Phys529B: Topics of Quantum Theory

Lecture 2: basic introduction to interacting fermions

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- Possible quantum states of fermions (Without spontaneously breaking symmetries)

- 1) Fermi Gases; ——> 2) Femi Liquids; ——> 3) non-Fermi liquids;

- 4) incompressible QH/FQH liquids (in magnetic fields); 5) Mott insulators (in lattices)…

- Possible quantum states of fermions (With broken symmetries)

- 1) Superconductors; 2) charge density waves/ Spin density waves (in lattices);…

- Possible quantum states of fermions (Without spontaneously breaking symmetries)

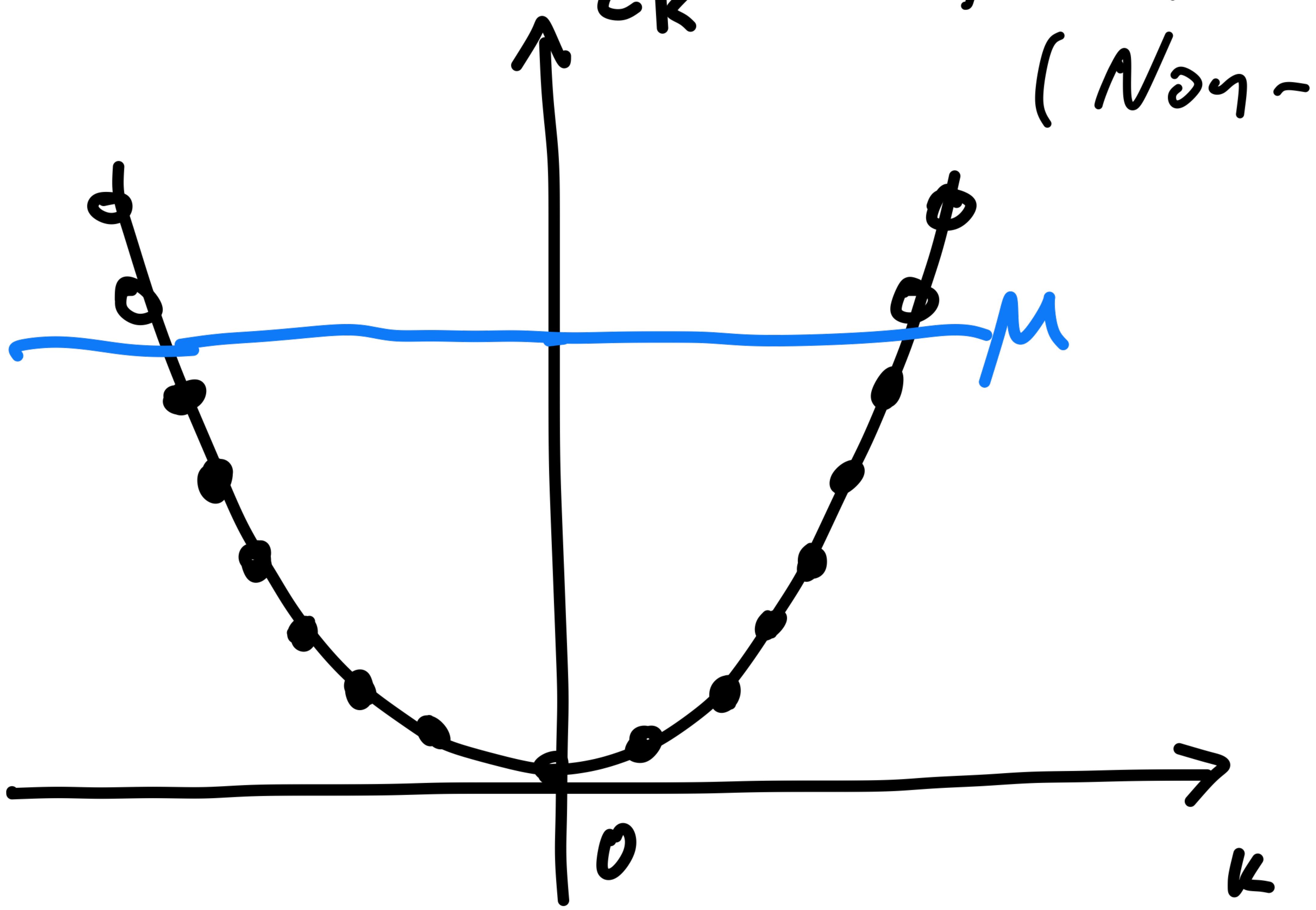
- 1) Fermi Gases; —–> 2) Femi Liquids; —–> 3) non-Fermi liquids;

