# Phys529B: Topics of Quantum Theory

## Lecture 13: identifying Non-Fermi liquids: 2D

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# Supplementary stuff on RGE

$$H = H(m, \lambda, \dots ; \Lambda) = H(\tilde{m}, \tilde{\lambda}, \dots, Z(\Lambda) ; \Lambda)$$

for any given $\Lambda$, there shall be a $\tilde{m}(\Lambda)$, $\tilde{\lambda}(\Lambda)$ . . .

so that $H$ leads to the same physics.

$$\frac{d\tilde{m}}{dt} = \beta_m(\tilde{m}, \tilde{\lambda}, \dots) \qquad \frac{dZ(\Lambda)}{dt} = \gamma(\tilde{m}, \tilde{\lambda}, Z)$$

$$\frac{d\tilde{\lambda}}{dt} = \beta_{\tilde{\lambda}}(\tilde{m}, \tilde{\lambda}, \dots) \qquad t = \ln\frac{\Lambda}{\Lambda_{UV}}$$

Renormalization Group Equations

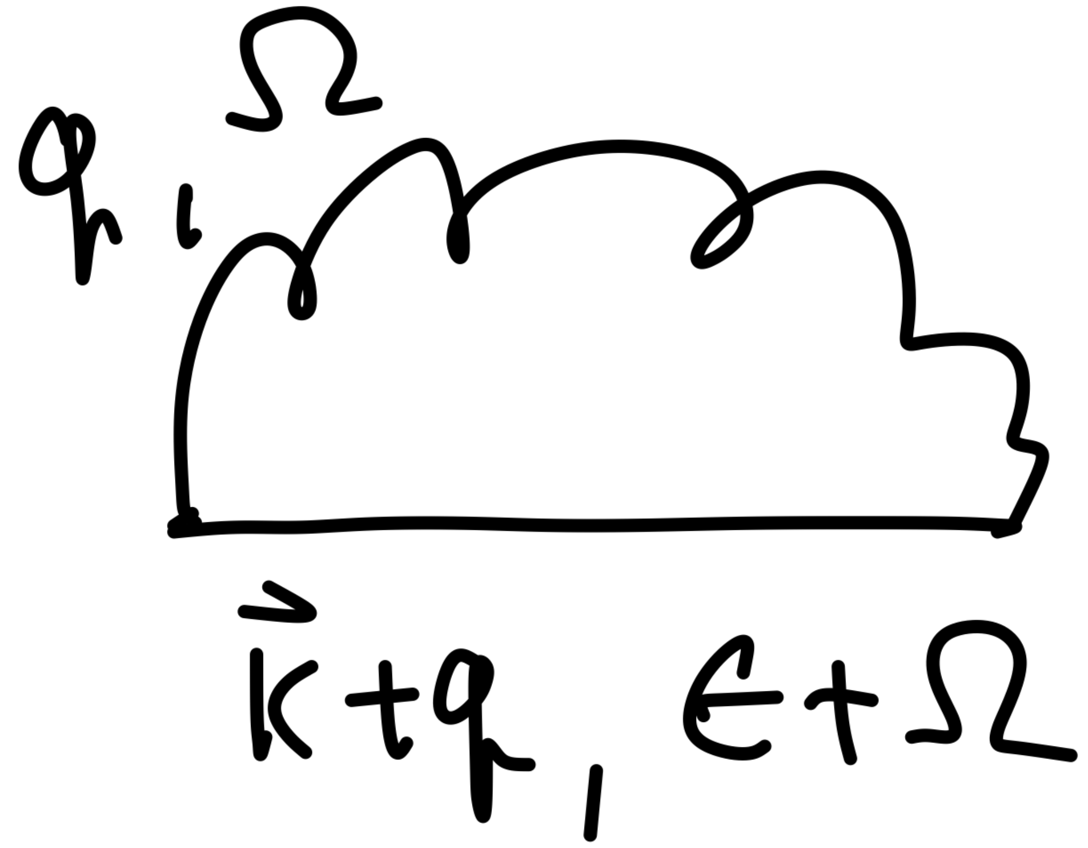
Running scale

Generally $\quad G = G_0 + G_0 \Sigma G \quad$ or $\quad G^{-1} = G_0^{-1} - \Sigma$

perturbation theory:

