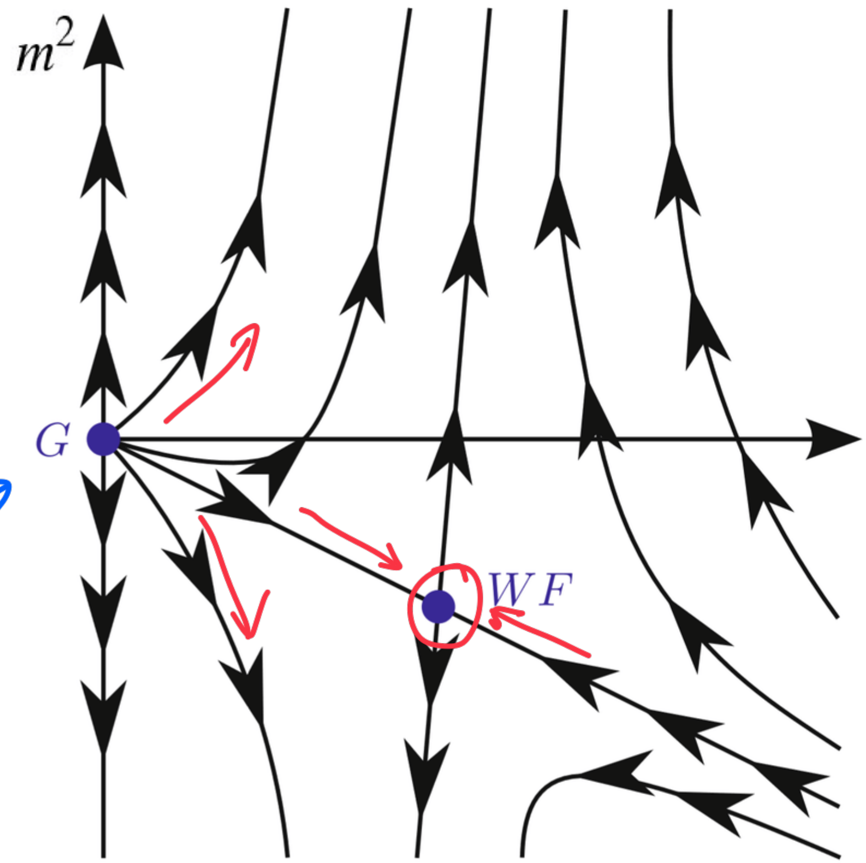
Phys525:
Quantum Condensed Matter Physics: Quantum Criticality Basics, Dynamics and Topological criticality

Episode 12:
Application: $z=2$ free particle/Strong interacting fixed points in BHM and in Quantum gases etc

RG flows (i.e. scale transformation) of a scalar model : Scale Symmetric Wilson-Fisher F.P. (d+1<4)

