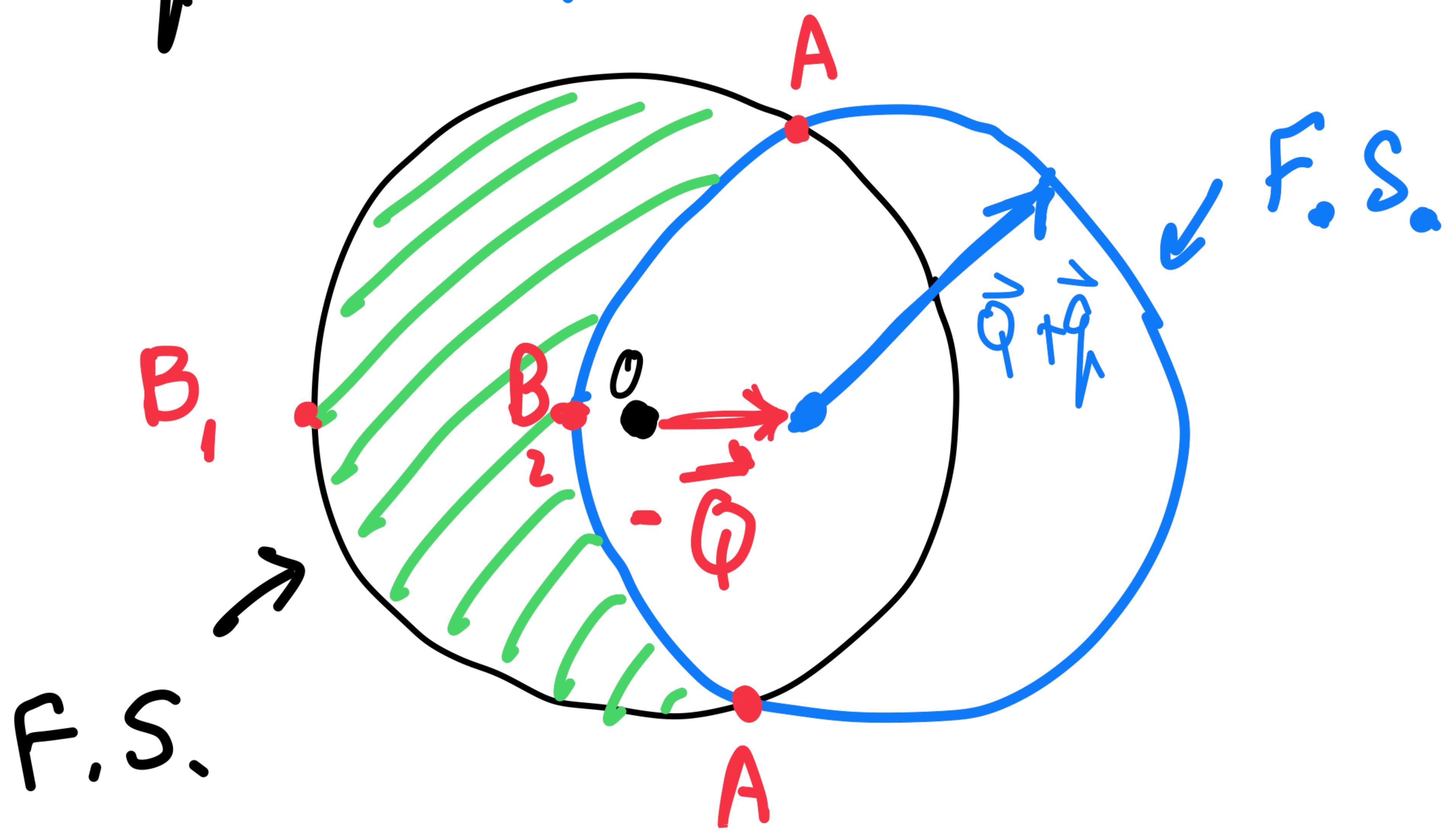
$|\vec{Q}| < 2k_F$



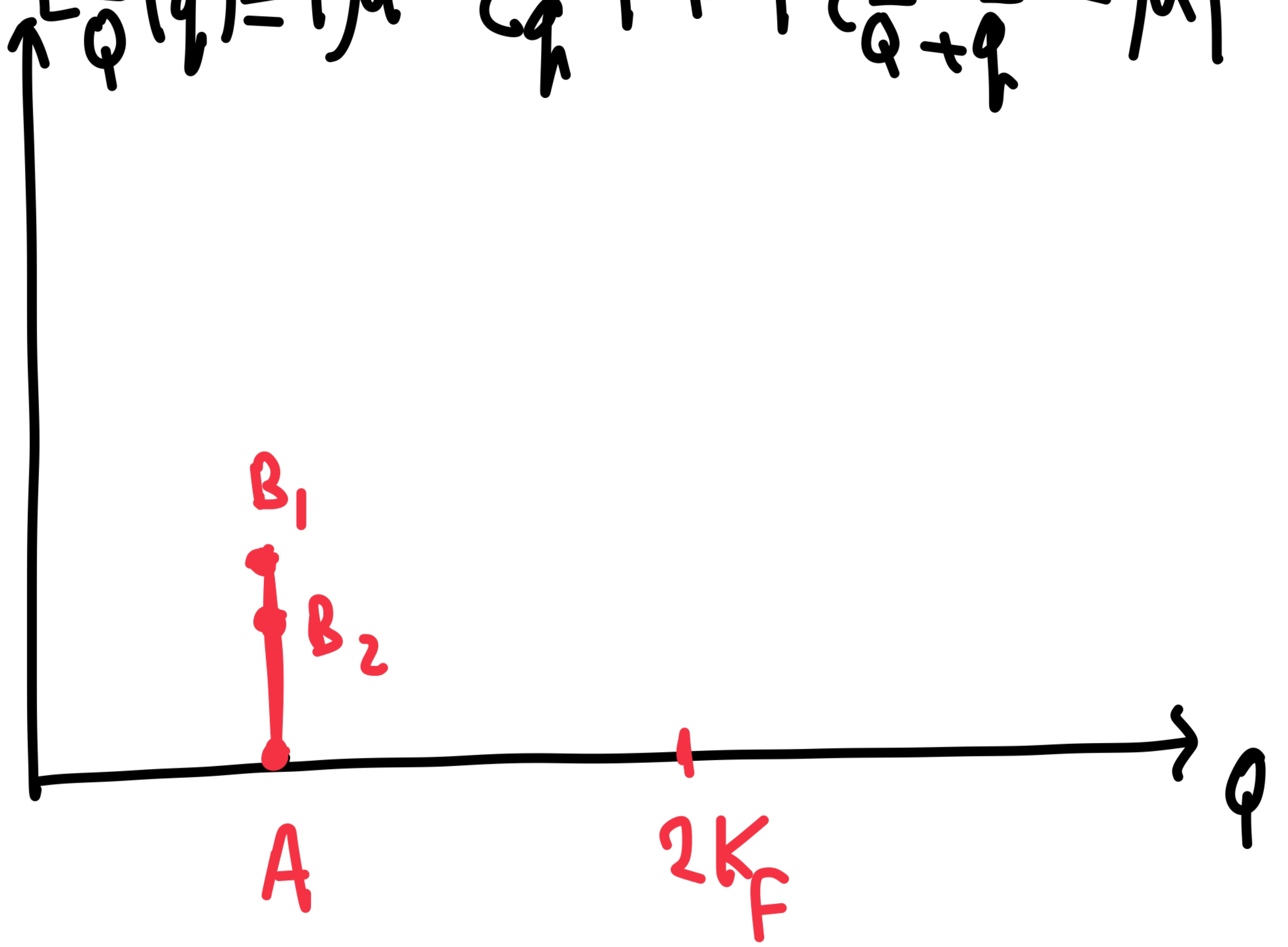
$|\vec{Q}| > 2k_F$



$\theta(\mu-\varepsilon_{\vec{q}})\,\theta(\varepsilon_{\vec{Q}+\vec{q}}-\mu)$ $\quad\mu=\varepsilon_{k_F}$