$$G_k = \langle 0 | -i T \psi_k(0) \psi_k^{\dagger}(0) | 0 \rangle$$

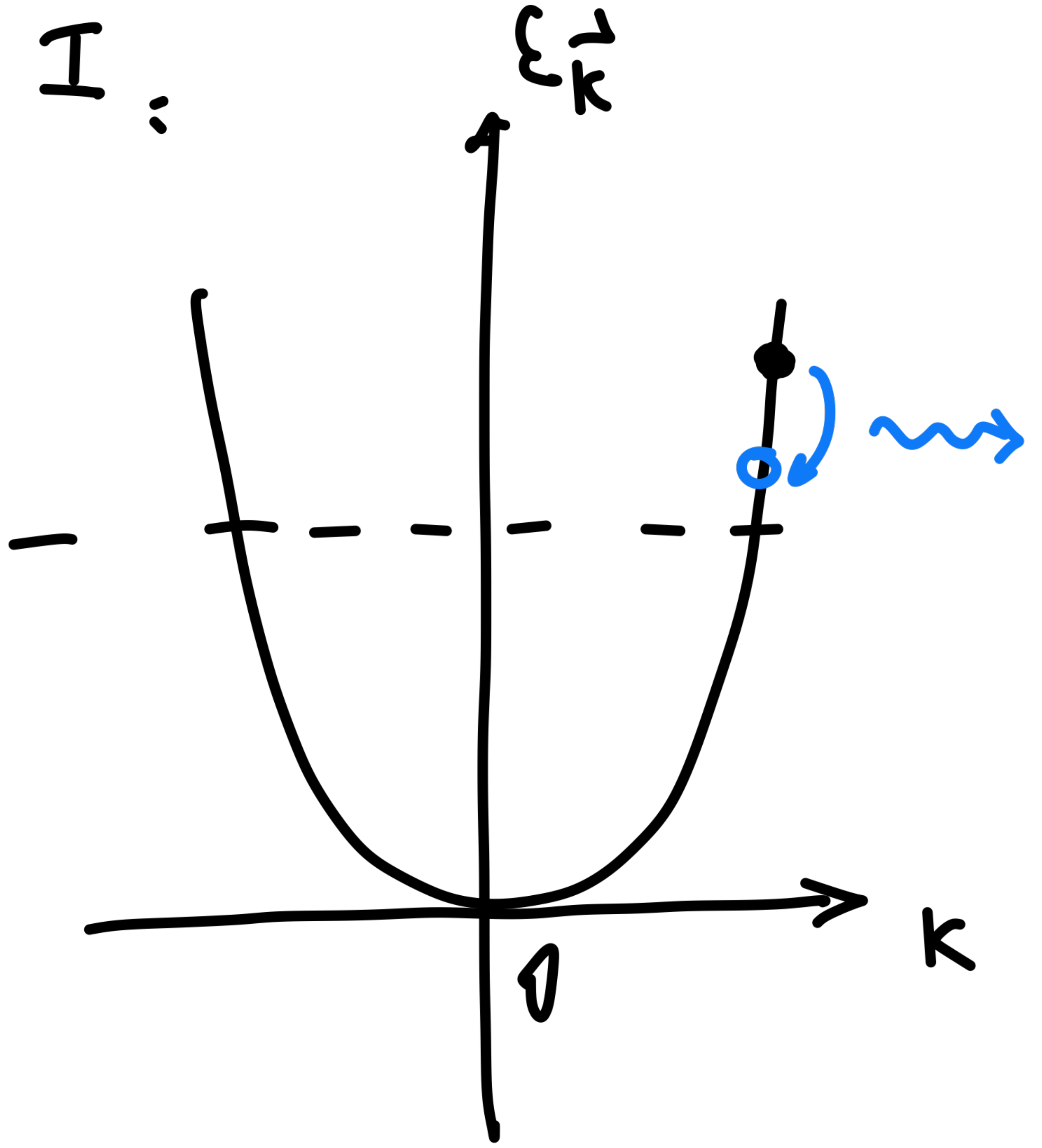
$$\delta G = G_0 \Sigma^{(2)} G_0, \quad \Sigma^{(2)}(\vec{k}, \epsilon) = i \int D_0(\vec{q}, \Omega)\, G_0(\vec{k}+\vec{q}, \epsilon+\Omega) \times g^2$$

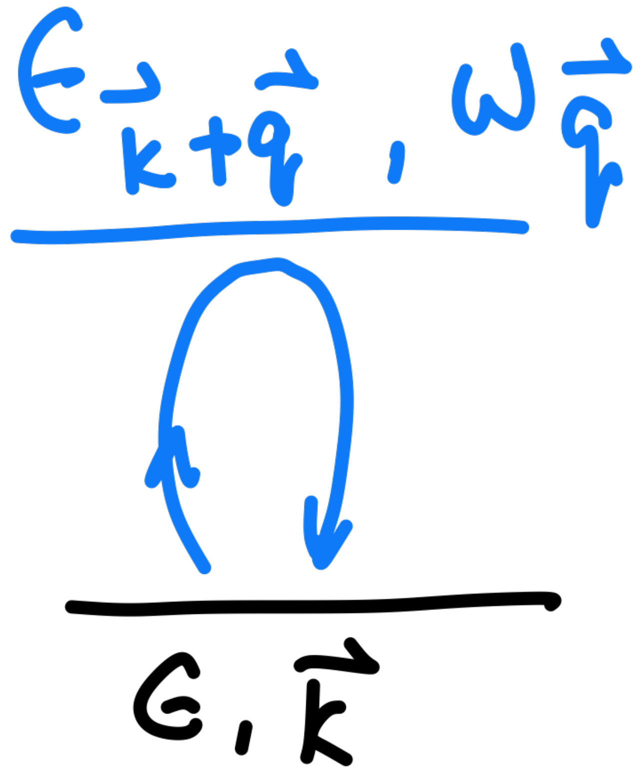



$$D_0(\vec{q}, \Omega) = \frac{1}{\Omega^2 - w_{\vec{q}}^2 + i\delta}$$

$$G_0(\vec{k}, \epsilon) = \frac{1}{\epsilon - (\epsilon_{\vec{k}} - \mu) + i\delta_{\vec{k}}}$$