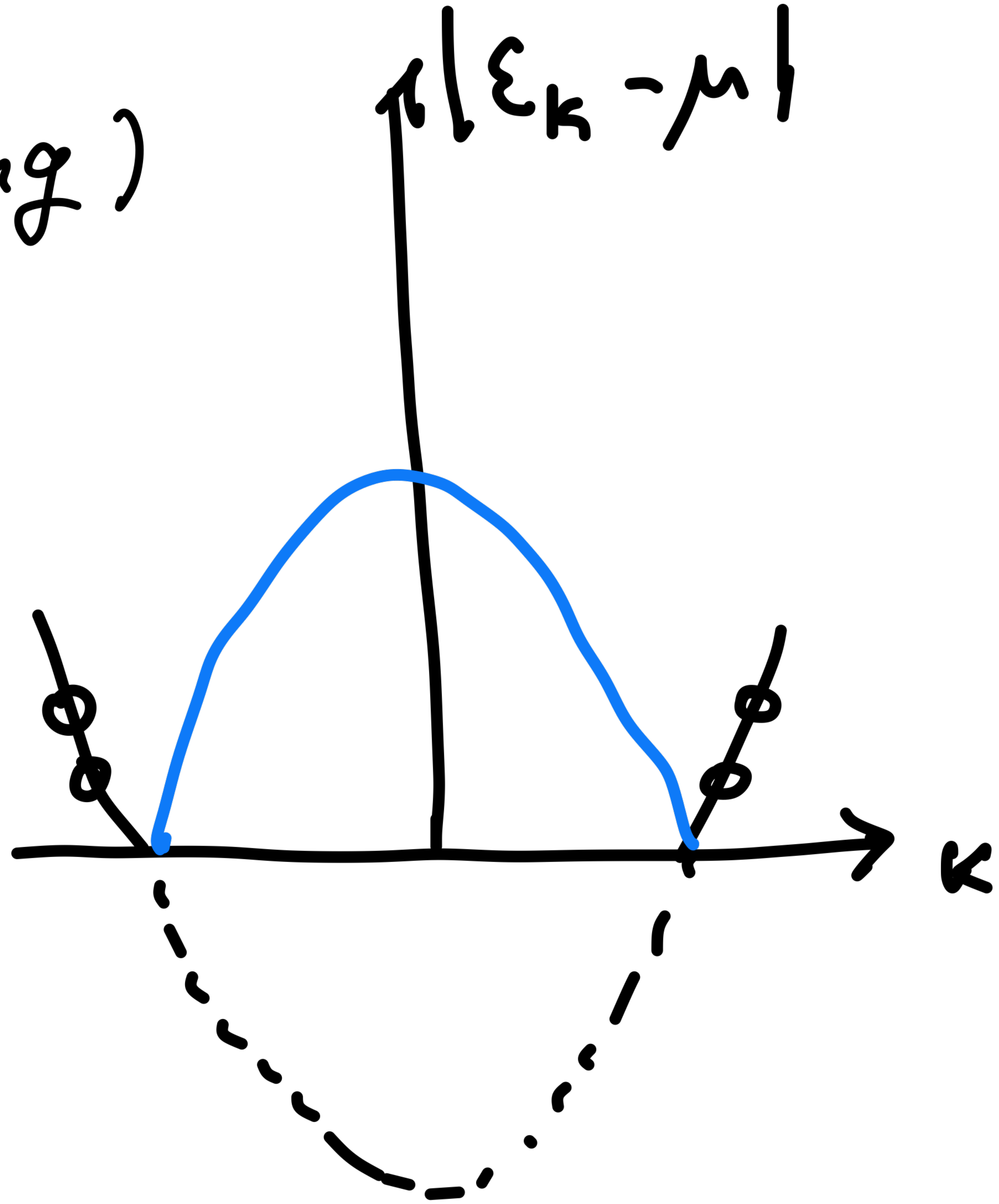
- 1) Fermi Gases (non-interacting): all low energy excitations are fermions.

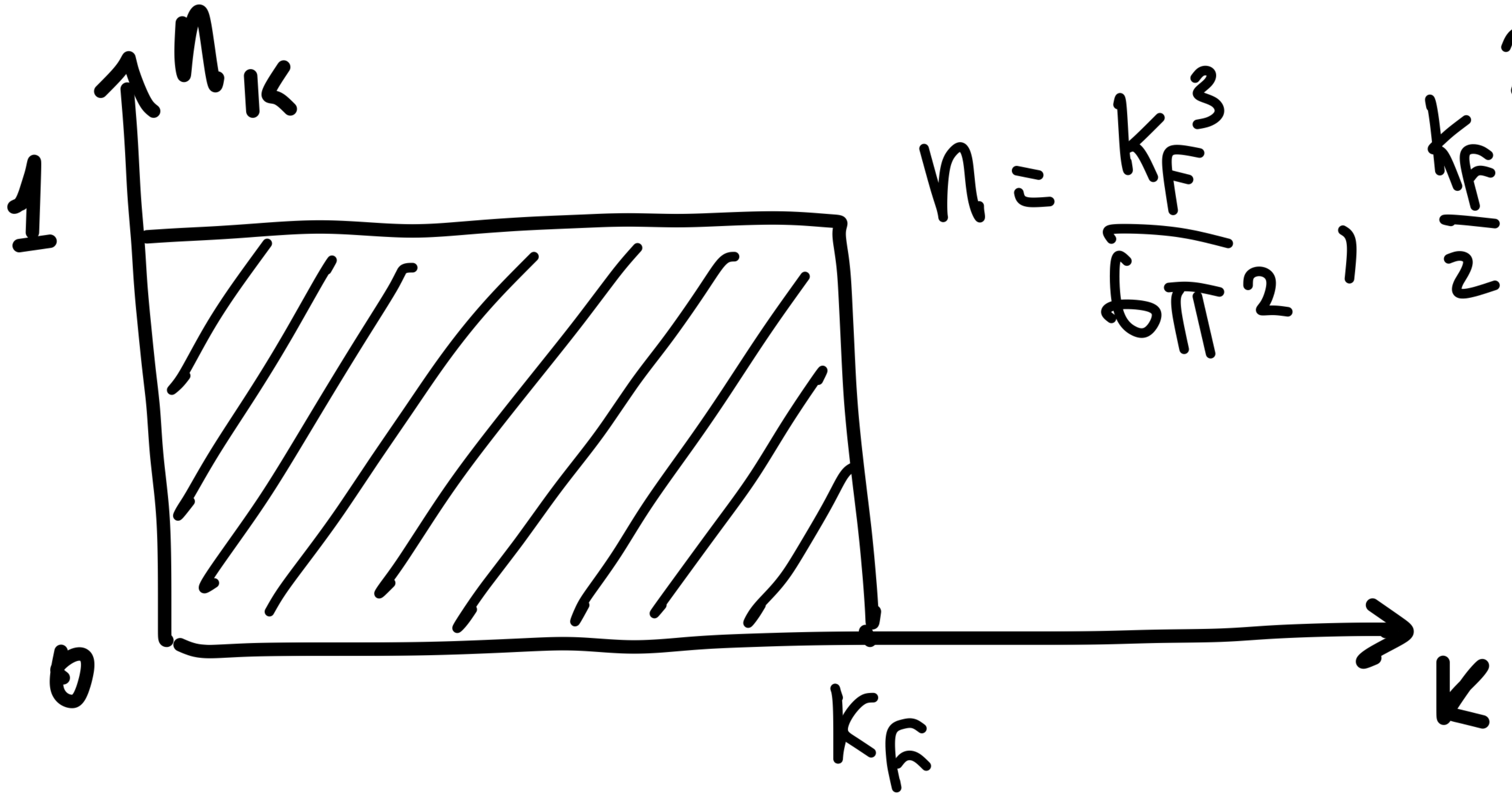
- 2) Femi Liquids (interacting): low energy excitations are emergent fermionic quasi-particles with spin-1/2 and with renormallzed properties; moreover there are emergent bosonic excitations.