$E_{\vec{Q}}(\vec{q})=|\mu-\varepsilon_{\vec{q}}|+|\varepsilon_{\vec{Q}+\vec{q}}-\mu|$



At A, $\varepsilon_{\vec{Q}}=0$

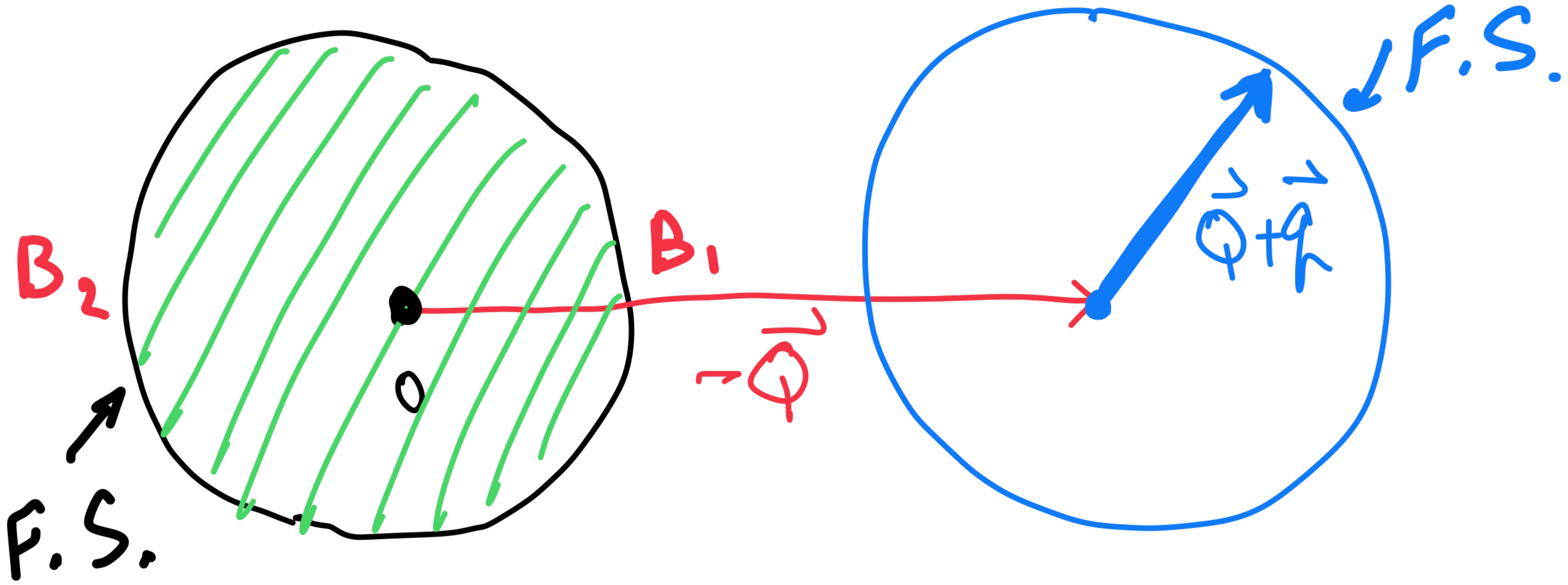
At B₁, $\varepsilon_{\vec{Q}}=\varepsilon_{k_F+Q}-\varepsilon_{k_F}\overset{\vec{Q}\to 0}{\simeq} Q\,k_F$

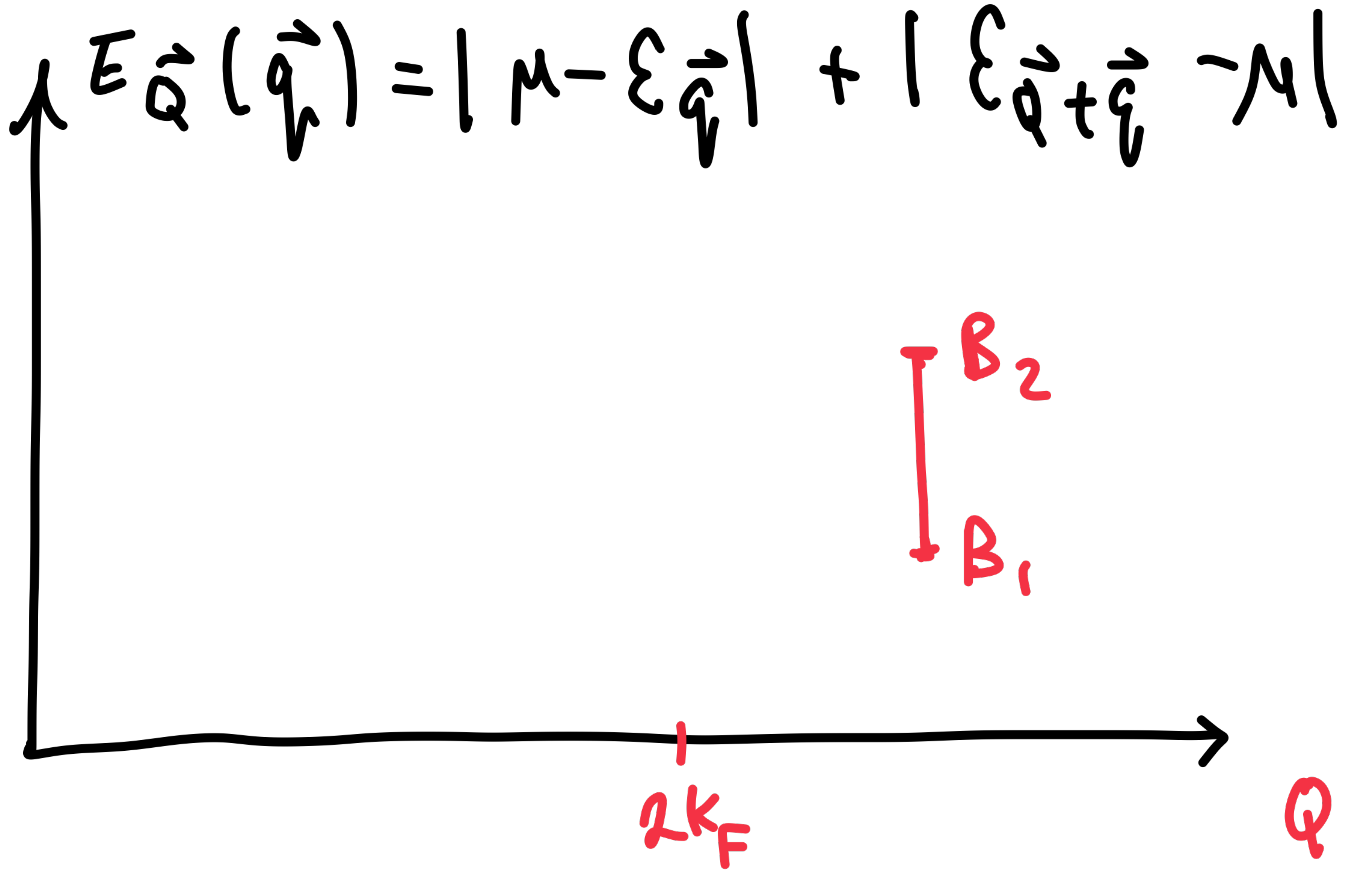
At B₂, $\varepsilon_{\vec{Q}}=\varepsilon_F-\varepsilon_{k_F-Q}\overset{\vec{Q}\to 0}{\simeq} Q\,k_F$

