

Phys529B: Topics of Quantum Theory

Lecture 12: identifying Non-Fermi liquids: 3D

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Supplementary stuff on RGE

$$H = H(m, \lambda, \dots; \Lambda) = H(\tilde{m}, \tilde{\lambda}, \dots, Z(\Lambda); \Lambda)$$

for any given Λ , there shall be a $\tilde{m}(\Lambda), \tilde{\lambda}(\Lambda) \dots$

so that H leads to the same physics.

$$\frac{d\tilde{m}}{dt} = \beta_m(\tilde{m}, \tilde{\lambda}, \dots)$$

$$\frac{dZ(\Lambda)}{d\Lambda} = \gamma(\tilde{m}, \tilde{\lambda}, Z)$$

$$\frac{d\tilde{\lambda}}{dt} = \beta_{\tilde{\lambda}}(\tilde{m}, \tilde{\lambda}, \dots)$$

$$t = \ln \frac{\Lambda}{\Lambda_{uv}}$$

Running scale

Renormalization Group Equations

Generally $G = G_0 + G_0 \Sigma G$ or $G^{-1} = G_0^{-1} - \Sigma$

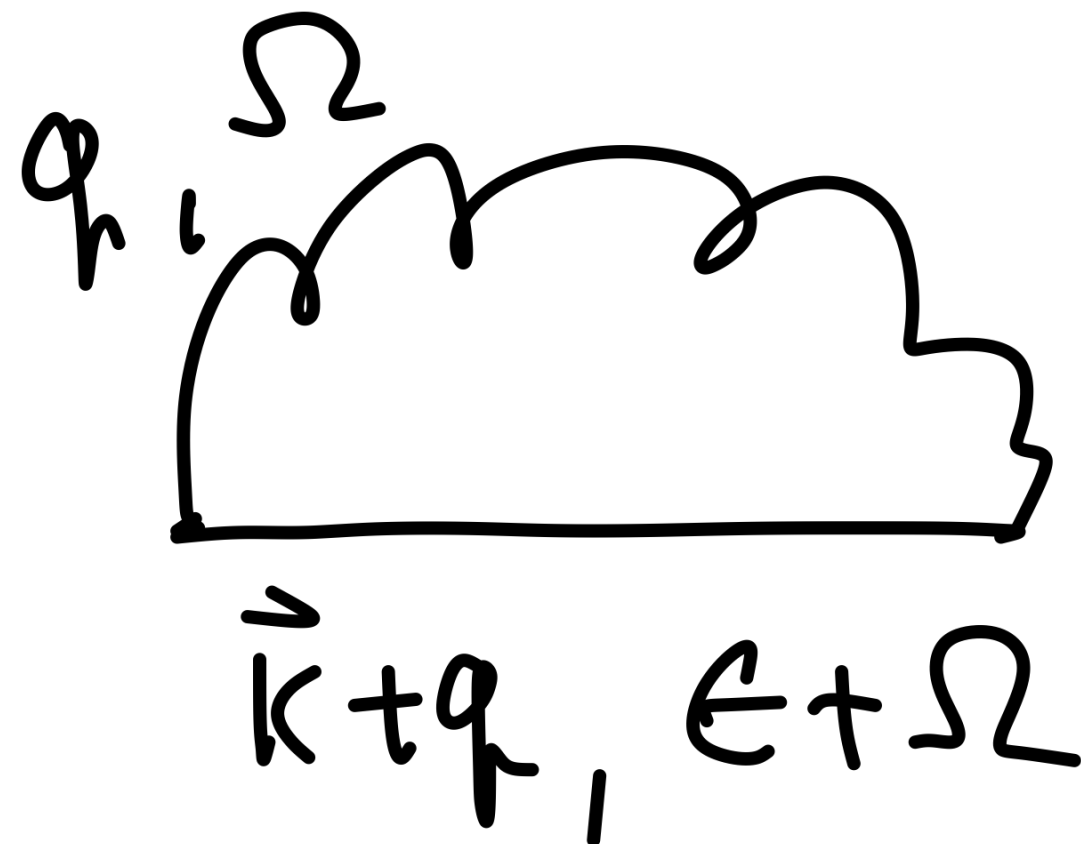
perturbation theory:



$$G_k = \langle 0 | -i T \psi_k(0) \psi_k^\dagger(0) | 0 \rangle$$

$$\delta G = G_0 \Sigma^{(2)} G_0,$$

$$\Sigma^{(2)}(\vec{k}, \epsilon) = i \int D_0(\vec{q}, \Omega) G_0(\vec{k} + \vec{q}, \epsilon + \Omega) \times g^2$$