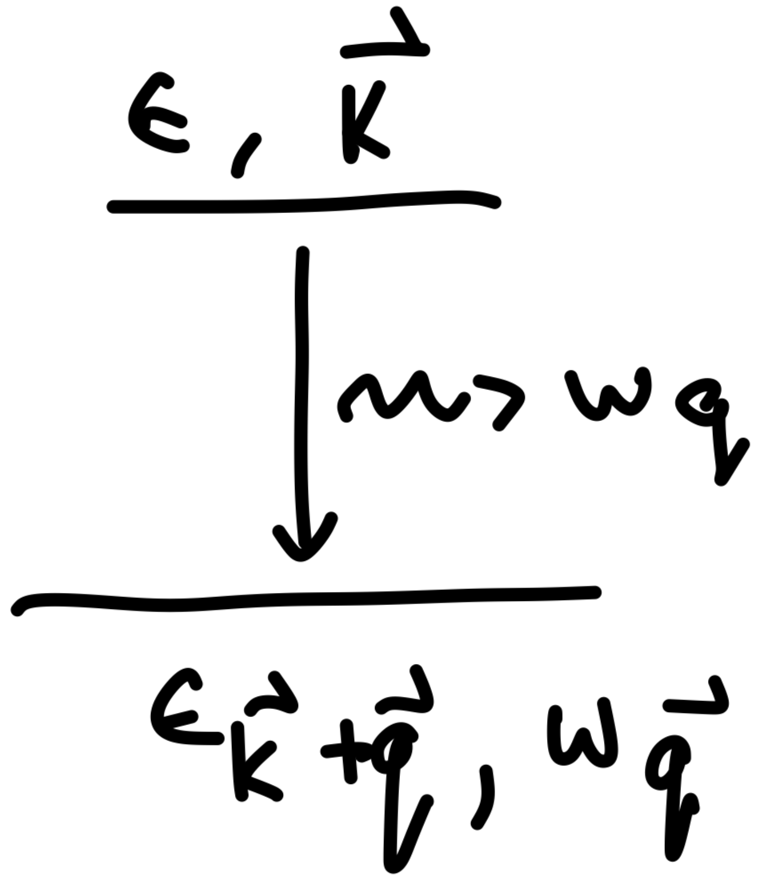
I :