- 3) non-Fermi liquids;: a) NO fermionic excitations at all at low energy sectors; fully bosonized; b) NO well-defined fermions at fermion surfaces (like a molasses); c) only anyons in low energy sectors (abelian or non-abelian); [fermionic quasi particles but no spin or chagres so that fermions are fractionalized]

Fermi Gas
(Non-interacting)

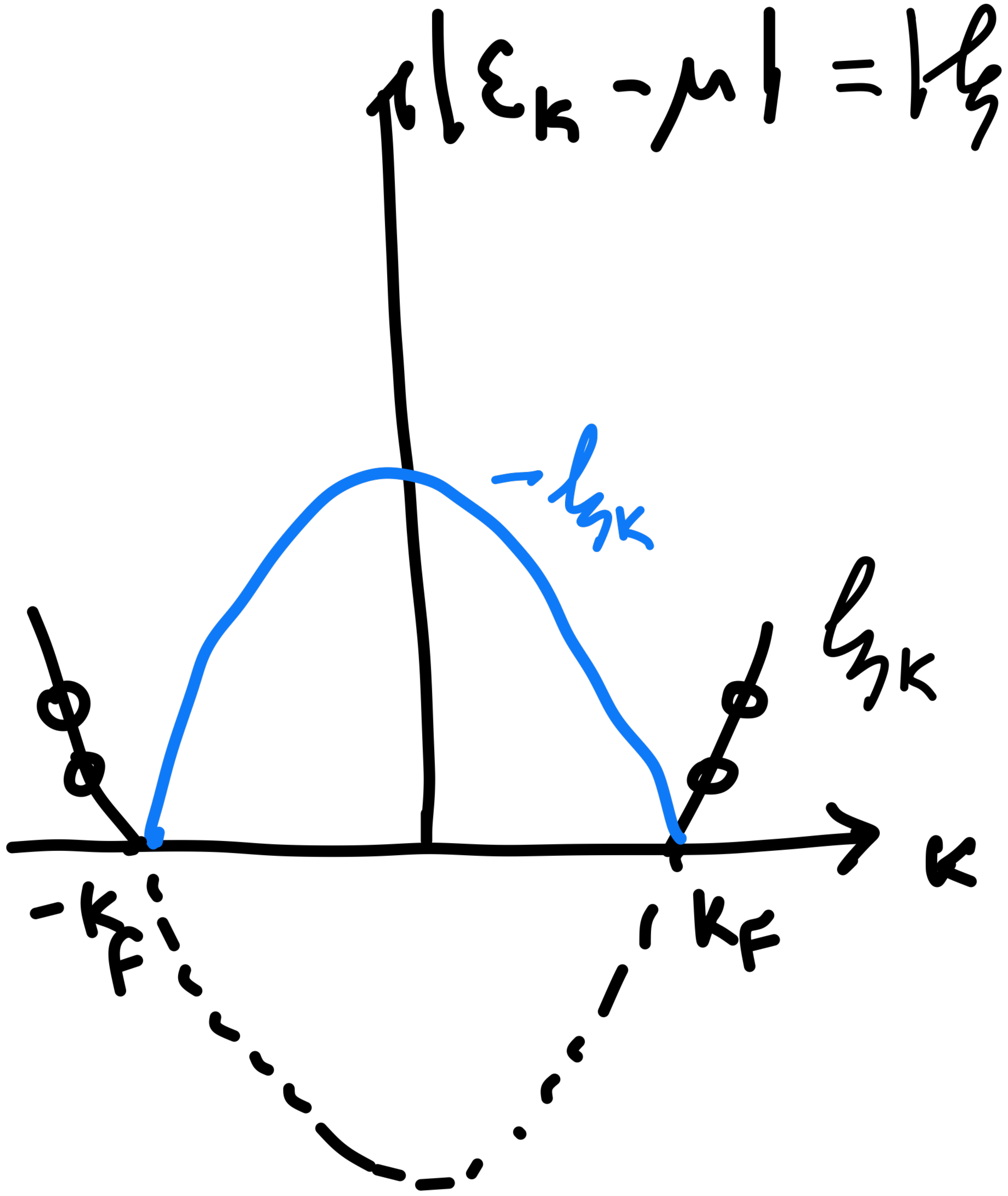






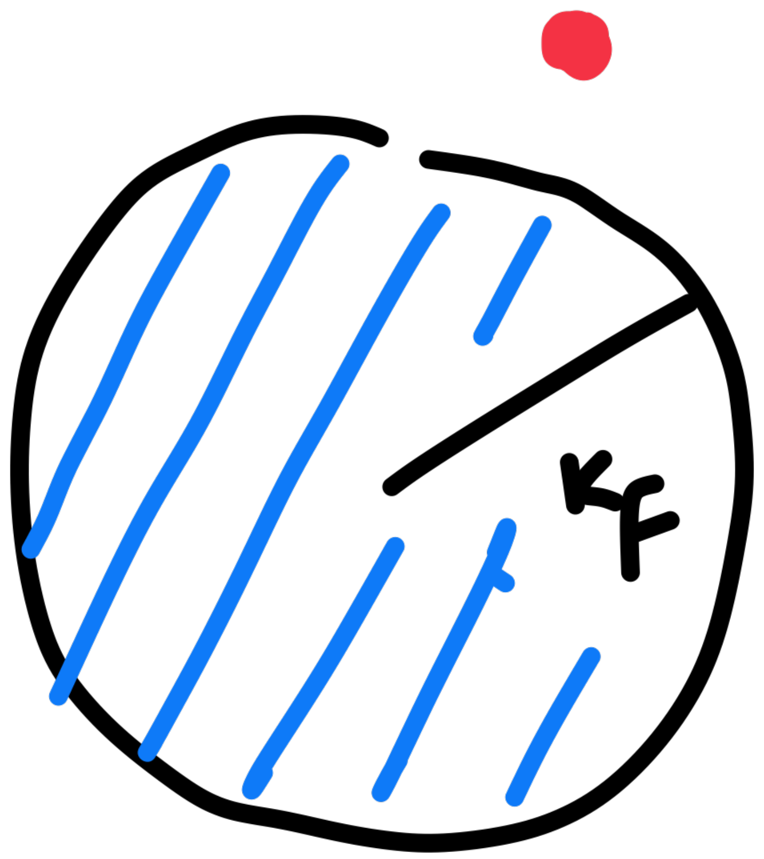
$$n = \frac{k_F^3}{6\pi^2}, \quad \frac{k_F^2}{2} = \mu$$

"Excitation Spectrum"



Fermionic
Excitation Spectrum