$d < 3$

mass



$\lambda = g$ interaction

$g\phi^4$

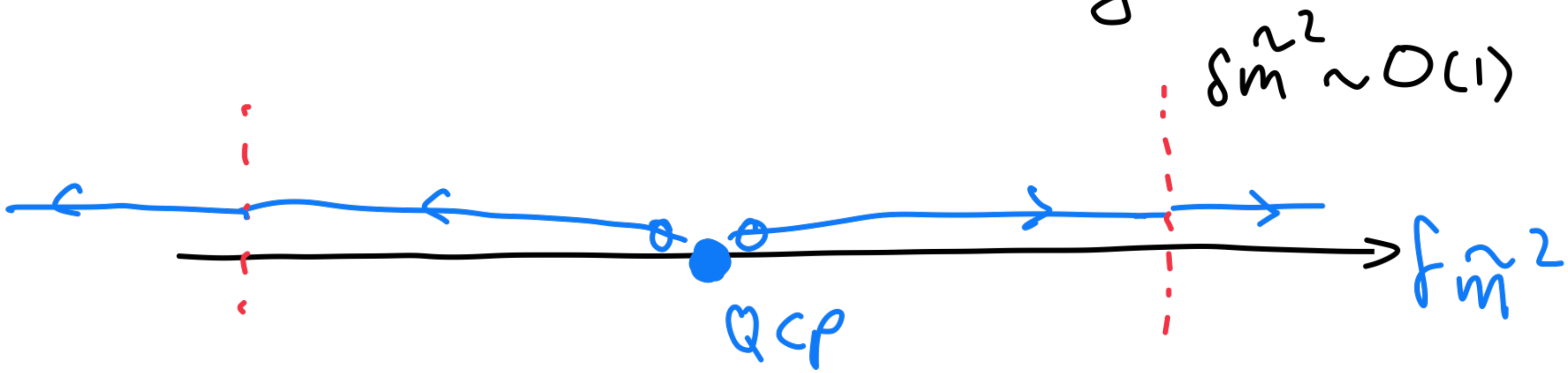
MF fixed pt

IR unstable

WF fixed pt $(d<3)$

I.R. Stable

RGE Vs Correlation length



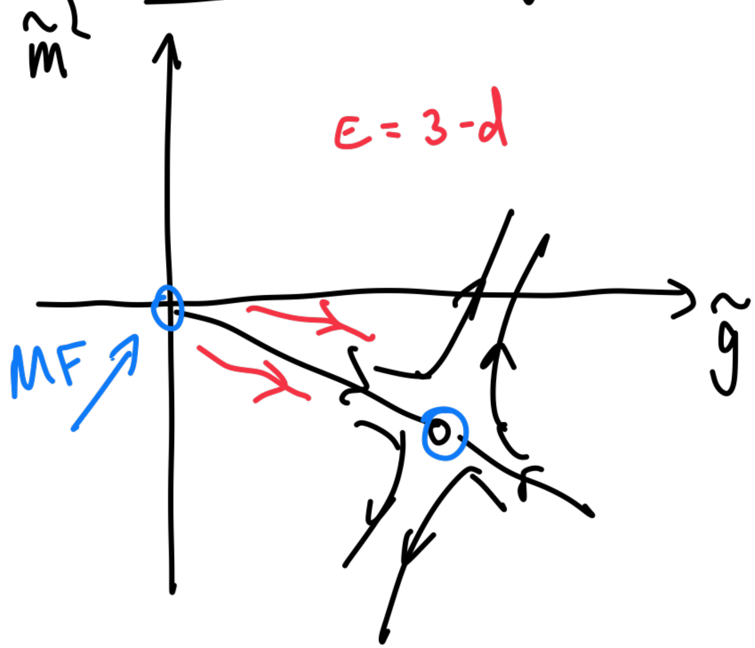
$\delta\tilde{m}^2 = 0 \longrightarrow$ QCP

$\delta\tilde{m}^2 \ll 1 \longrightarrow$ Quantum Critical Regime – QCR

$\delta\tilde{m}^2 \sim 1 \longrightarrow$ Scale Symmetry fully broken

$$\delta\tilde{m}^2(L=\xi_c) \simeq 1$$

Correlation length



Near WF fixed pt

$$\frac{d}{dt}\begin{bmatrix}\delta\tilde{m}^2\\ \delta\tilde{g}\end{bmatrix} = \begin{bmatrix}-2+e_d\epsilon, & -a_d\\ 0 & \epsilon\end{bmatrix}\begin{bmatrix}\delta\tilde{m}^2\\ \delta\tilde{g}\end{bmatrix}$$

$$\frac{\delta\tilde{m}^2(L)}{\delta\tilde{m}^2(\Lambda_{UV})} \sim \left(\frac{L}{a}\right)^{(2-e_d\epsilon)},$$

$$\xi_c \simeq |J-J_c|^{-\frac{1}{2-e_d\epsilon}}$$

$$\delta \quad |J-J_c|/J_c$$

$$\epsilon \to 0, \quad \xi_c \sim |J-J_c|^{-\frac{1}{2}}$$

MF Approximation

$$\xi_c \sim \delta\tilde{m}^{2^{-1/2}} \sim |J-J_c|^{-\frac{1}{2}}$$

$|\delta\tilde{m}^2| \sim |J-J_c|$ for BHM

RGE/ Scale Transformation

$$\xi_c \sim |\delta\tilde{m}^2|^{-\frac{1}{2-e_d\epsilon}} \sim |J-J_c|^{-\frac{1}{2-e_d\epsilon}}$$

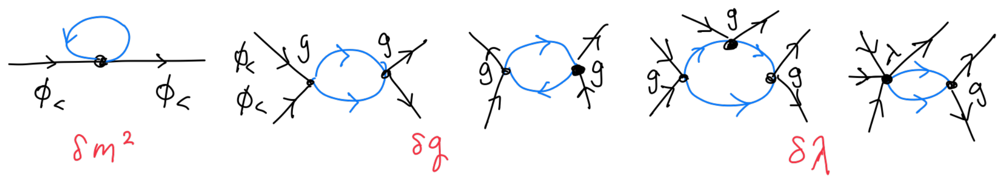
$|\delta\tilde{m}^2| \sim |J-J_c|$ for BHM

Homework SET 3 (due March 15, 2021)

During the lectures 5A, 5B, 6A, 6B, we introduced scale symmetries, scale transformation and renormalization equations. We also briefly discussed the implications on QCPs in the BHM. in this homework set, you will have the opportunities to fill in some of the small gaps in my lectures so to better understand how things work. All calculations in similar studies require knowledge of QFT although I will try o formulate the problem in a way that you can approach the conclusions with minimal knowledge and minimal analysis.

Q1: In this problem, we are going to under the QCP in BHM using the RGEs in QFT. Some of these discussions will be very useful when we discuss quantum dynamics near QCPs.