$$D_0(\vec{q}, \Omega) = \frac{1}{\Omega^2 - \omega_q^2 + i\delta}$$

$$G_0(\vec{k}, \epsilon) = \frac{1}{\epsilon - (\epsilon_k - \mu) + i\delta_k}$$

$$\dots$$

$$\sum(\vec{k}, \epsilon) = \int_{\vec{q}} \left[\frac{\Theta(\vec{k} + \vec{q})}{\omega_{\vec{q}} + \tilde{\epsilon}_{\vec{k} + \vec{q}} - (\epsilon + i\delta)} - \frac{\bar{\Theta}(\vec{k} + \vec{q})}{(\epsilon - i\delta) + \omega_{\vec{q}} - \tilde{\epsilon}_{\vec{k} + \vec{q}}} \right] \times \frac{g^2}{2\omega_{\vec{q}}}$$

$$\frac{\partial \sum(\vec{k}, \epsilon)}{\partial \epsilon} \Big|_{\omega \rightarrow 0} \stackrel{\rightarrow 0}{\approx} \int_{\vec{q}} \left[\frac{\Theta(\vec{k} + \vec{q})}{(\omega_{\vec{q}} + \tilde{\epsilon}_{\vec{k} + \vec{q}})^2} + \frac{\bar{\Theta}(\vec{k} + \vec{q})}{(\omega_{\vec{q}} - \tilde{\epsilon}_{\vec{k} + \vec{q}})^2} \right] \frac{g^2}{2\omega_{\vec{q}}}$$

$$\tilde{\epsilon}_{\vec{k}} = \epsilon_{\vec{k}} - \mu$$

$$\sum_{\vec{q}}(\vec{k}, \epsilon) = \int \left[\underbrace{\frac{\Theta(\vec{k} + \vec{q})}{\omega_{\vec{q}} + \tilde{\epsilon}_{\vec{k} + \vec{q}} - (\epsilon + i\delta)}}_{\Sigma_I} - \underbrace{\frac{\bar{\Theta}(\vec{k} + \vec{q})}{(\epsilon - i\delta) + \omega_{\vec{q}} - \tilde{\epsilon}_{\vec{k} + \vec{q}}}}_{\Sigma_{II}} \right] \times \frac{g^2}{2\omega_{\vec{q}}}$$

$$3D \quad \sum_I(\vec{k}, \epsilon) = \frac{g^2}{8\pi^2} \int d\vec{q} \, q \int_0^1 dx \frac{-1}{\underbrace{q v_s + v_F q x - (\epsilon + i\delta)}_I}$$

$$I = \frac{-1}{q v_F} \left[\ln \left| \frac{q(v_s + v_F) - \epsilon}{q v_s - \epsilon} \right| + i\pi \Theta(\epsilon - q v_s) \Theta((v_F + v_s)q - \epsilon) \right]$$

3D

$$\Sigma_{\pm}(\vec{k}, \epsilon) = \frac{g^2}{v_F} \int \frac{dq}{8\pi^2} \left[\ln \left| \frac{q(v_F + v_s) - \epsilon}{qv_s - \epsilon} \right| - i\pi \theta(\epsilon - qv_s) \theta(q(v_F + v_s) - \epsilon) \right]$$

$$\begin{aligned} \frac{\partial \text{Re} \Sigma_{\pm}(\vec{k}_F, \epsilon)}{\partial \epsilon} &= \frac{g^2}{8\pi^2} \int dq \left[\frac{1}{q(v_F + v_s) - \epsilon} - \frac{1}{qv_s - \epsilon} \right] \frac{1}{v_F} \\ &= - \ln \frac{\Lambda_{uv}}{|\epsilon|} \frac{1}{(v_F + v_s)v_s} \times \frac{g^2}{8\pi^2} \end{aligned}$$

$$\text{Im} \Sigma_{\pm}(\vec{k}_F, \epsilon) = - \theta(\epsilon) \epsilon \left[\frac{1}{v_s(v_F + v_s)} \right] \times \frac{g^2}{8\pi^2}$$

Putting things together for 3D

$$G_R = G \Theta(\omega) + G^* \Theta(-\omega)$$

$$\frac{\partial \text{Re} \Sigma_R(k_F, \epsilon)}{\partial \epsilon} \sim -g^2 \ln \frac{\Lambda}{|\epsilon|}$$

$$\text{Im} \Sigma_R(k_F, \epsilon) \sim \frac{1}{v_F} g^2 |\epsilon| \pi \quad (\text{even function})$$