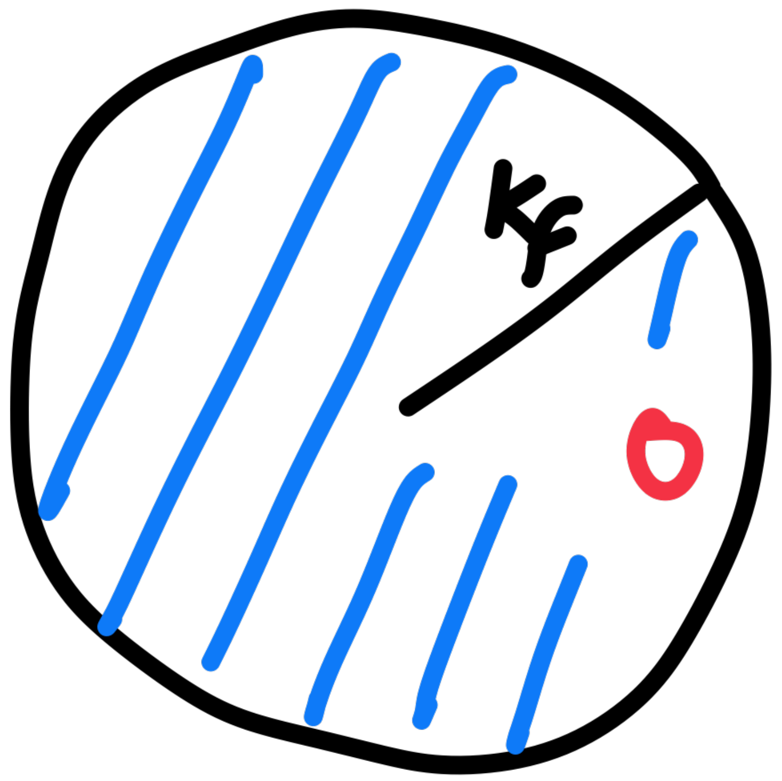
$\xi_k = \epsilon_k - \mu$



Particle-like excitations

$\xi_k = \varepsilon_k - \mu > 0$

Forward propagating



Hole-like excitations

$-\xi_k = \mu - \varepsilon_k > 0$

Backward propagating

Dynamics (Time Ordered Green's function)

$$G(k,t) = -i \langle g.s.| T \psi_k(t) \psi_k^+(0) | g.s. \rangle$$

$$T \psi_k(t) \psi_k^+(0) = \psi_k(t) \psi_k^+(0) \theta(t) - \psi_k^+(0) \psi_k(t) \theta(-t)$$

$$G(k,t) = -i e^{-i\xi_k t} \theta(k-k_F)\, \theta(t) + i e^{-i\xi_k t} \theta(k_F - k)\, \theta(-t)$$