$E_{\vec{Q}}(\vec{q})$ depends on $\vec{q}$ as well.

with gapless continuum

$\Theta(\mu - \varepsilon_{\vec{q}})\,\Theta(\varepsilon_{\vec{Q}+\vec{q}} - \mu)$ $\quad \mu = \varepsilon_{k_F}$





At B_2, $\varepsilon_{\vec{Q}} = \varepsilon_{k_F+Q} - \varepsilon_{k_F} = \frac{Q^2}{2} + Qk_F$

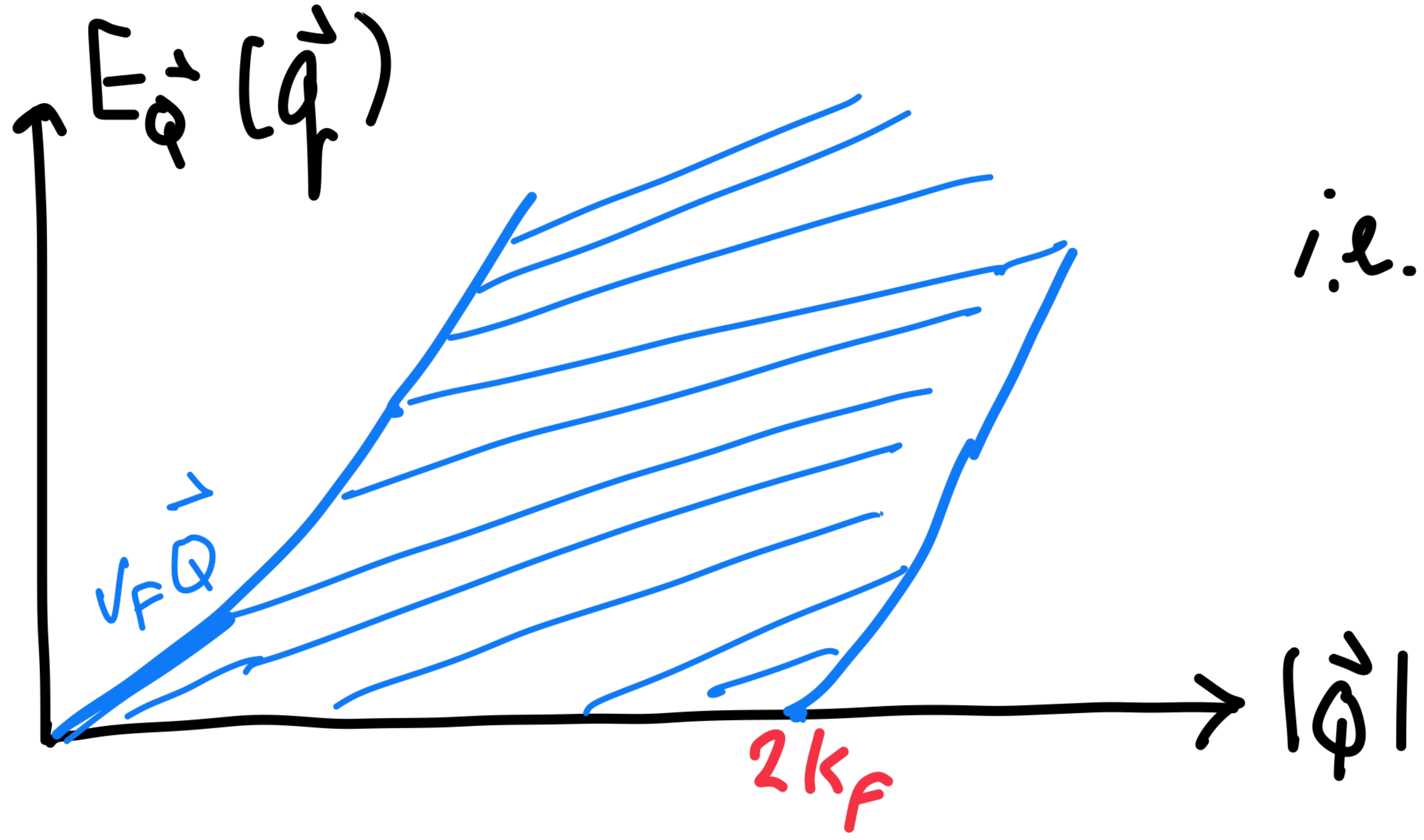
At B_1, $\varepsilon_{\vec{Q}} = \varepsilon_{Q-k_F} - \varepsilon_{k_F} = \frac{Q^2}{2} - Qk_F$

$|\vec{Q}| \geq 2k_F$

(Only $Q = 2k_F$, $\varepsilon_{\vec{Q}} = 0$ at B_1)

$E_{\vec{Q}}(\vec{q})$ is a gaped Continuum.