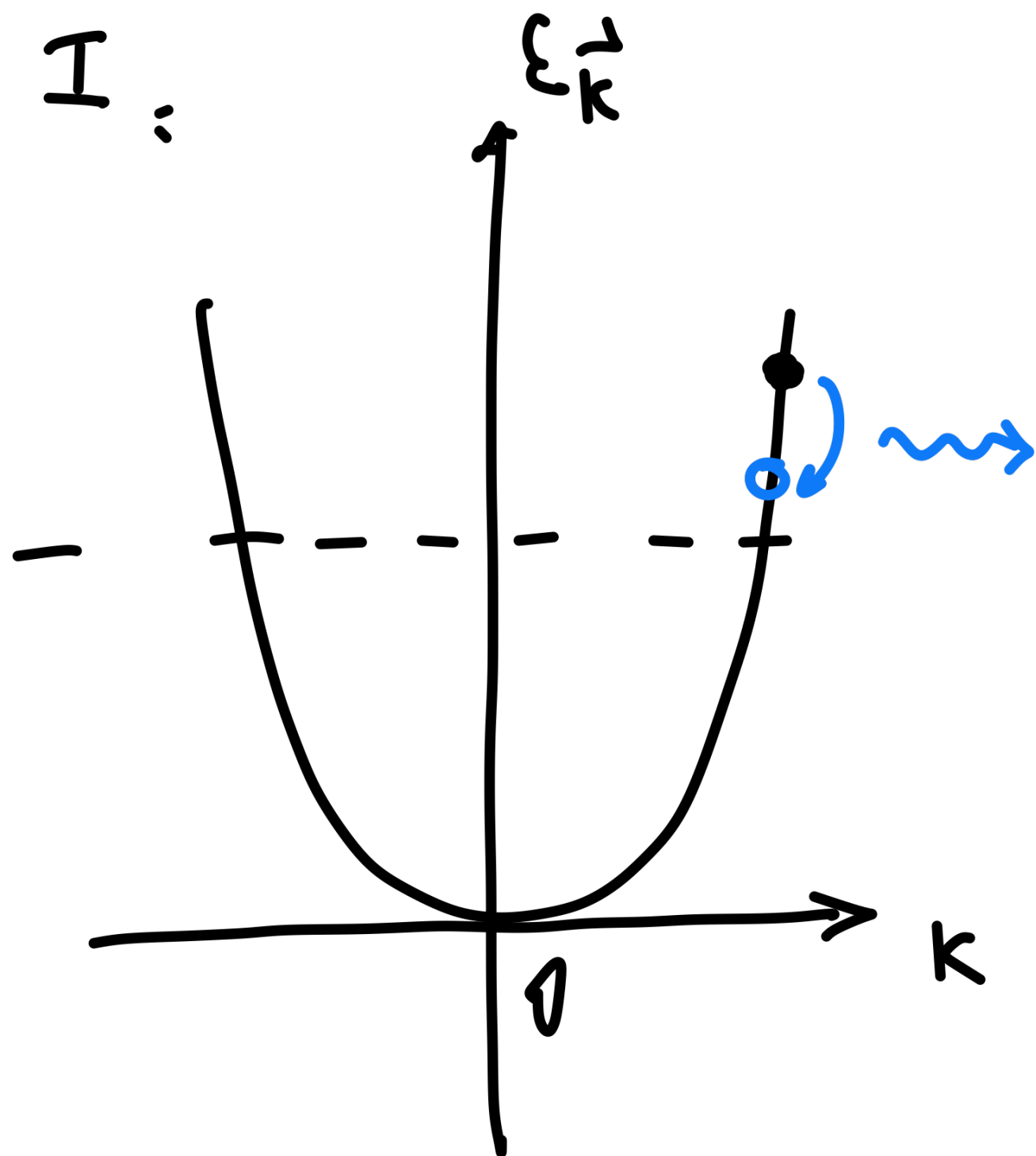
$$\Gamma(k_F) \approx \frac{1}{1 + \frac{1}{v_F} \frac{g^2}{4\pi^2} \ln \frac{\Lambda}{|\epsilon|}} \bigg|_{|\epsilon| \rightarrow 0} \rightarrow 0 \quad (v_s = 1)$$

$$\dots$$

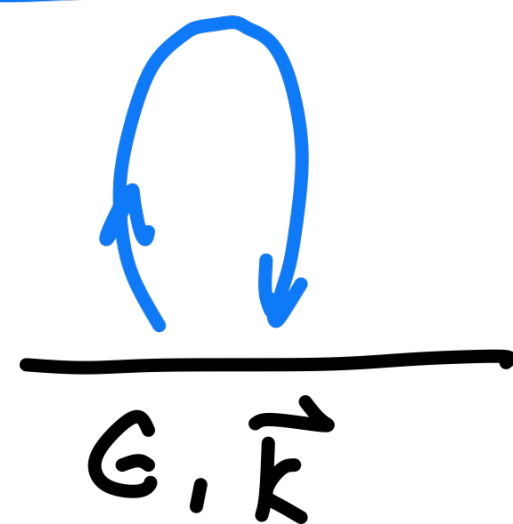
$$\sum(\vec{k}, \epsilon) = \int_{\vec{q}} \left[\frac{\Theta(\vec{k} + \vec{q})}{\omega_{\vec{q}} + \tilde{\epsilon}_{\vec{k} + \vec{q}} - (\epsilon + i\delta)} - \frac{\bar{\Theta}(\vec{k} + \vec{q})}{(\epsilon - i\delta) + \omega_{\vec{q}} - \tilde{\epsilon}_{\vec{k} + \vec{q}}} \right] \times \frac{g^2}{2\omega_{\vec{q}}}$$