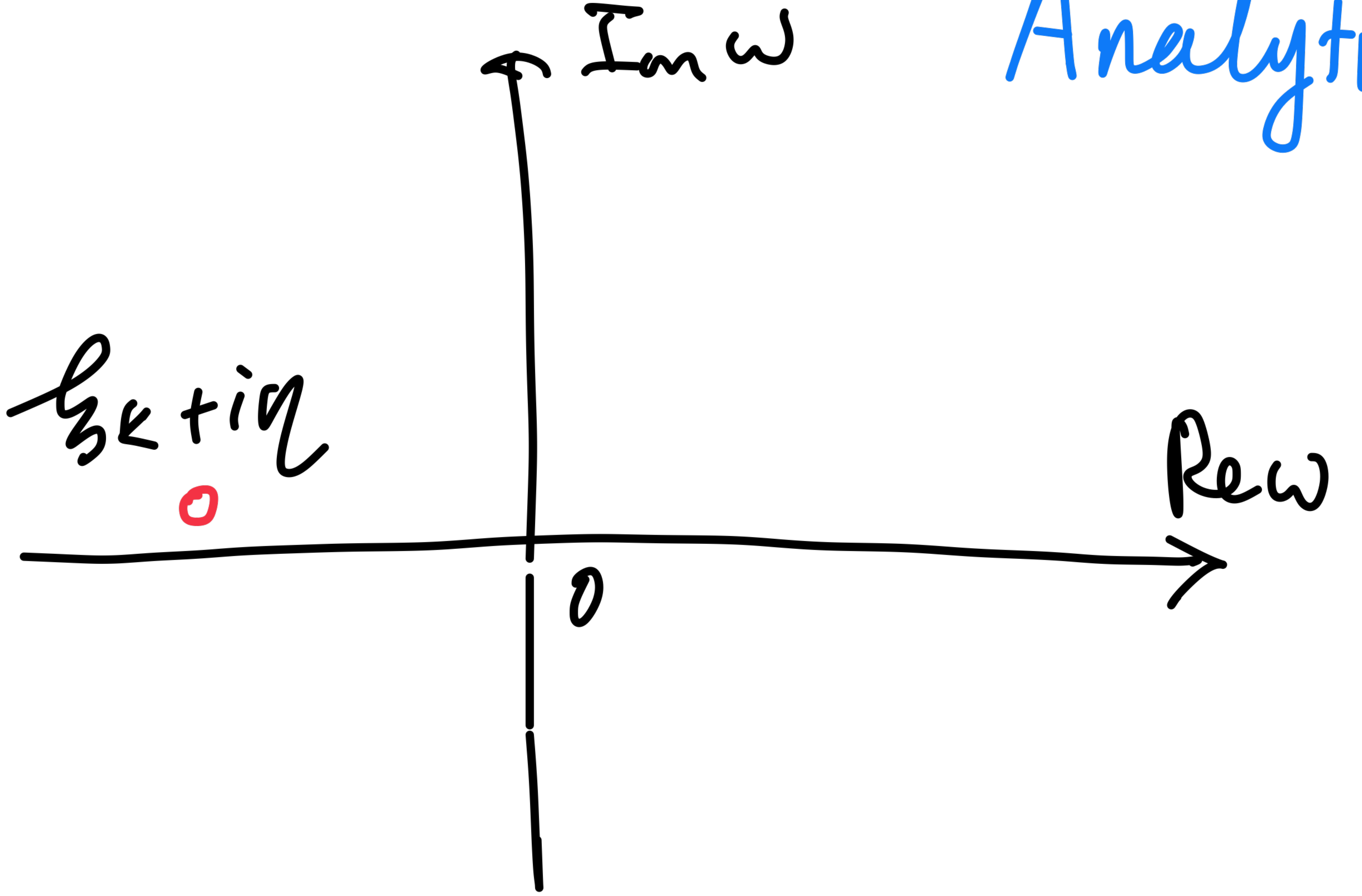
$$\left(\xi_k = \epsilon_k - \mu, \quad \mu = \frac{k_F^2}{2} \right)$$

$$G(k,\omega) = \frac{1}{\omega - \xi_k + i\eta_k}, \quad \eta_k = \delta \operatorname{sign} \xi_k \quad (\delta > 0)$$

or $\delta \operatorname{sign} \omega$

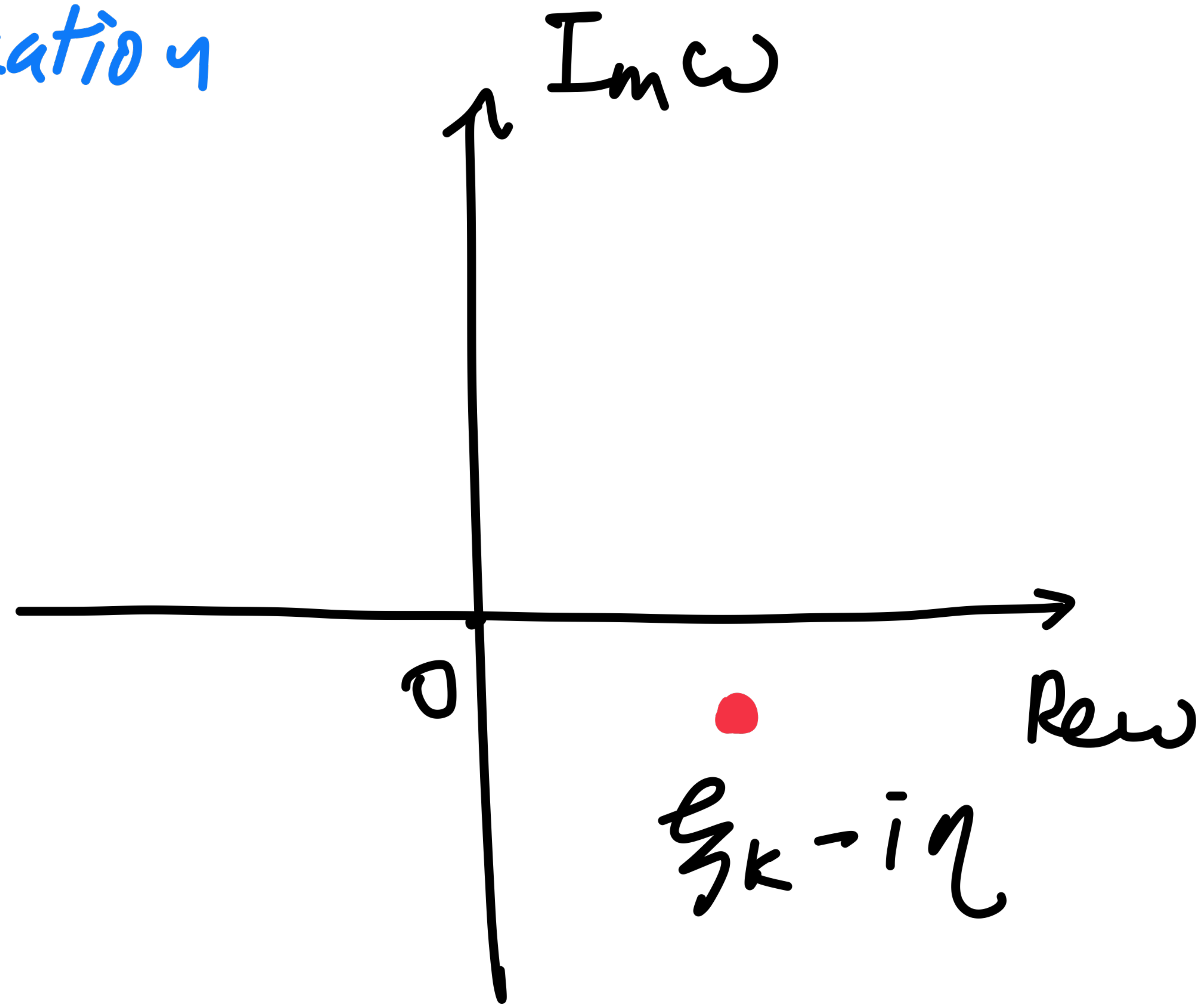
for fixed $\vec{k}$, $G(\omega, \vec{k})$ in Complex plane of ω has Simple poles