(Gapless only at $|\vec{Q}| = 2k_F$)



Continuum for bosonic states

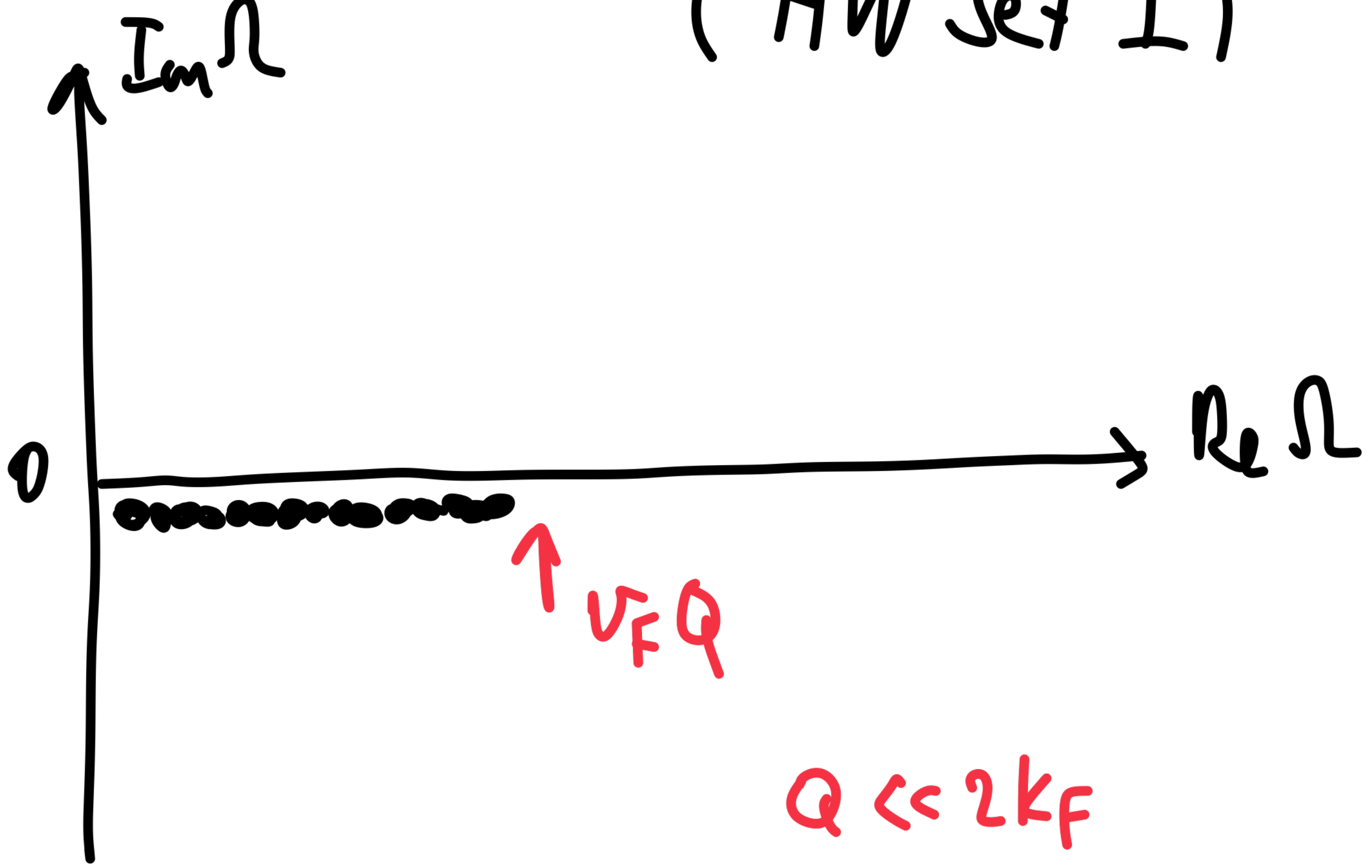
i.e. No bosonic "particle"s in F. G.

Unlike $G(\vec{k},\omega)$, $G_B(\vec{Q},\Omega)$ doesn't have isolated simple poles; instead it has a cut, i.e. infinite number of poles.

$$G_B(\vec{Q},t) = -i\langle g.s.|T\,\phi_{\vec Q}(t)\,\phi^\dagger_{\vec Q}(0)\,|g.s.\rangle$$

$$G_B(\vec{Q},\Omega) \simeq \int \frac{d\vec{q}}{(2\pi)^3}\frac{d\varepsilon}{2\pi}\; G(\vec{Q}+\vec{q},\Omega+\varepsilon)\,G(\vec{q},\varepsilon)\,\mathbb{Z}(\vec{Q},\vec{q})$$

After analytical Continuation, $G_B(\vec{q},\Omega)$
(HW Set I)