$$\frac{\partial \sum(\vec{k}, \epsilon)}{\partial \epsilon} \Big|_{\omega \rightarrow 0} \stackrel{\rightarrow 0}{\approx} \int_{\vec{q}} \left[\frac{\Theta(\vec{k} + \vec{q})}{(\omega_{\vec{q}} + \tilde{\epsilon}_{\vec{k} + \vec{q}})^2} + \frac{\bar{\Theta}(\vec{k} + \vec{q})}{(\omega_{\vec{q}} - \tilde{\epsilon}_{\vec{k} + \vec{q}})^2} \right] \frac{g^2}{2\omega_{\vec{q}}}$$

$$\tilde{\epsilon}_{\vec{k}} = \epsilon_{\vec{k}} - \mu$$



$$\frac{\epsilon_{\vec{k}+\vec{q}}, \omega_{\vec{q}}}{\epsilon, \vec{k}}$$



Virtual / $\text{Re} \Sigma(\epsilon, \vec{k})$

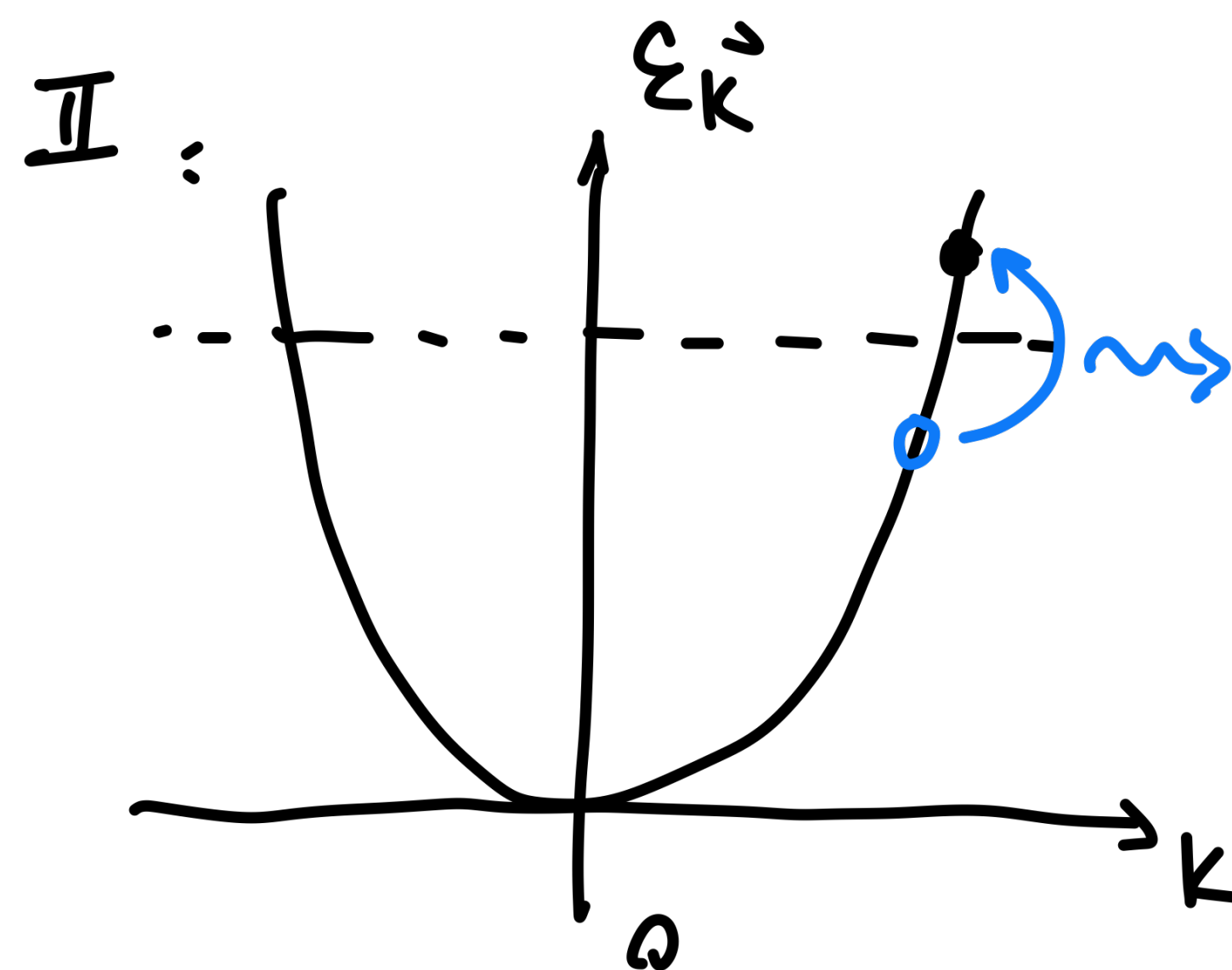
$$\delta(\epsilon - \omega_{\vec{q}} - \tilde{\epsilon}_{\vec{k}+\vec{q}}) \theta(\epsilon_{\vec{k}+\vec{q}})$$