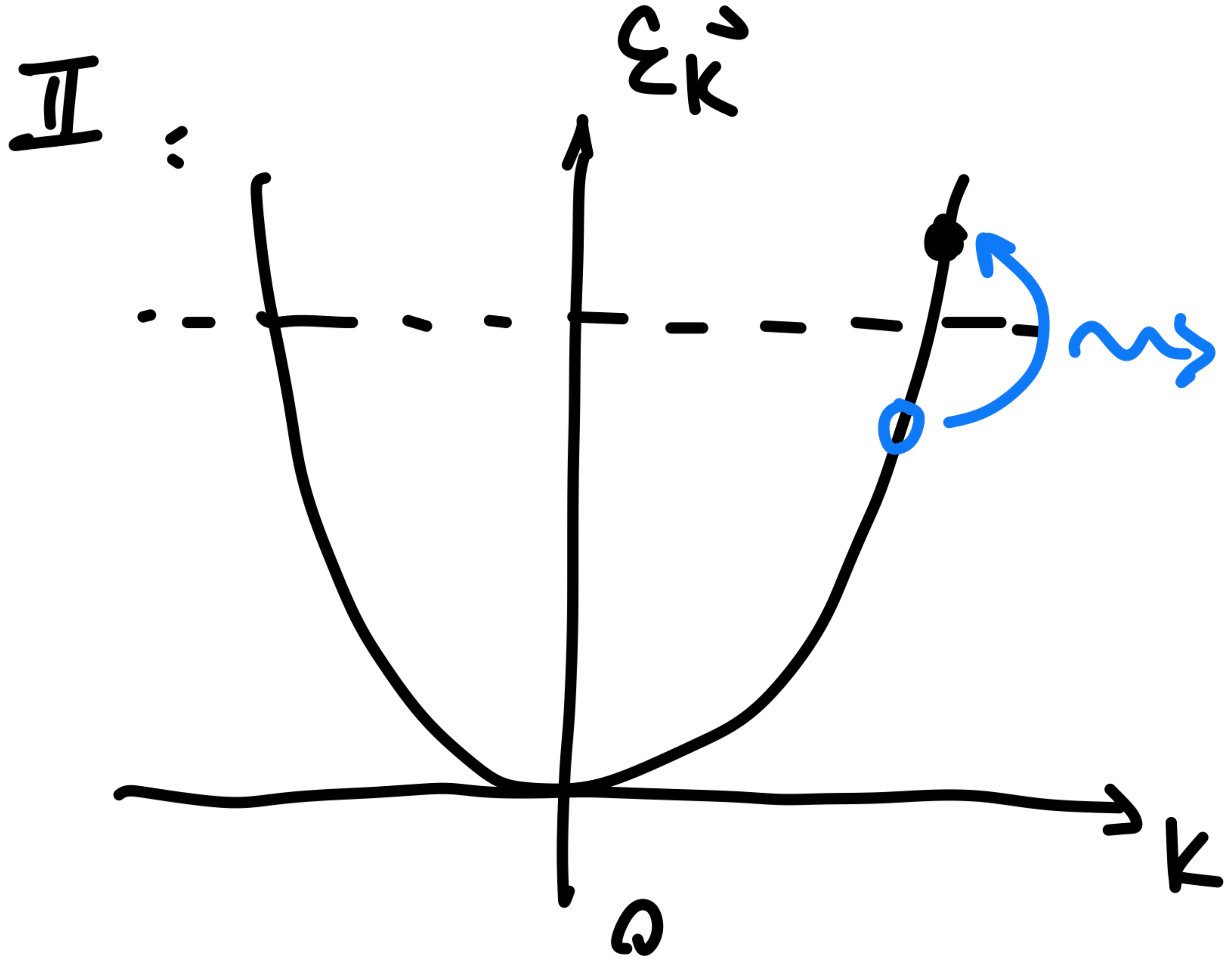

Virtual / $\mathrm{Re}\,\Sigma(\epsilon, \vec{k})$

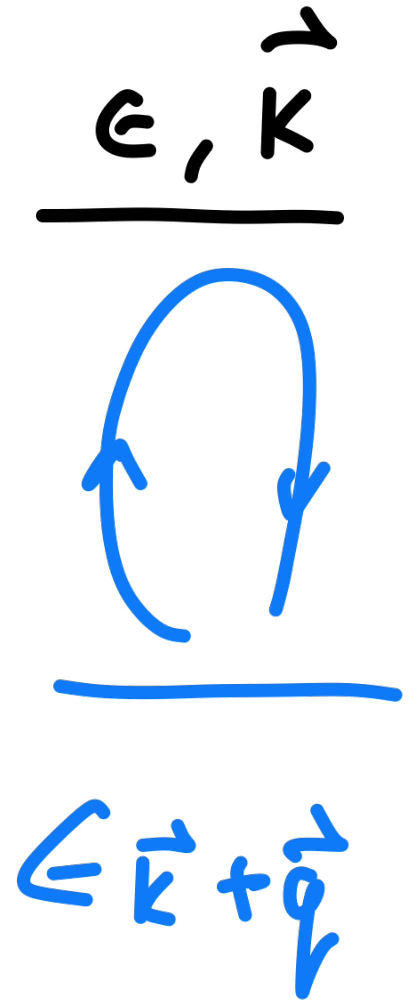
$$\delta(\epsilon - \omega_q - \tilde{\epsilon}_{\vec{k}+\vec{q}})\ \theta(\underbrace{\xi_{\vec{k}+\vec{q}}})$$



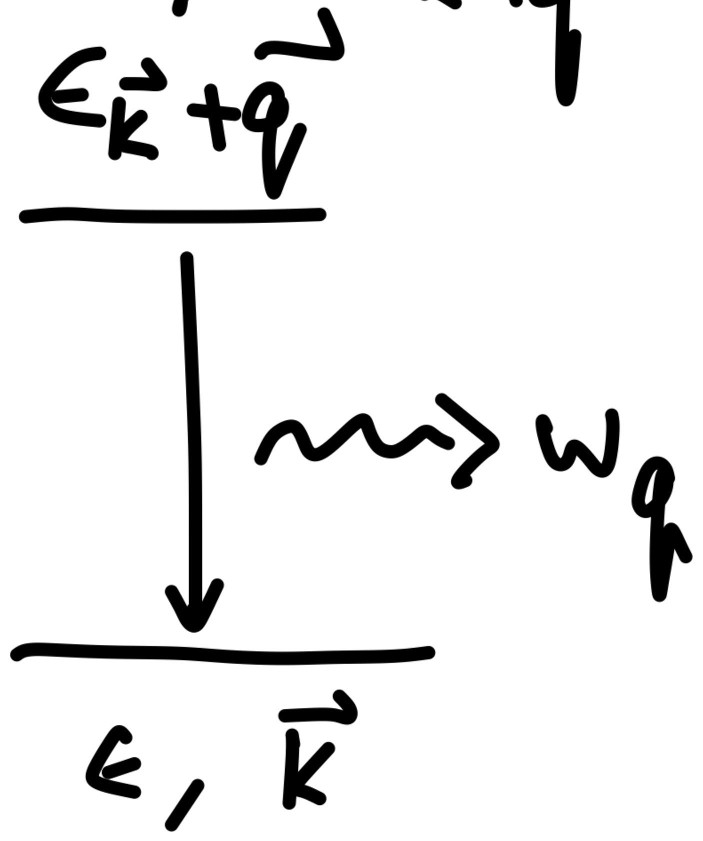

Real / $\mathrm{Im}\,\Sigma(\epsilon, \vec{k})$

II :




$$\delta(\epsilon + \omega_q - \tilde{\epsilon}_{\vec{k}+\vec{q}})\ \theta(\underbrace{-\xi_{\vec{k}+\vec{q}}})$$