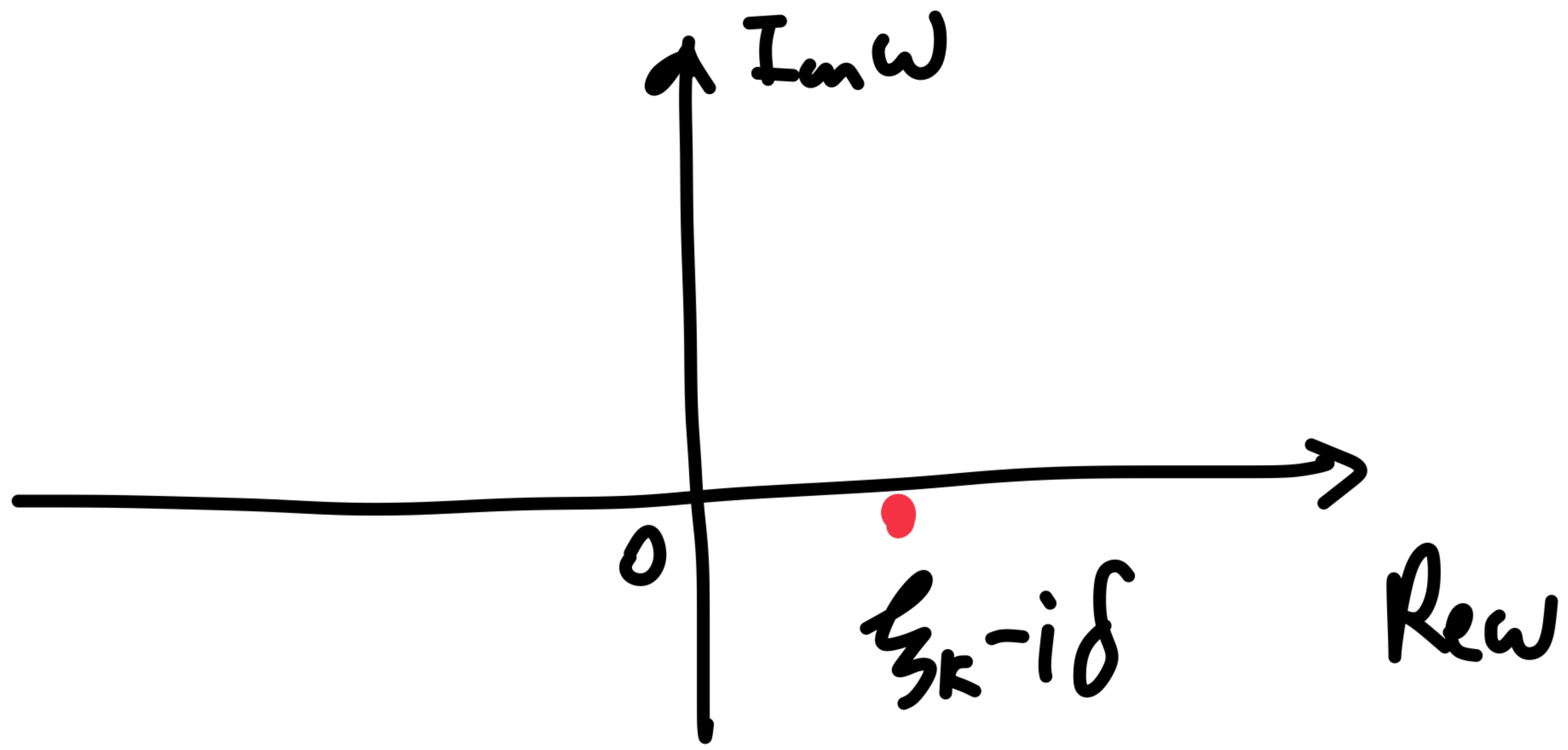


$Q \ll 2k_F$

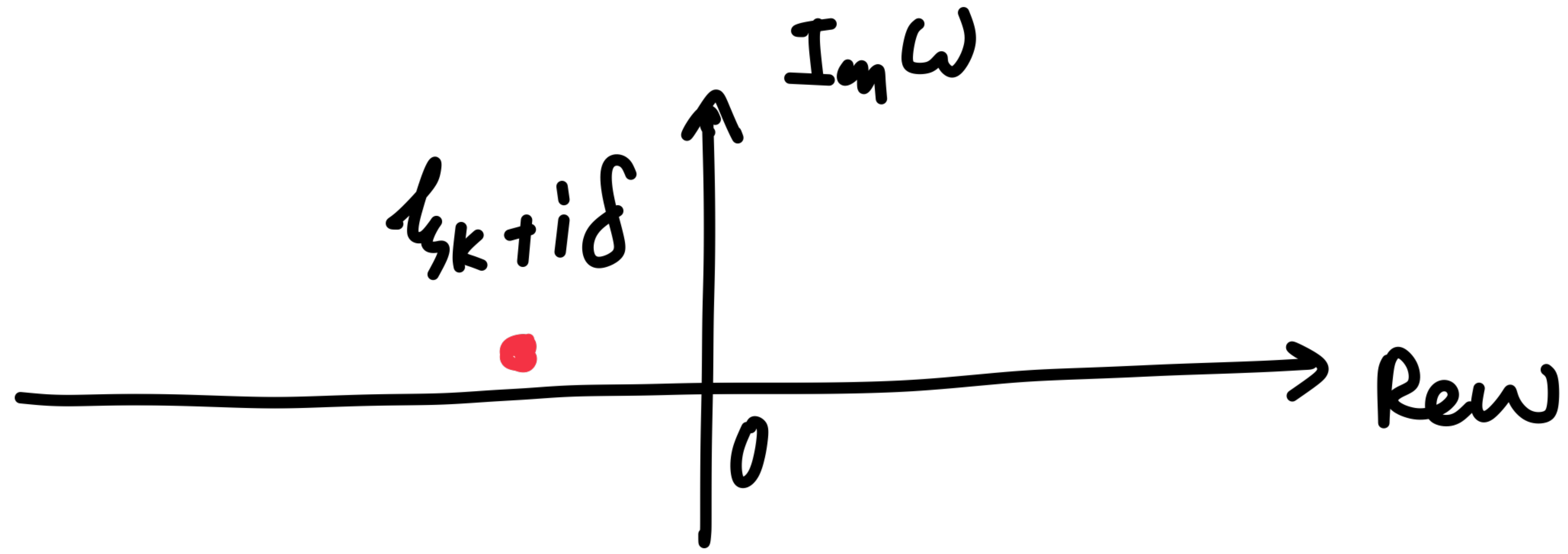
$Q > 2k_F$

"No-particle feature"

Comparison to $G(\vec{k},\omega)$

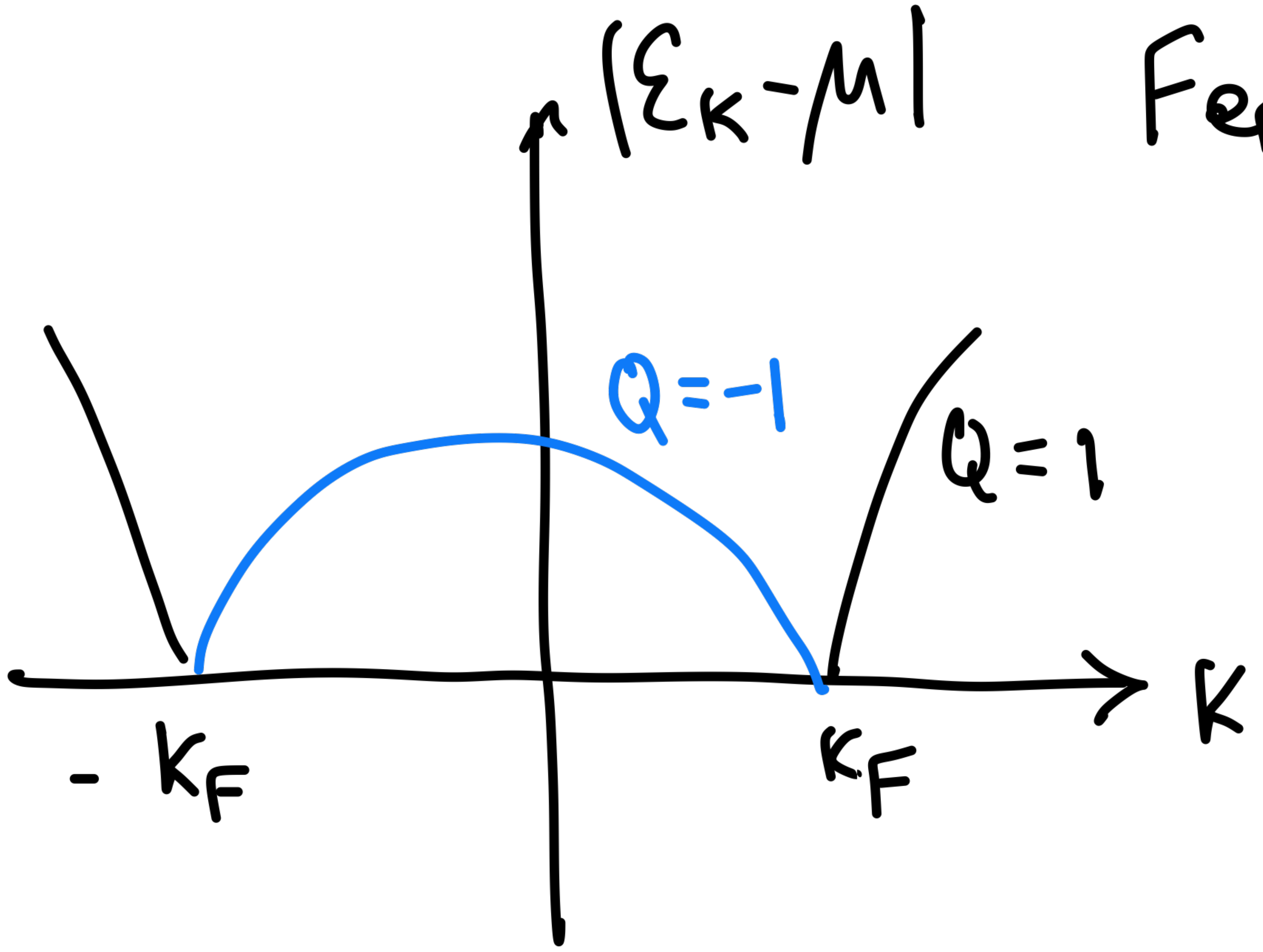


$$\frac{1}{\pi}\,\mathrm{Im}\,G(\vec{k},\omega) = \delta(\omega - \xi_{\vec{k}})$$