$$\frac{\epsilon, \vec{k}}{\epsilon_{\vec{k}+\vec{q}}, \omega_{\vec{q}}}$$

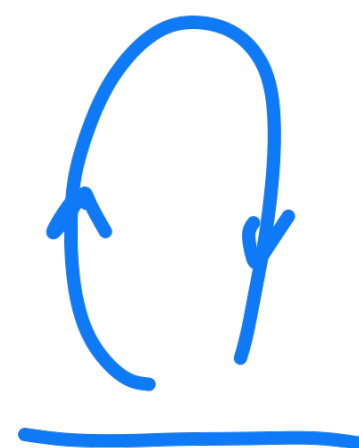
Real / $\text{Im} \Sigma(\epsilon, \vec{k})$

$$\delta(\epsilon + \omega_{\vec{q}} - \tilde{\epsilon}_{\vec{k}+\vec{q}}) \theta(-\epsilon_{\vec{k}+\vec{q}})$$

$$\frac{\epsilon_{\vec{k}+\vec{q}}}{\epsilon, \vec{k}}$$



$$\frac{\epsilon, \vec{k}}{\epsilon_{\vec{k}+\vec{q}}}$$



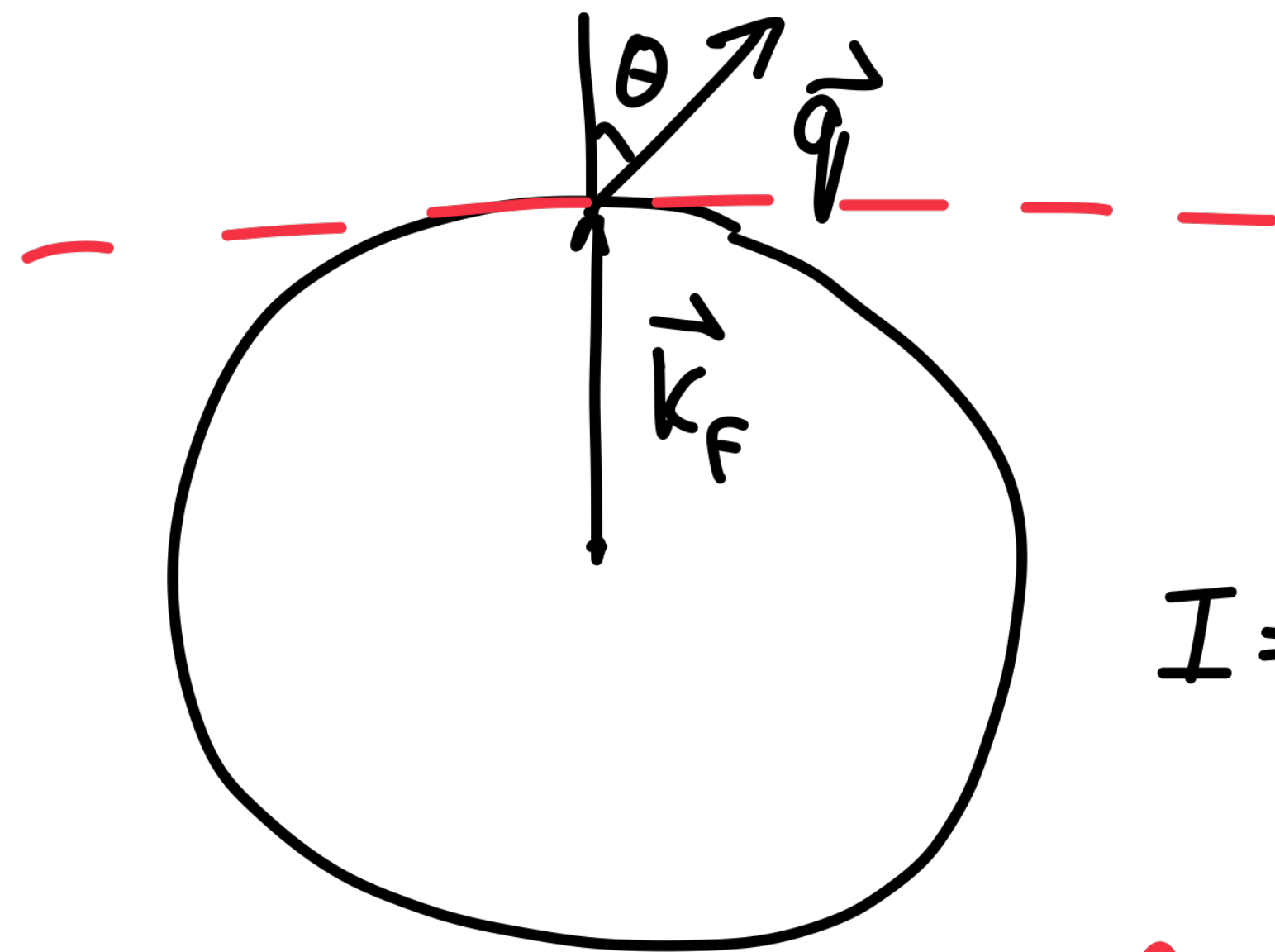
$$\epsilon_{\vec{k}+\vec{q}}$$

$$\Sigma(\vec{k}, \epsilon) = \int_{\vec{q}} \left[\underbrace{\frac{\Theta(\vec{k} + \vec{q})}{\omega_{\vec{q}} + \tilde{\epsilon}_{\vec{k} + \vec{q}} - (\epsilon + i\delta)}}_{\Sigma_I} - \underbrace{\frac{\bar{\Theta}(\vec{k} + \vec{q})}{(\epsilon - i\delta) + \omega_{\vec{q}} - \tilde{\epsilon}_{\vec{k} + \vec{q}}}}_{\Sigma_{II}} \right] \times \frac{q^2}{2\omega_{\vec{q}}}$$