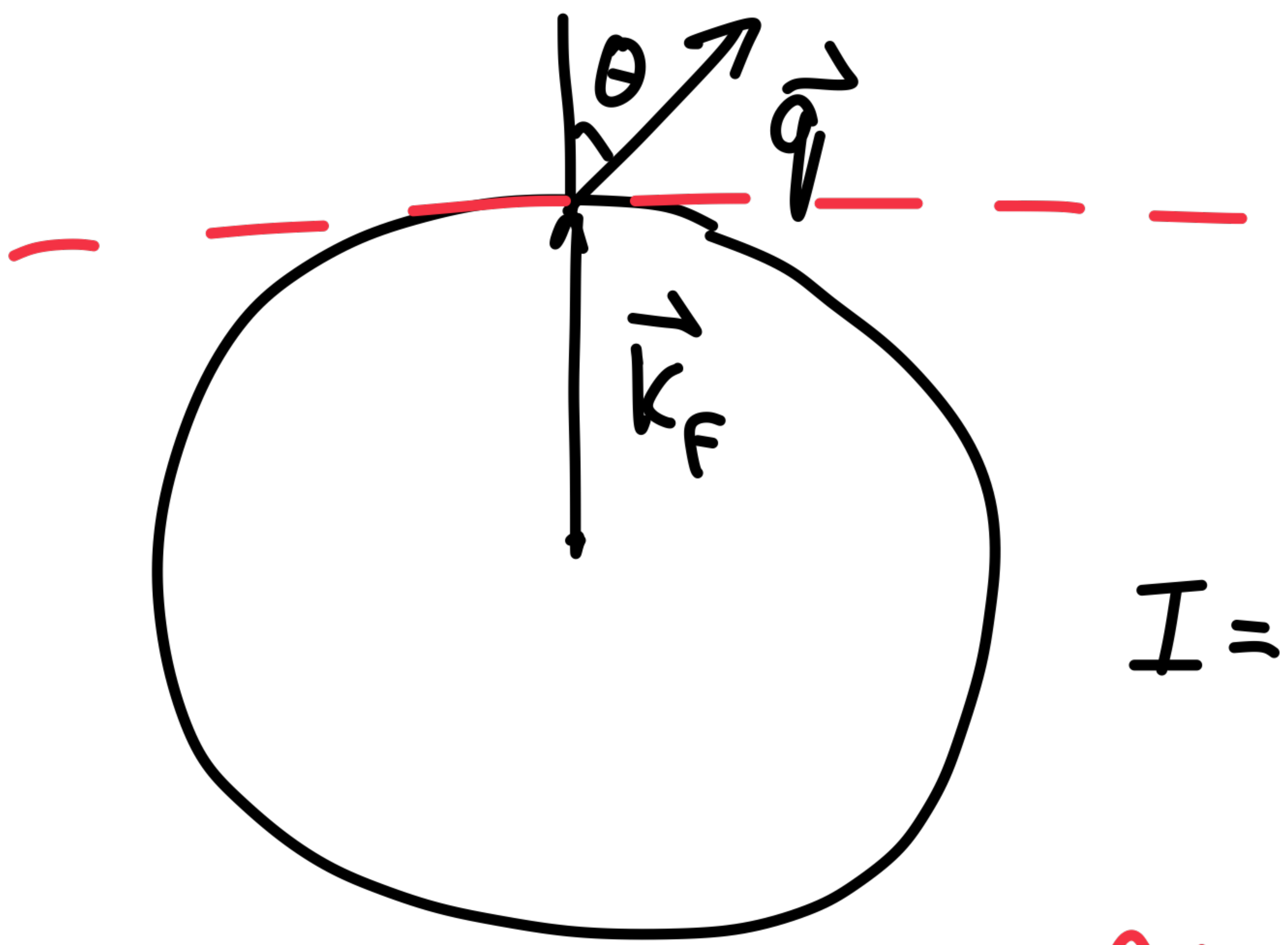
$$\Sigma(\vec{k},\epsilon) = \int_{\vec{q}} -\left[ \underbrace{\frac{\Theta(\vec{k}+\vec{q})}{\omega_{\vec{q}} + \tilde{\epsilon}_{\vec{k}+\vec{q}} - (\epsilon + i\delta)}}_{\Sigma_{I}} - \underbrace{\frac{\bar{\Theta}(\vec{k}+\vec{q})}{(\epsilon - i\delta) + \omega_{q} - \tilde{\epsilon}_{\vec{k}+\vec{q}}}}_{\Sigma_{II}} \right] \times \frac{g^2}{2\omega_q}$$

2D

$$\vec{k} \parallel k_F$$

$$\Sigma_{I}(\vec{k},\epsilon) = \frac{g^2}{8\pi^2} \int dq \int_0^{2\pi} d\theta \, \underbrace{\frac{-1 \quad \Theta(\epsilon_{\vec{k}+\vec{q}})}{q v_s + q v_F \cos\theta - (\epsilon + i\delta)}}_{I}$$