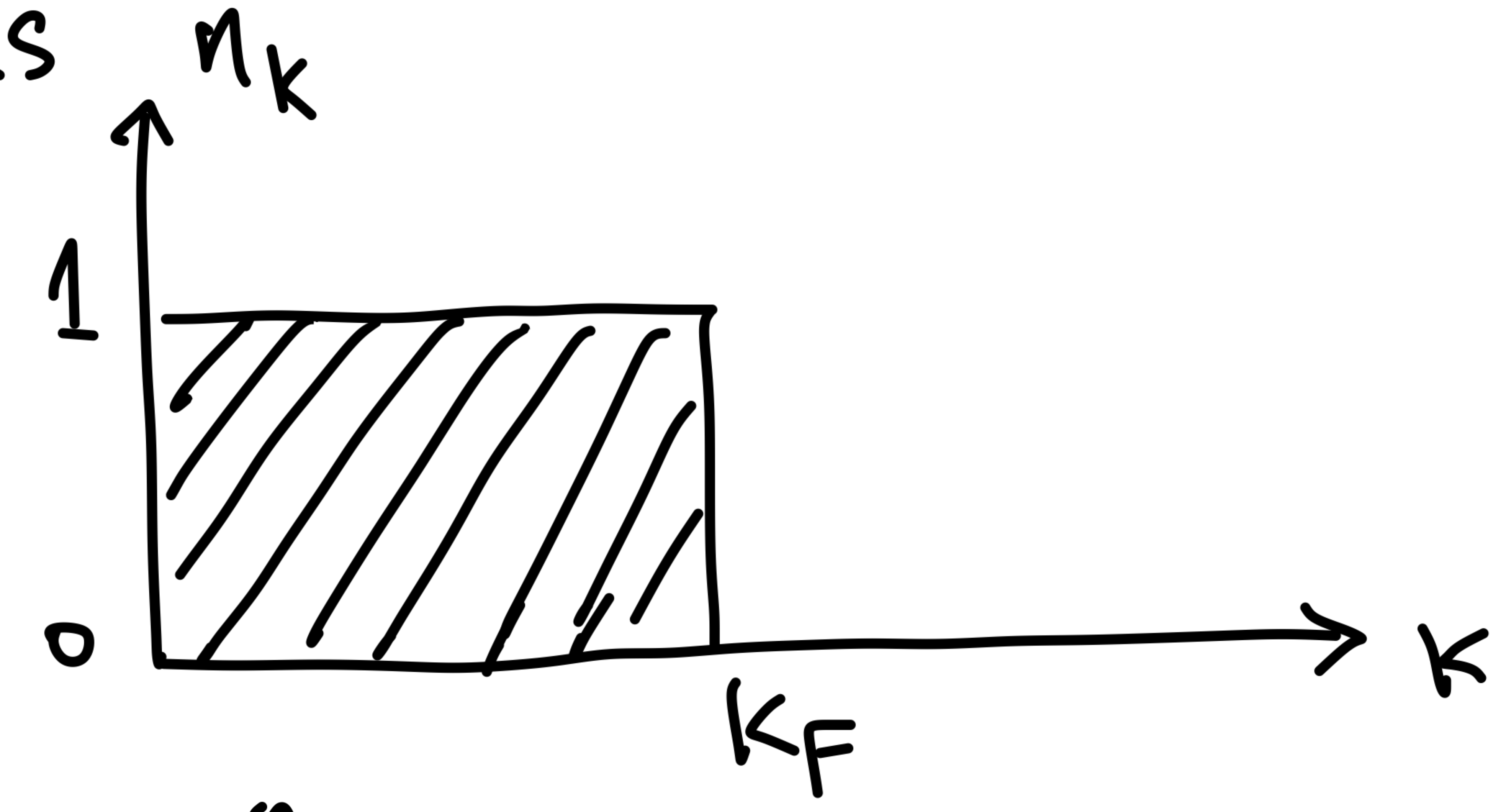


"Particle feature"