2D

$$\Sigma_I(\vec{k}, \epsilon) = \frac{q_F^2}{8\pi^2} \int d\vec{q} \int_0^{2\pi} d\theta \frac{-1}{\underbrace{q V_s + q V_F \cos\theta}_{I} - (\epsilon + i\delta)} \Theta(\epsilon_{\vec{k} + \vec{q}})$$

2D Kinematics



$$\tan \frac{\theta}{2} = x$$

$$d\theta = \frac{2}{1+x^2} dx$$

$$\theta \in [0, \frac{\pi}{2}], \quad \bar{V}_F = V_F + V_B$$

$$I = -4 \int_0^1 \frac{dx}{V_+ q - \epsilon - (\epsilon + V_- q) x^2 - i\delta} \quad \theta(\epsilon)$$

$$\text{Re } I = - \frac{2 \theta(q - \frac{\epsilon}{V_+})}{\sqrt{(V_+ q - \epsilon)(\epsilon + V_- q)}} \left\{ \ln \left| \frac{\sqrt{V_+ q - \epsilon} + \sqrt{\epsilon + V_- q}}{\sqrt{V_+ q - \epsilon} - \sqrt{\epsilon + V_- q}} \right| \right\} + \dots$$

$V_+ = V_F + V_B$
 $V_- = V_F - V_B$

$$\text{Im } I = - \theta(q - \frac{\epsilon}{V_+}) \theta(\epsilon) \left[2\pi \frac{1}{\sqrt{(V_+ q - \epsilon)(\epsilon + V_- q)}} \right]$$

$$\left\{ \begin{aligned} \text{Re } I &= - \frac{2 \Theta(q - \frac{\epsilon}{v_+})}{\sqrt{(v_+ q - \epsilon)(\epsilon + v_- q)}} \left\{ \ln \left| \frac{\sqrt{v_+ q - \epsilon} + \sqrt{v_- q + \epsilon}}{\sqrt{v_+ q - \epsilon} - \sqrt{v_- q + \epsilon}} \right| \right\} + \dots & \times \frac{g^2}{8\pi^2} \\ \text{Im } I &= - \Theta(q - \frac{\epsilon}{v_+}) \Theta(\epsilon) 2\pi \frac{1}{\sqrt{(v_+ q - \epsilon)(\epsilon + v_- q)}} & \times \frac{g^2}{8\pi^2} \end{aligned} \right.$$

Take the limit $v_B \rightarrow 0$, $v_+ = v_-$

$$\text{Im } \Sigma_R = - \frac{g^2}{4\pi} \int_1^{\frac{\Lambda}{\epsilon}} \frac{d\tilde{q}}{\tilde{q}} \left\{ \frac{1}{\sqrt{(\tilde{q} - 1)(1 + \tilde{q})}} \right\}, \quad \tilde{q} = \frac{v_+ q}{\epsilon}$$

$$\tilde{I}\left(\frac{v_F}{v_B}\right) = \ln \frac{\Lambda}{|\epsilon|}$$

$$\text{Re} \Sigma = \frac{g^2}{8\pi^2} \int_0^1 \frac{1}{\tilde{\epsilon}} d\tilde{q} \left\{ \frac{1}{\sqrt{\tilde{q}-1} \sqrt{\tilde{q}+1}} \ln \left| \frac{\sqrt{\tilde{q}-1} + \sqrt{\tilde{q}+1}}{\sqrt{\tilde{q}-1} - \sqrt{\tilde{q}+1}} \right| \right\} + \dots$$

$$\frac{\partial \text{Re} \Sigma}{\partial \epsilon} = I(\epsilon, \frac{V_F}{V_B}) = - \frac{g^2}{4\pi^2} \frac{1}{\epsilon} \ln \left(\frac{V_F}{V_B} \right) > 0$$