# 2D Kinematics




$$\operatorname{tg}\frac{\theta}{2} = x \qquad d\theta = \frac{2}{1+x^2}\,dx$$

$$\theta \in \left[0, \frac{\pi}{2}\right], \quad \bar{V}_F = V_F + V_B$$

$$I = -4\int_0^1 \frac{dx}{V_+ q - \epsilon - (\epsilon + V_- q)x^2 - i\delta}\,\Theta(\epsilon)$$

$$\operatorname{Re} I = -\frac{2\,\Theta\left(q - \frac{\epsilon}{V_+}\right)}{\sqrt{(V_+ q - \epsilon)(\epsilon + V_- q)}}\left\{\ln\left|\frac{\sqrt{V_+ q - \epsilon} + \sqrt{\epsilon + V_- q}}{\sqrt{V_+ q - \epsilon} - \sqrt{\epsilon + V_- q}}\right|\right\} + \ldots, \qquad V_+ = V_F + V_B, \quad V_- = V_F - V_B$$

$$\operatorname{Im} I = -\,\Theta\left(q - \frac{\epsilon}{V_+}\right)\Theta(\epsilon)\left[2\pi\,\frac{1}{\sqrt{(V_+ q - \epsilon)(\epsilon + V_- q)}}\right]$$

$$\begin{cases} \mathrm{Re}\, I = -\dfrac{2\,\theta\left(q-\frac{\epsilon}{v_+}\right)}{\sqrt{(v_+q-\epsilon)(\epsilon+v_-q)}} \left\{ \ln \left| \dfrac{\sqrt{v_+q-\epsilon}+\sqrt{v_-q+\epsilon}}{\sqrt{v_+q-\epsilon}-\sqrt{v_-q+\epsilon}} \right| \right\} + \dots & \times \dfrac{g^2}{8\pi^2} \\[2ex] \mathrm{Im}\, I = -\,\theta\left(q-\frac{\epsilon}{v_+}\right)\theta(\epsilon)\, 2\pi \dfrac{1}{\sqrt{(v_+q-\epsilon)(\epsilon+v_-q)}} & \times \dfrac{g^2}{8\pi^2} \end{cases}$$

Take the limit $v_B \to 0$, $v_+ = v_-$

$$\mathrm{Im}\sum_R = -\frac{g^2}{4\pi} \underbrace{\int_1^{\frac{\Lambda}{\epsilon}} d\tilde{q} \left\{ \frac{1}{\sqrt{(\tilde{q}-1)\times(1+\tilde{q})}} \right\}}_{\tilde{I}\left(\frac{v_F}{v_B}\right) = \ln\frac{\Lambda}{|\epsilon|}}, \quad \tilde{q} = \frac{v_+ q}{\epsilon}$$

$$\mathrm{Re}\,\Sigma = \frac{g^2}{8\pi^2}\int_0^{\Lambda/\epsilon} d\tilde{q}\,\left\{\frac{1\;\theta(\tilde{q}-1)}{\sqrt{\tilde{q}-1}\,\sqrt{\tilde{q}+1}}\,\ln\left|\frac{\sqrt{\tilde{q}-1}+\sqrt{\tilde{q}+1}}{\sqrt{\tilde{q}-1}-\sqrt{\tilde{q}+1}}\right|\right\}$$

$$+\ldots$$

$$\frac{\partial\,\mathrm{Re}\,\Sigma}{\partial\,\varepsilon} = I\left(\varepsilon, \frac{V_F}{V_B}\right) = -\frac{g^2}{4\pi^2}\,\frac{1}{\epsilon}\,h\left(\frac{V_F}{V_B}\right) > 0$$

Better Way to evaluate $\frac{\partial \, \mathrm{Re}\, \Sigma(\vec{k}_F, \epsilon)}{\partial \epsilon}$

$$\left.\frac{\partial \, \mathrm{Re}\, \Sigma(\vec{k}_F, \epsilon)}{\partial \epsilon}\right|_{\epsilon \to 0} = -2g^2 \int dq \int_0^{\frac{\pi}{2}} d\theta \, \underbrace{\frac{1}{(v_B q - v_F q \cos\theta)^2}} \longleftrightarrow \int_0^1 dx \, \frac{1+x^2}{(v_+ - v_- x^2)^2} \, \frac{1}{q^2}$$

$$= -2g^2 \int \frac{dq}{q^2} \int_0^1 dx \, \underbrace{\frac{1+x^2}{(v_+ - v_- x^2)^2}}_{I}$$

$$\int_0^1 dx \, \frac{1}{(V_+ - V_- x^2)^2} = -\frac{\partial}{\partial V_+} \int_0^1 dx \, \frac{1}{V_+ - V_- x^2} = -\frac{\partial}{\partial V_+}\left[\frac{1}{\sqrt{V_+ V_-}} \ln \frac{\sqrt{\frac{V_+}{V_-}}+1}{\sqrt{\frac{V_+}{V_-}}-1}\right]$$

$$= +\frac{1}{2}\sqrt{\frac{V_+}{V_-}}\left\{\ln \frac{\sqrt{\frac{V_+}{V_-}}+1}{\sqrt{\frac{V_+}{V_-}}-1} + \cdots\right\} \quad \text{if } \frac{V_+}{V_-} - 1 \ll 1$$

$$Z^{-1} = 1 - \left.\frac{\partial \Sigma(\vec{k}_F, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon \to 0} = 1 + 2g^2 \cdot \frac{1}{\varepsilon} \cdot h\left(\frac{V_B}{V_F}\right)$$