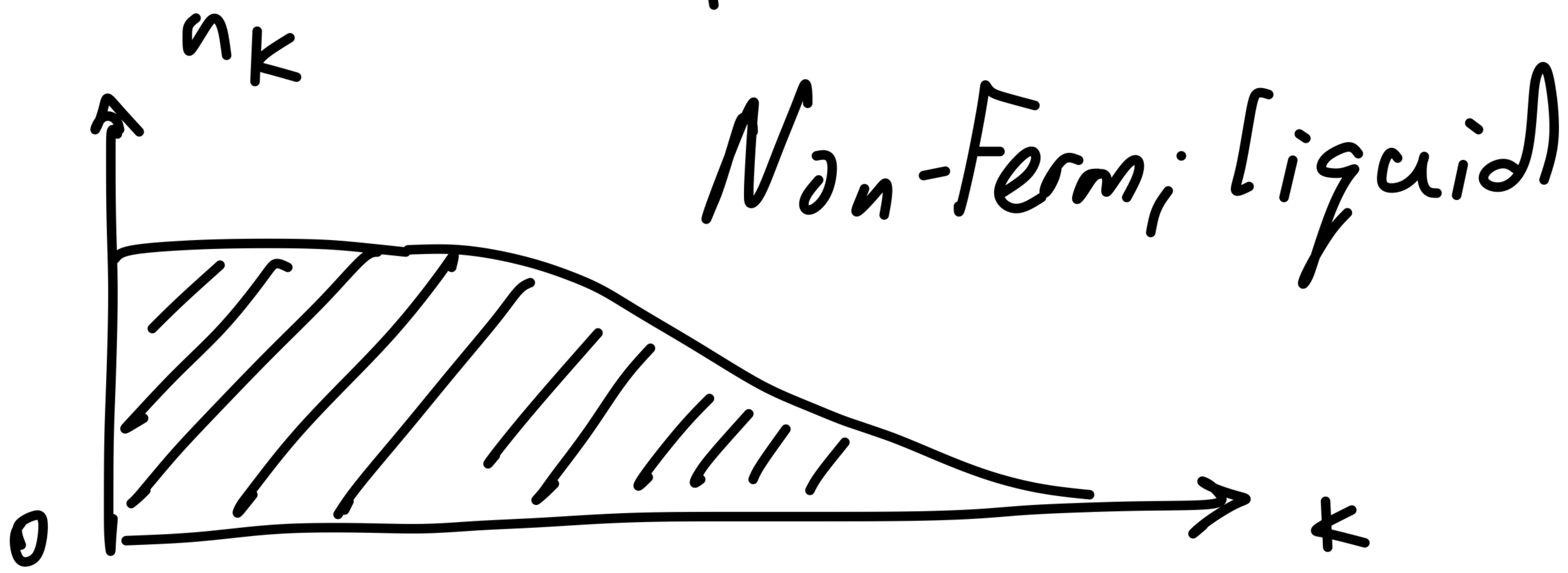
Fermi Gas



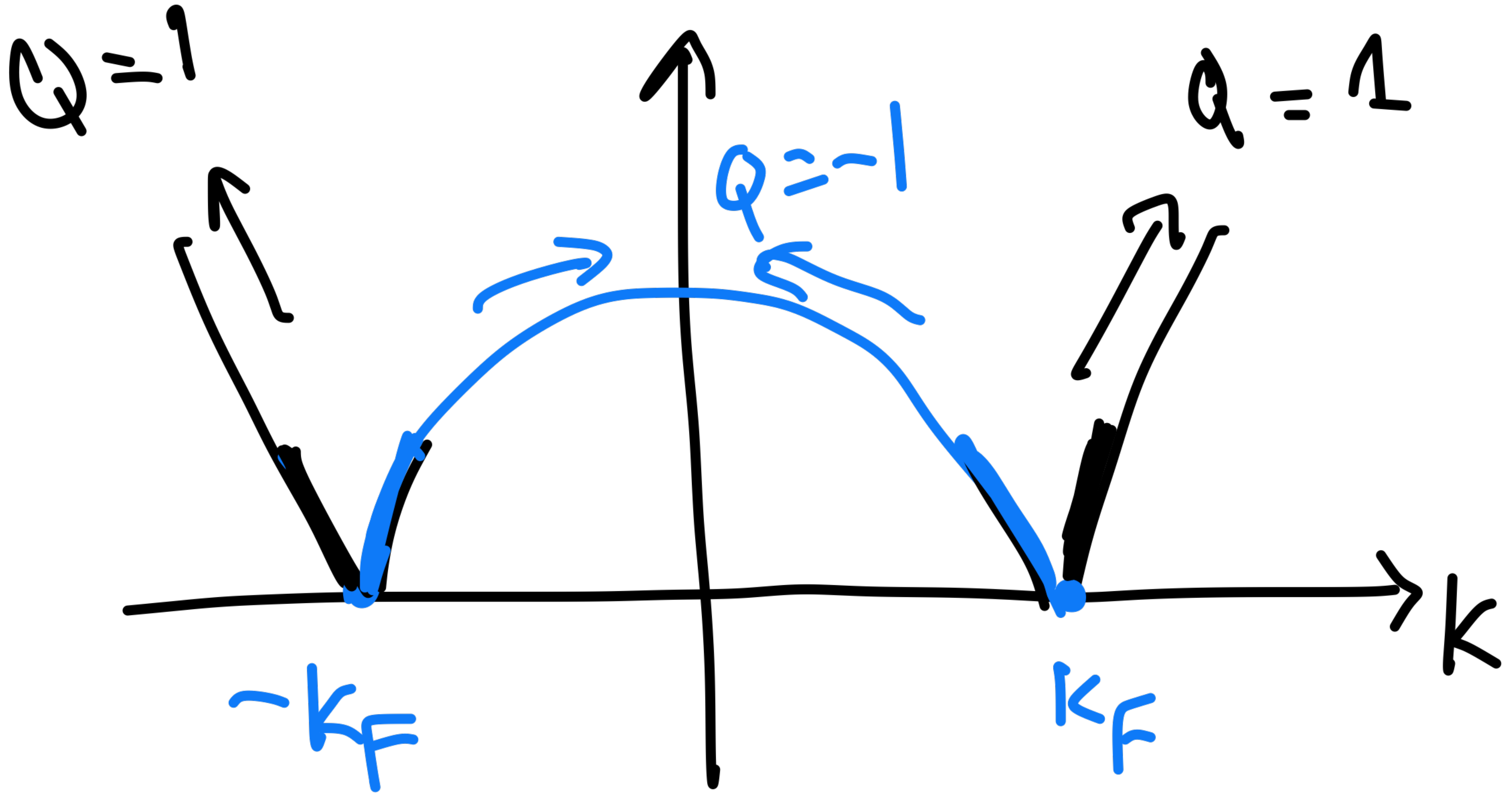
$|\varepsilon_k - \mu|$
$Q=-1$
$Q=1$
k
$-k_F$
k_F