Analytical Continuation



poles with $\xi_k < 0$

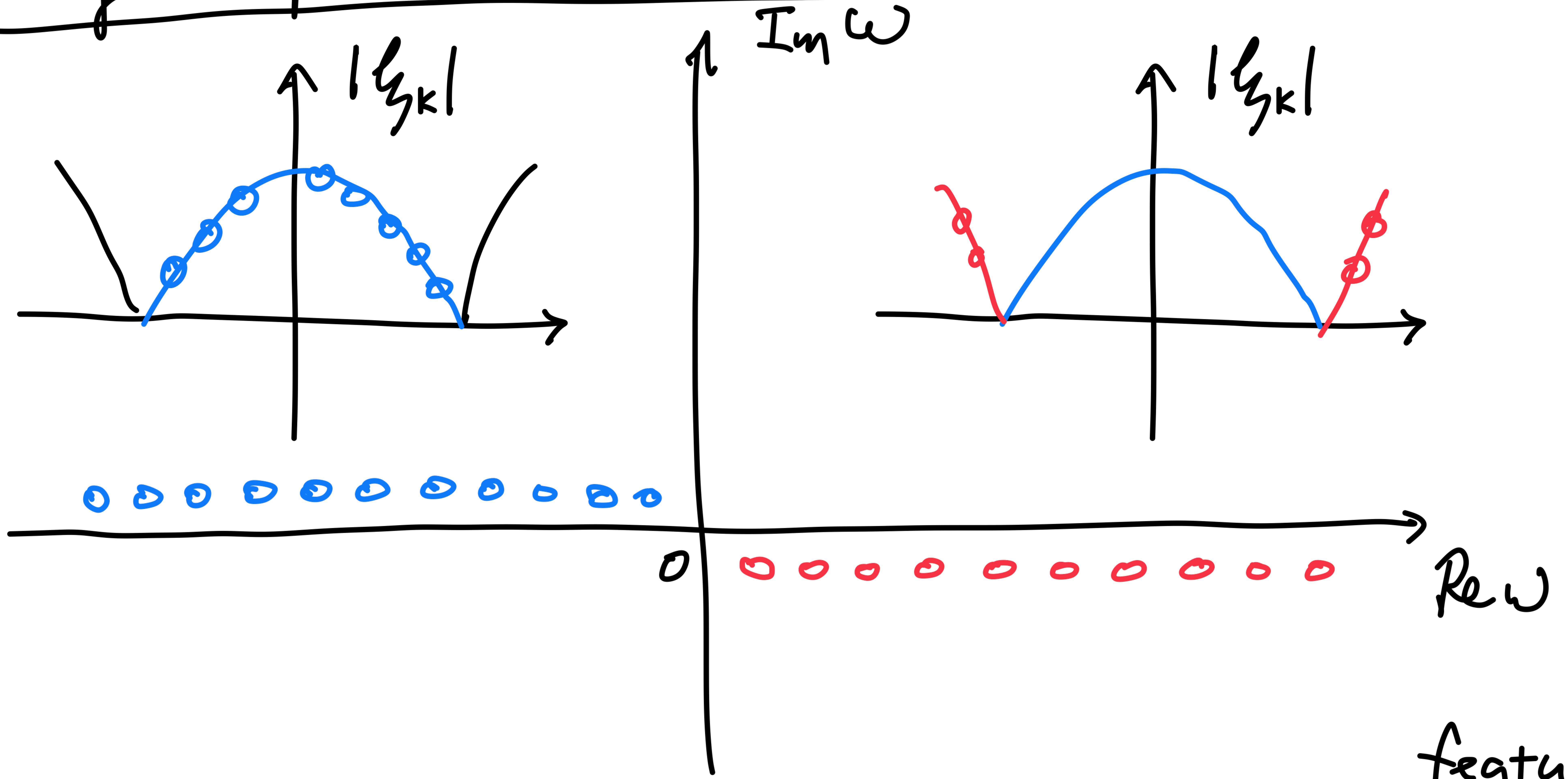
(backward propagating)



poles with $\xi_k > 0$

(forward propagating)

Putting all poles in the same sheet $\quad G(\vec{k}, \omega)$



$$\boxed{Z_{\vec{k}} = \mathrm{Res}\, G(\vec{k}, \omega = \xi_k + i\eta_k) = \mathbb{1}}$$

