Phys529 Homework Set 1 (Posted Feb 1, 5pm; Due Feb 25, 5pm)

One of the hallmark signatures of the Fermi Liquid Theory (FLT) is the emergent bosonic field in the low energy sector historically associated with **zero sound.**
The other one is the discontinuity of the occupation number at the Fermi surface associated with **Z.** This short homework set is to help you further understand the origins and consequences of these phenomena.

Q1: In 3D, while for non-interacting fermions, all low energy degrees of freedom are carried by fermions, in Fermi liquids there are emergent bosons as elementary excitations. There bosons can be thought as collective ***particle-hole pairs***. In 3D non-interacting Fermi gases, these pairs simple form a continuum rather than being elementary excitations. (Please see discussions in L3, L6.) And the Green's function for boson fields does not have isolated poles; rather it has cuts in the complex plane therefore doesn't have the characteristics of elementary excitations. Below you will study boson field dynamics in 1D and 2D and illustrate the differences between 1D and 2D. (3D limit was discussed in L6).

On a very general ground, one can argue that the boson field shall be described by

$$\omega = v_F Q[1 + f_d(\tilde{g})], d = 1,2.$$

where Q is the total momentum associated with the excitations and g-tilde is a dimensionless coupling constant one introduce to parametrize a short range 4-fermion interaction.

a) Using the dimension analysis, find out how to construct dimensional constants in 1D and 2D given a local interaction of the form: $H_I = g_d \int d^d\mathbf{r}\psi^\dagger\psi\psi^\dagger\psi, d = 1,2.$ where spin indices have been muted.

b) Following the general discussions in L6, calculate the time ordered Green's function for boson fields in both 1D and 2D. Start with non-interacting case where it reduces to a loop diagram of free fermions. Identify the frequency range where the imaginary part becomes zero. Hint: First understand the kinematics of the particle -hole continuum.

c) Now turn on interactions and apply the ladder summation to identify the isolated poles in the Green's function by forcing the real and imaginary part of the inversed Green's function to be zero. This leads to the dimensionless function f in 1D and 2D.

d) What are the main differences between 1D and higher dimensions 2D and 3D ?

Q2: Non Fermi liquid behaviour in the **Hatsugai-Kohmoto Toy** model model

In this problem, you will need to show if the Green's functions has poles both below and above the fermi surfaces with the same residue **Z=1/2**, there shall be no discontinuity in the occupation number at the fermi surface. (In this problem, you assume **U** is smaller than the bandwidth of **2t**.)

a) Following our discussions in L8 etc, the Green's function has the following structure near the Fermi surface or k=k_F:

$$G_\sigma(k,\epsilon) = \frac{Z_p}{\epsilon - (\epsilon_k + U - \mu) + i\delta} + \frac{Z_h}{\epsilon - (\epsilon_k - \mu) - i\delta}$$

where $\sigma = \uparrow, \downarrow, Z_p + Z_h = 1$. Find out the values for Zs and verify the poles following the explicit definition of the time ordered Green's function. The average over the ground state can be done for each k independently as the Hamiltonian is a sum of commuting terms defined at different k-points (see L8 notes); the ground state is an ensemble of huge number of micro-states or a mixture.

Hint:

At a Fermi surface $k = k_F, \epsilon_k + U > \mu > \epsilon_k$. Or in regime II (blue) in L8. (And it is true for for an arbitrary filling factor !). You can also define the fermi surface by tracing out zeros of the zero frequency Green's function above in k-space.

b) Show that the occupation number discontinuity near a Fermi surface at k=k_F vanishes for the above HK model using the deformation into the complex frequency plane.