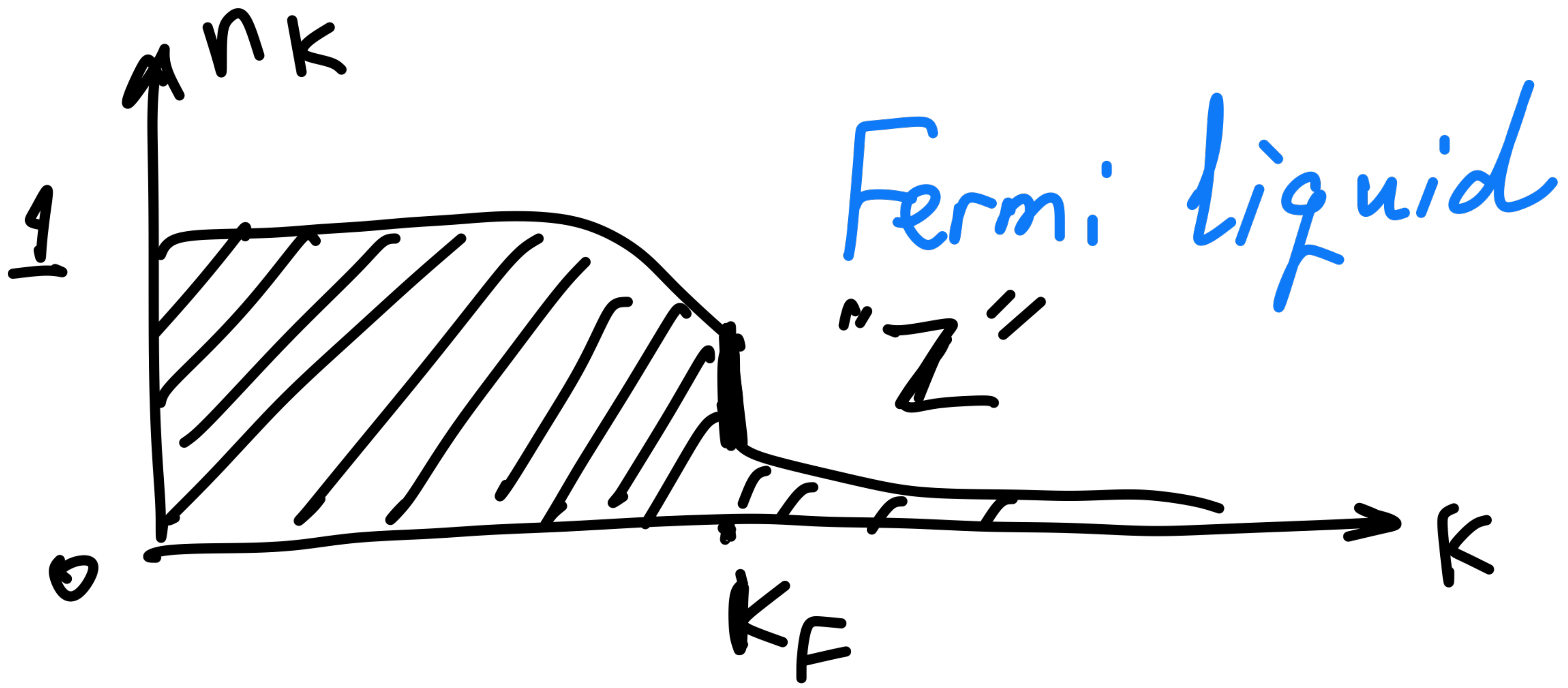
n_k
1
0
k_F
k



n_k
Non-Fermi liquid
0
k



$Q=1$
$Q=-1$
$Q=1$
k
$-k_F$
k_F



n_k
Fermi liquid
"Z"
1
0
k_F
k

- Fermi Liquid theory (nice discussions in AGD, chapter 1 and 4)

- 1) there is a finite step in the occupation number at exactly K_F. This defines a Fermi surface. $n_{k_F-0} - n_{k_F+0} = Z$

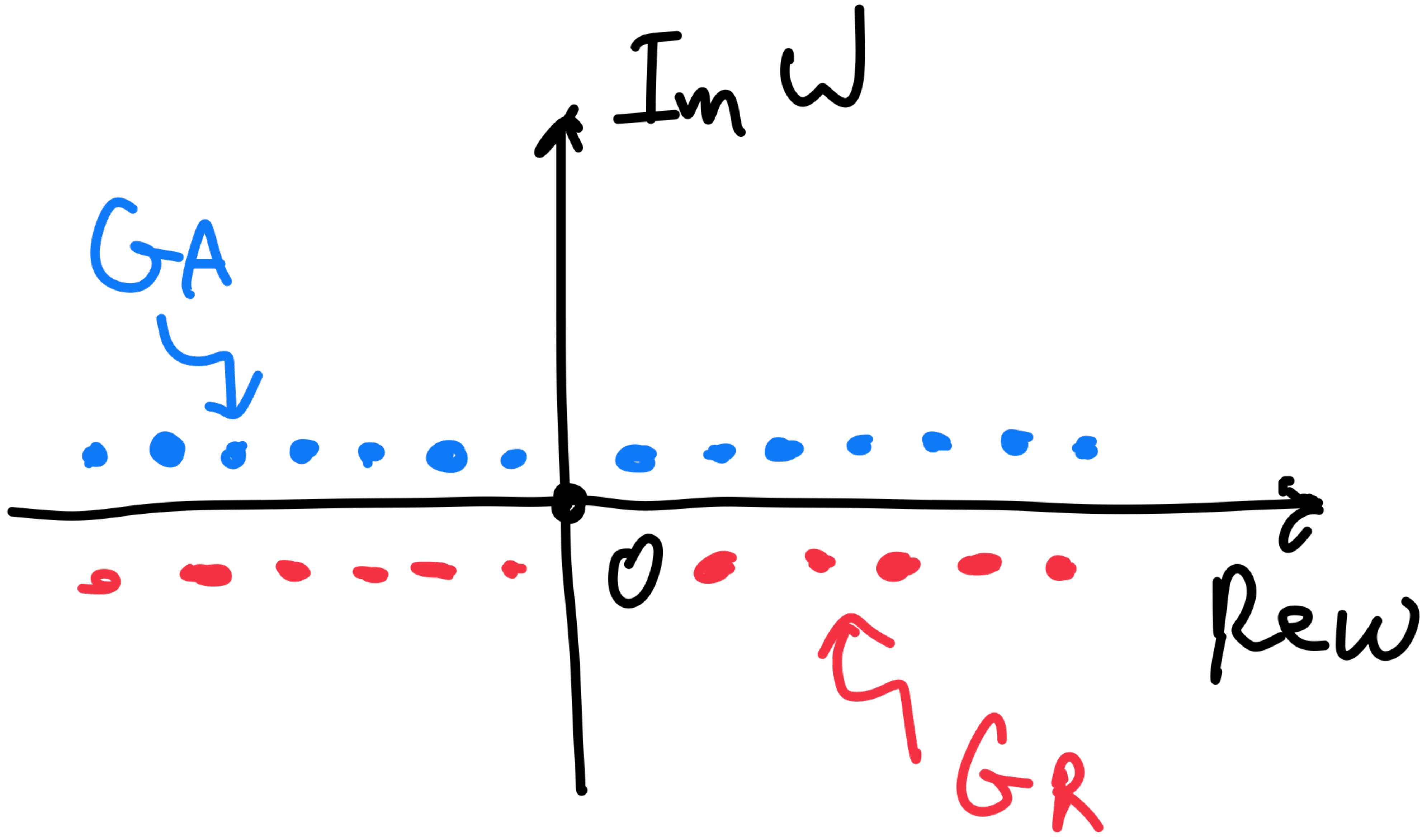
- 2) quasi-particles are of finite life time and become well defined once near Fermi surface,I.e. in the low energy sector. $\dfrac{1}{\tau_k} = \gamma_k \ll |\xi_k|$

- 3) apart from mass renormalization, wave function renormalization Z occurs at Fermi surface.

- 4) there are low energy emergent bosonic particles.

- 5) for a fixed k, time ordered 'G" is not analytical in either lower or upper frequency planes. However, retarded (advanced) green functions are analytical in lower (upper) plane for any k.(a proof in Lehmann Rep.)

Very Useful phenomenology F. G



$$G(\omega,\vec{k}) = \underset{\uparrow}{G_R(\omega,\vec{k})}\,\theta(\omega) + \underset{\uparrow}{G_A(\omega,\vec{k})}\,\theta(-\omega)$$

analytical in upper half — analytical in lower half

$$G(\omega,\vec{K}) = \frac{1}{\omega - \xi_K + i\delta\,\mathrm{sig}\,\omega} = \overbrace{\frac{1}{\omega - \xi_K + i\delta}}^{G_R}\,\theta(\omega) + \overbrace{\frac{1}{\omega - \xi_K - i\delta}}^{G_A}\,\theta(-\omega)$$

($\delta \to 0$ for F. G.)

"Fermi Liquid": pedagogical discussion ← "following Lehmann Rep"

$$G(\omega, k) = G_A(\omega, k)\,\Theta(-\omega) + G_R(\omega, k)\,\Theta(\omega)$$

A.L. in Lower half   A.L. in Upper half

A "Quasi-particles (holes)" → "poles in Lower (Upper) half-plane for G_A (G_R) away from the Real axis"

B "$\gamma_k \ll \xi_k$ or $\frac{\gamma_k}{\xi_k} \xrightarrow{k \to k_F} 0$" → "$\delta n_{k=k_F} = Z$"

"Residual of poles"