Better Way to evaluate

$$\frac{\partial \text{Re} \Sigma(\vec{k}_F, \epsilon)}{\partial \epsilon}$$

$$\left. \frac{\partial \text{Re} \Sigma(\vec{k}_F, \epsilon)}{\partial \epsilon} \right|_{\epsilon \rightarrow 0} = 2g^2 \int d^3q \int_0^{\frac{\pi}{2}} d\theta \frac{1}{(v_B q - v_F q \cos \theta)^2} \rightarrow \int_0^1 dx \frac{1+x^2}{(v_+ - v_- x^2)^2} \frac{1}{q^2}$$

$$= -2g^2 \int \frac{d^3q}{q^2} \int_0^1 dx \frac{1+x^2}{(v_+ - v_- x^2)^2} \quad \text{I}$$

$$\int_0^1 dx \frac{1}{(v_+ - v_- x^2)^2} = -\frac{\partial}{\partial v_+} \int_0^1 dx \frac{1}{v_+ - v_- x^2} = -\frac{\partial}{\partial v_+} \left[\frac{1}{\sqrt{v_+ v_-}} \ln \frac{\sqrt{\frac{v_+}{v_-} + 1}}{\sqrt{\frac{v_+}{v_-} - 1}} \right]$$

$$= + \frac{1}{2} \sqrt{\frac{v_+}{v_-}} \left\{ \ln \frac{\sqrt{\frac{v_+}{v_-} + 1}}{\sqrt{\frac{v_+}{v_-} - 1}} + \dots \right\} \quad \text{if } \frac{v_+}{v_-} - 1 \ll 1$$

$$Z^{-1} = 1 - \left. \frac{\partial \Sigma(\vec{k}_F, \epsilon)}{\partial \epsilon} \right|_{\epsilon \rightarrow 0} = 1 + 2g^2 \cdot \frac{1}{\epsilon} \cdot \ln\left(\frac{v_B}{v_F}\right)$$