# Putting everything together 2D

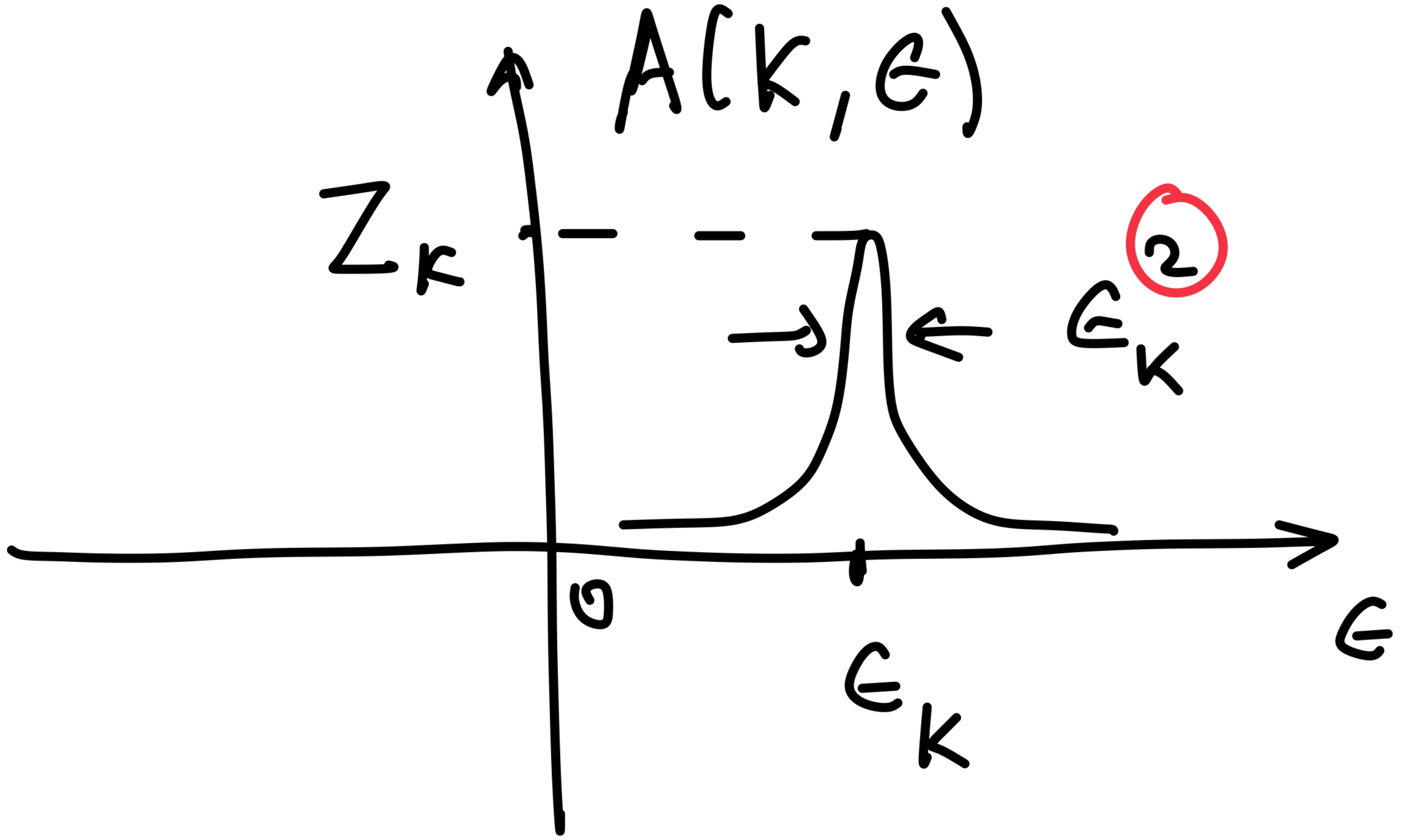
$$Z = \frac{1}{1 - \left.\frac{\partial \Sigma}{\partial \epsilon}\right|_{\epsilon \to 0}} = \frac{1}{1 + \frac{g^2}{\epsilon} \#}$$