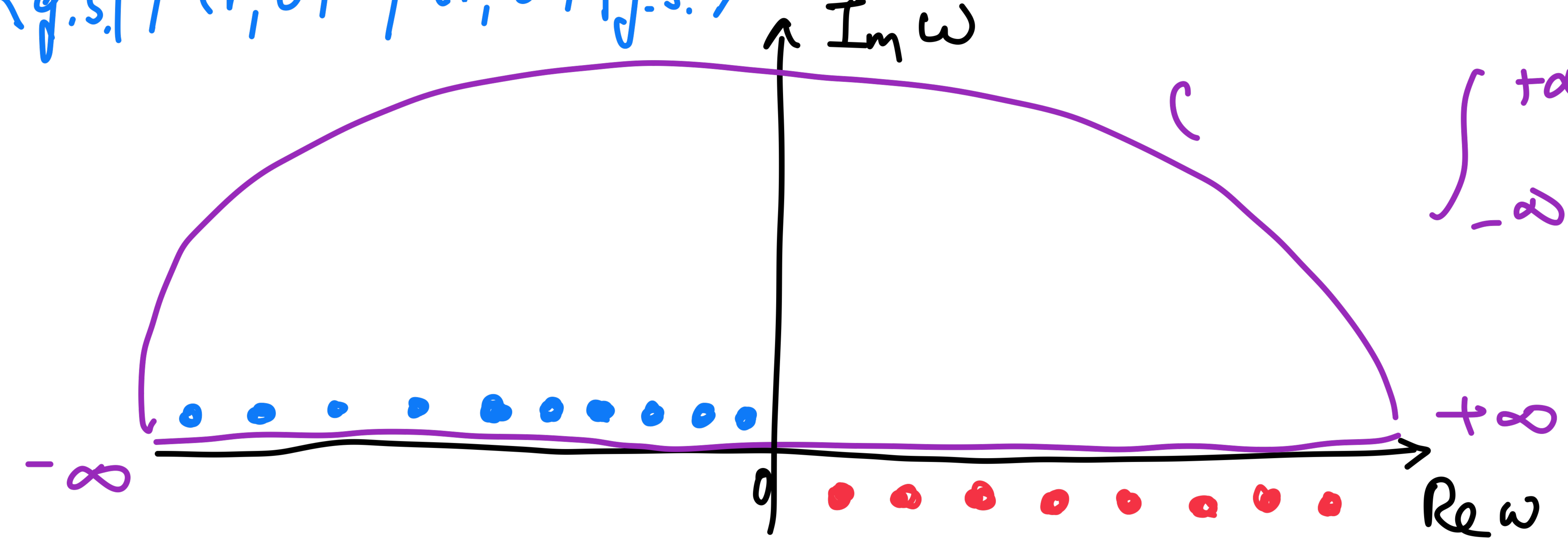
featuring fermi Gas Only

A simple application

$$\varepsilon > 0$$

$$iG(\vec{r},\vec{r};t=0^-) = \frac{i}{V}\sum_k G(k,\, t=-\varepsilon) = \frac{i}{V}\sum_k \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, G(k,\omega)\, e^{+i\omega\varepsilon}$$

$$\langle g.s.|\psi^\dagger(r,0)\,\psi(r,0^-)|g.s.\rangle$$



$$\int_{-\infty}^{+\infty} = \oint_C$$

$$iG(\vec{r},\vec{r};t=0^-) = \int\frac{d^3\vec{k}}{(2\pi)^3}\,\Theta(k_F-k), \quad \text{or} \quad \boxed{n = \frac{k_F^3}{6\pi^2}, \quad \frac{k_F^2}{2} = \mu}$$

- Summary:

- 1) in F.G., there are well defined Fermionic excitations i.e. for a given momentum k, the energy is unique.

- 2) there appear as poles in time ordered correlation functions;

- 3) poles in the upper half plane =“hole” like; poles in lower half planes=“particle” like.

- 4) These are the only fundamental excitations; there are no other fundamental excitations with well defined energy-momentum excitations.