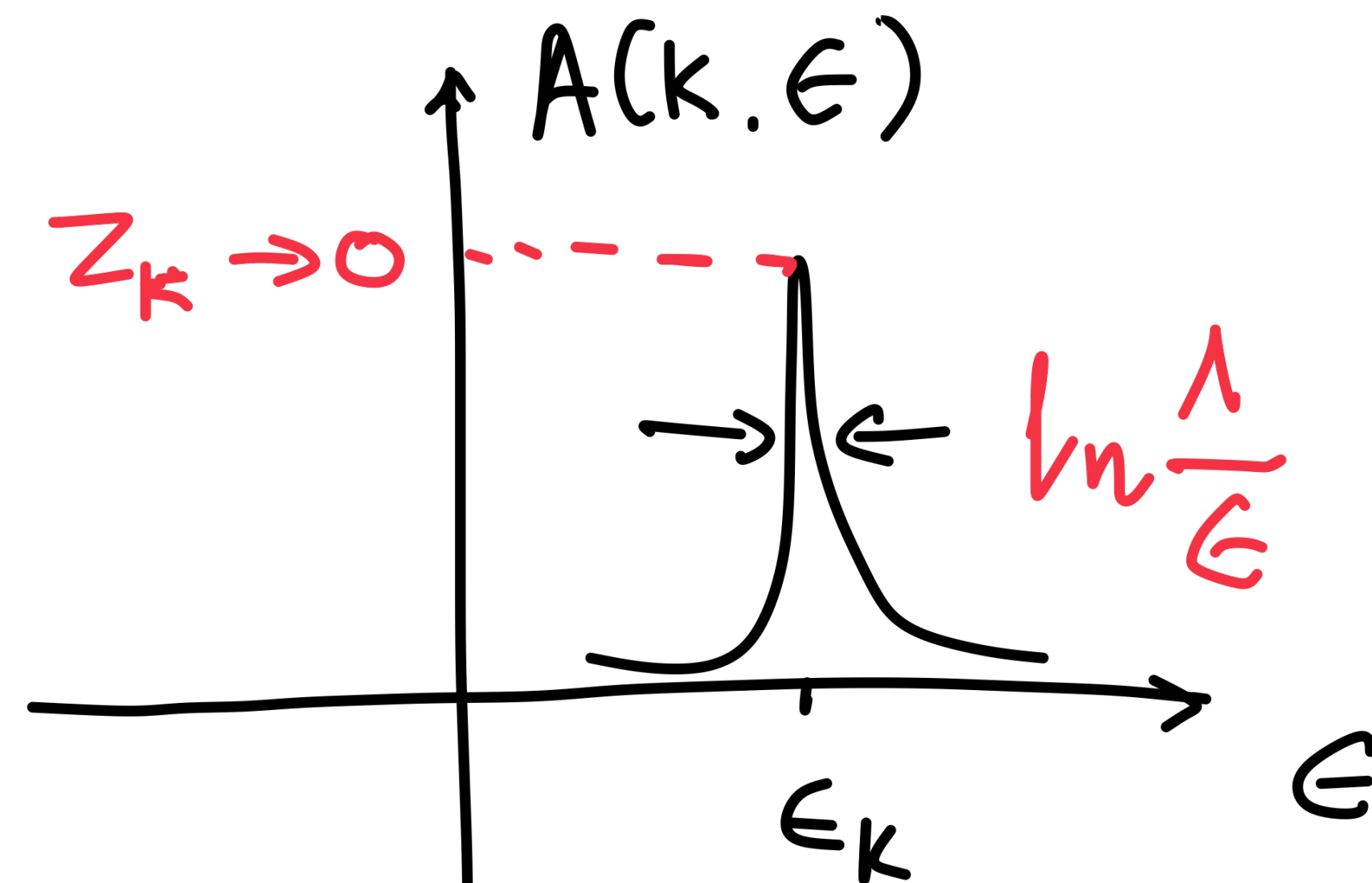
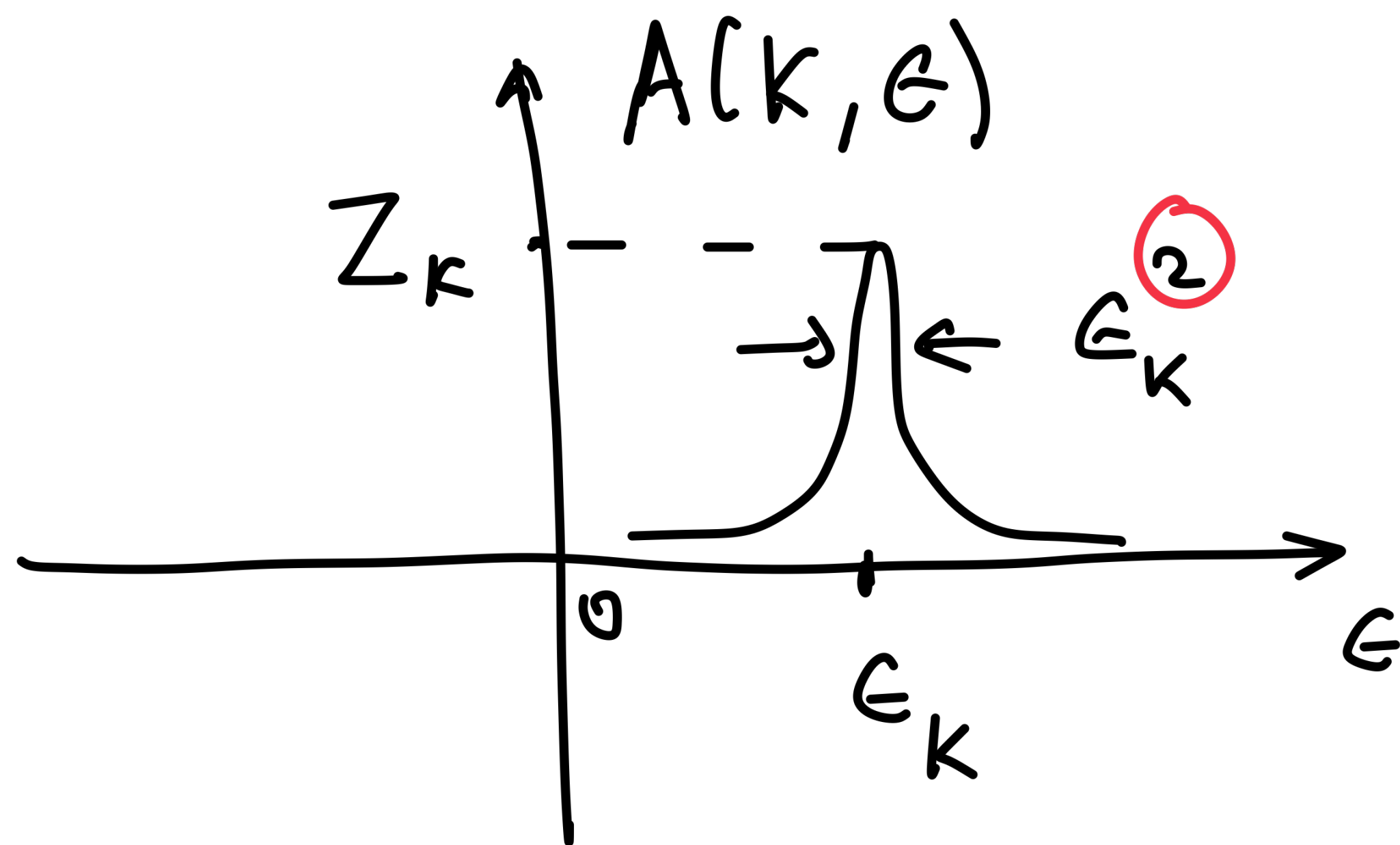
Putting everything together 2D

$$Z = \frac{1}{1 - \left. \frac{\partial \Sigma}{\partial \epsilon} \right|_{\epsilon \rightarrow 0}} = \frac{1}{1 + \frac{g^2}{\epsilon}} \#$$

$$\text{Im } \Sigma_R = -g^2 \ln \frac{1}{|\epsilon|} \# (\text{weak-}\epsilon \text{ dependence})$$

$$\frac{\text{Im } \Sigma_R}{\epsilon} \xrightarrow{\epsilon \rightarrow 0} \infty$$

Fermi Liquid $A(k, \epsilon) = -\frac{1}{\pi} \text{Im } G(k, \epsilon)$
 vs Non Fermi liquid 2D



$$Z_k^{-1} \rightarrow 1 + g^2 \frac{1}{\epsilon} \#$$

[Suggested by Perturbations]