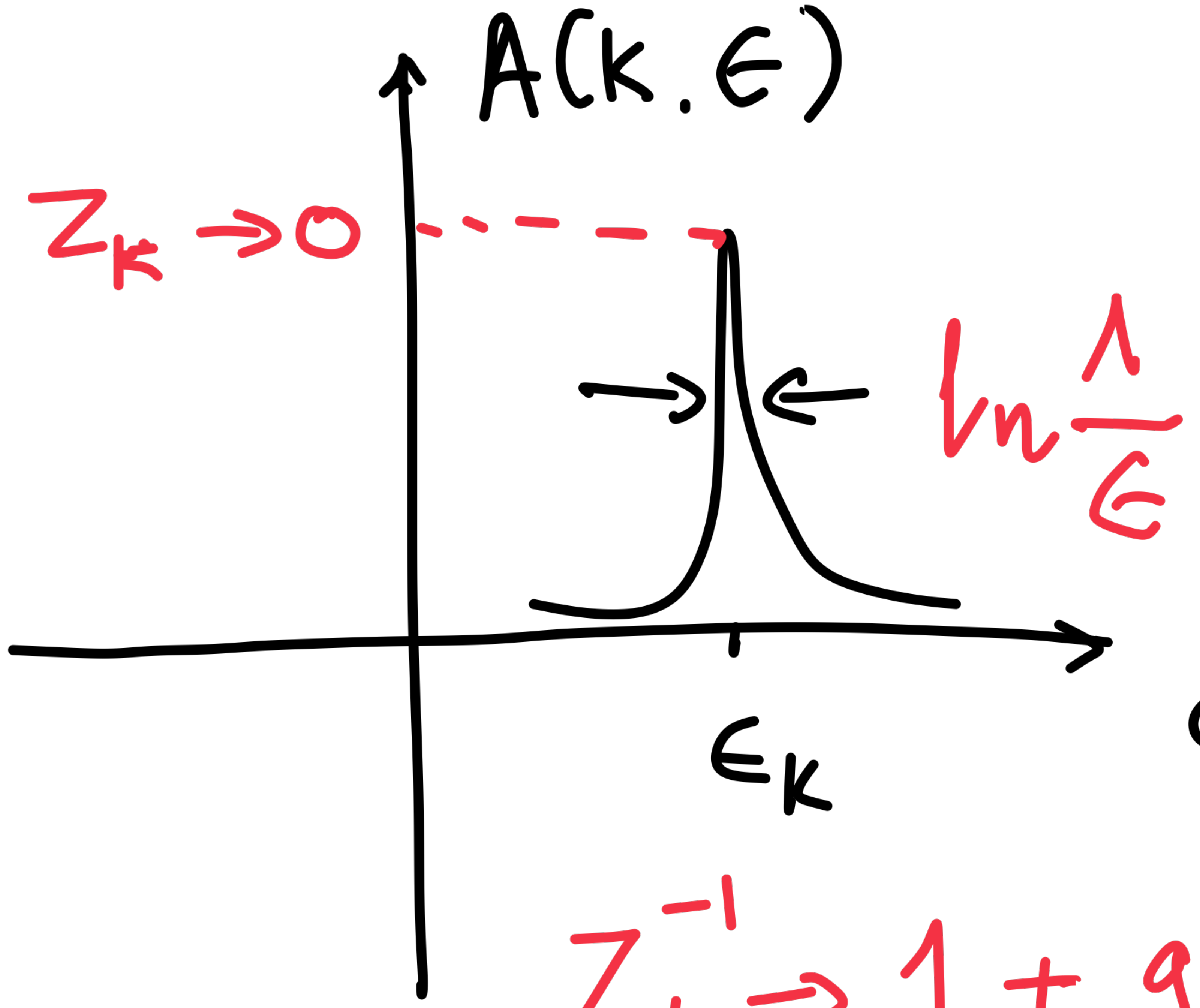
$$\mathrm{Im}\,\Sigma_R = -g^2 \ln \frac{\Lambda}{|\epsilon|} \# \text{ (weak-}\epsilon\text{ dependence)}$$

$$\frac{\mathrm{Im}\,\Sigma_R}{\epsilon} \xrightarrow{\epsilon \to 0} \infty$$

Fermi Liquid $\qquad A(k,\epsilon) = -\frac{1}{\pi} \operatorname{Im} G(k,\epsilon)$

vs Non Fermi Liquid 2D




$Z_k^{-1} \to 1 + g^2 \frac{1}{\epsilon}$ ※

[Suggested by Perturbations]

## What to do beyond perturbation theory : RGE

idea

$$P = P(g;\Lambda) = \Lambda^{p}\tilde{P}(\tilde{g}), \qquad H = H(g;\Lambda)$$

$$0 = \frac{\partial P}{\partial t} = pP + \Lambda^{p}\frac{\partial \tilde{P}(\tilde{g})}{\partial \tilde{g}}\beta(\tilde{g}), \qquad \beta(\tilde{g}) = \frac{\partial \tilde{g}}{\partial t}$$

$$\beta(\tilde{g}) = -p\frac{\tilde{P}}{\frac{\partial \tilde{P}}{\partial \tilde{g}}} \quad \swarrow \text{ a function of } \tilde{g}.$$

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$$\frac{d\tilde{\lambda}}{dt} = \beta_{\tilde{\lambda}}(\tilde{m}, \tilde{\lambda}, \ldots) \qquad t = \ln\frac{\Lambda}{\Lambda_{uv}}$$

Renormalization Group Equations

Running scale