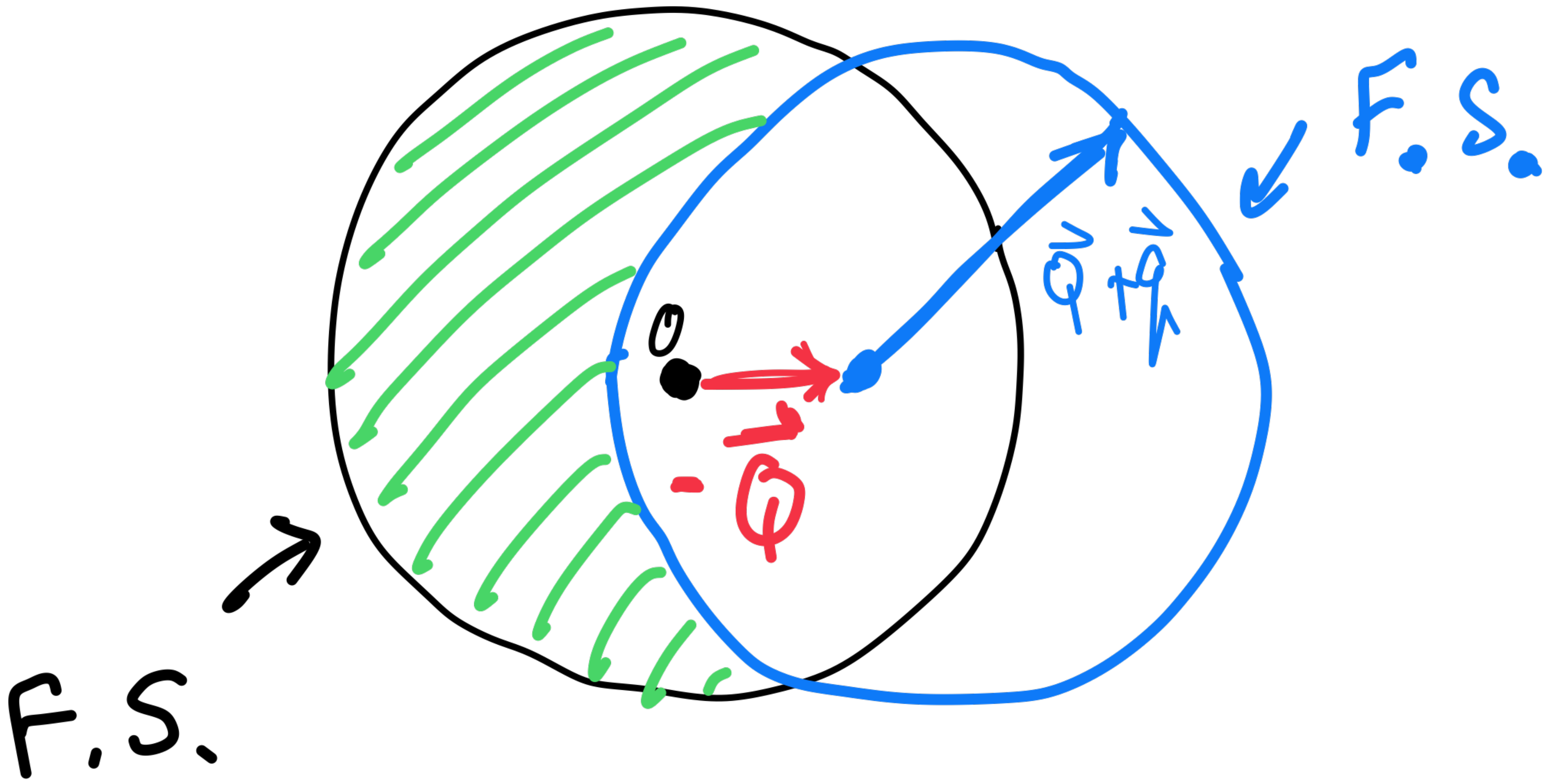
- 5) non-interacting time ordered green’s function is analytical in the upper half plane when k >K_F (I.e. poles in lower half plane) but analytical in the lower half plane when k>k_F (poles in upper half plane). In general, it is non-analytical in either upper or lower half planes.

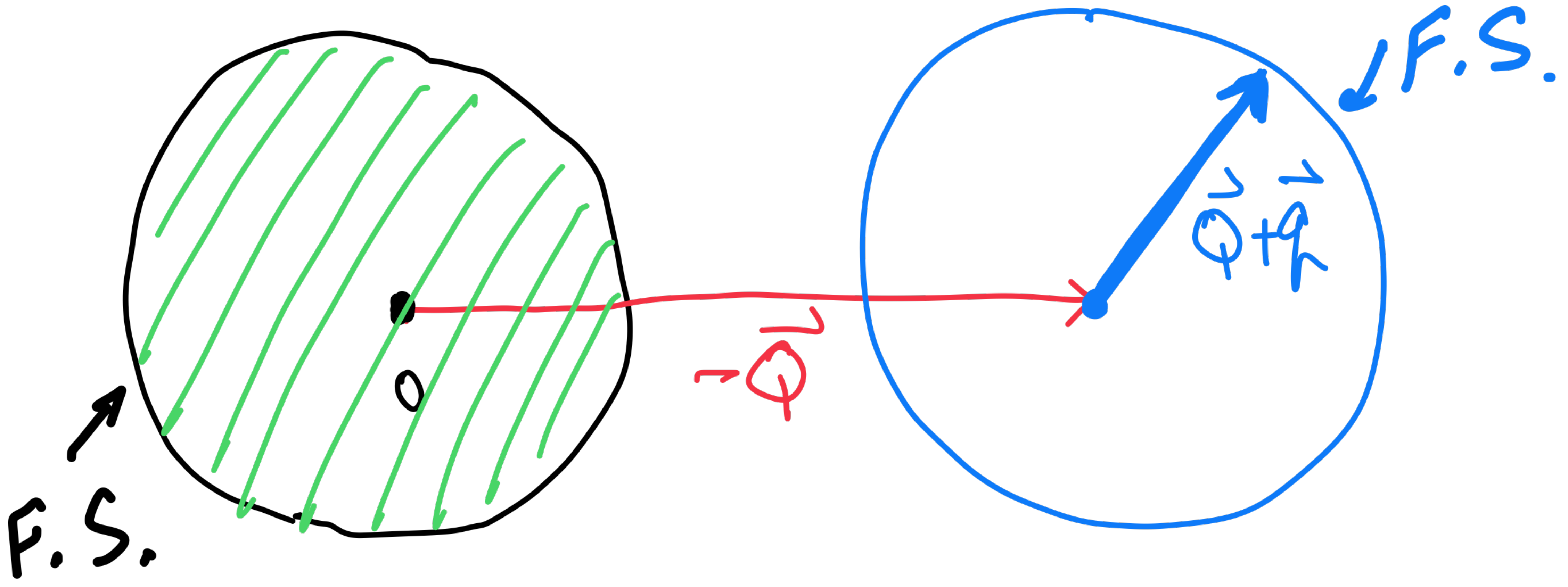
- 6) it is possible to construct retarded (advanced) Green’s functions that are analytical in upper half (lower half) plane for any k. (See more discussions when we add interactions).

Composite bosonic states as a continuum

$$\phi^{\dagger}_{Q} = \sum_{\vec{q}} A_{Q,\vec{q}}\, \psi^{\dagger}_{\vec{Q}+\vec{q}}\, \psi_{\vec{q}}$$

("$Q=0$ Sector")
charge neutral Sector





$$|\vec{q}| < k_F, \quad |\vec{Q}+\vec{q}| > k_F$$

$|\vec{Q}| < 2k_F$ $\qquad$ $|\vec{Q}| > 2k_F$