1) Using the momentum-frequency shell approach outlined in 6A and 6B, find out the renormalized Larangian explicitly in **(2+1)D** after performing the integration over the fields within the shell. Expressed your result in the linear order of the width of the shell and in the leading non-vanishing order of **g.**



$$\longrightarrow \quad G(\omega,\vec{p}) = \langle \varphi^*_>(\omega,\vec{p})\,\varphi_>(\omega,\vec{p})\rangle_{Z_>} = \frac{1}{\omega^2+\vec{p}^{\,2}+m^2}$$

$$\delta\mathcal{L}_< = \delta m^2 \frac{1}{2!}\phi^*_<\phi_< + \frac{\delta g}{4!}(\phi^*_<\phi_<)^2 + \frac{\delta\lambda}{6!}(\phi^*_<\phi_<)^3$$

$$\delta m^2 = +A g\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{d\omega}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\,G(\omega,\vec{p}), \quad A=\frac{1}{6}$$

$$\delta g = -B g^2\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{d\omega}{(2\pi)}\frac{d\vec{p}}{(2\pi)^2}\,G(\omega,\vec{p})G(\omega,\vec{p}), \quad B=\frac{3}{8}$$

$$\delta\lambda = +C g^3\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{d\omega}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\,G(\omega,\vec{p})G(\omega,\vec{p})G(\omega,\vec{p})$$

$$- D\lambda g\int_{\Lambda-\delta\Lambda}^{\Lambda}\frac{d\omega}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\,G(\omega,\vec{p})G(\omega,\vec{p}), \quad C=\frac{5}{6}, \quad D=\frac{1}{4}$$

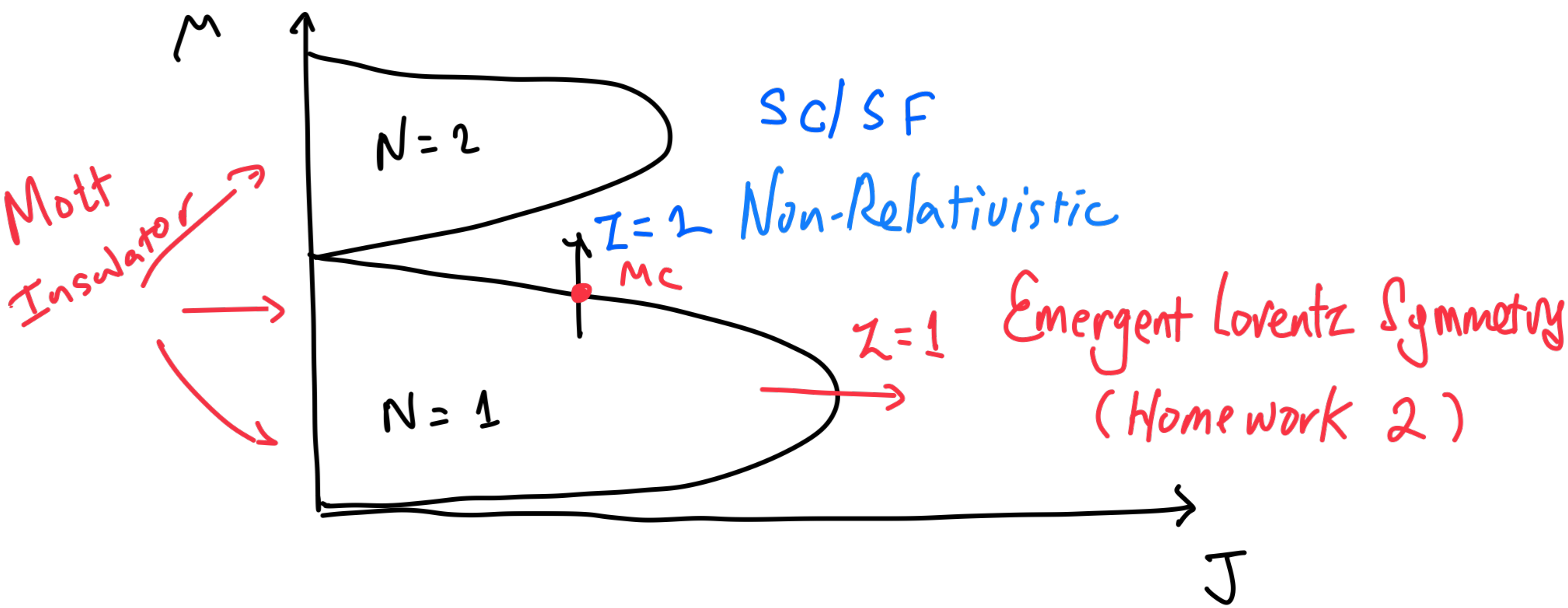
A, B, C, D are numerical factors depending on diagram topology as well as the exchange factors and are given for this problem.

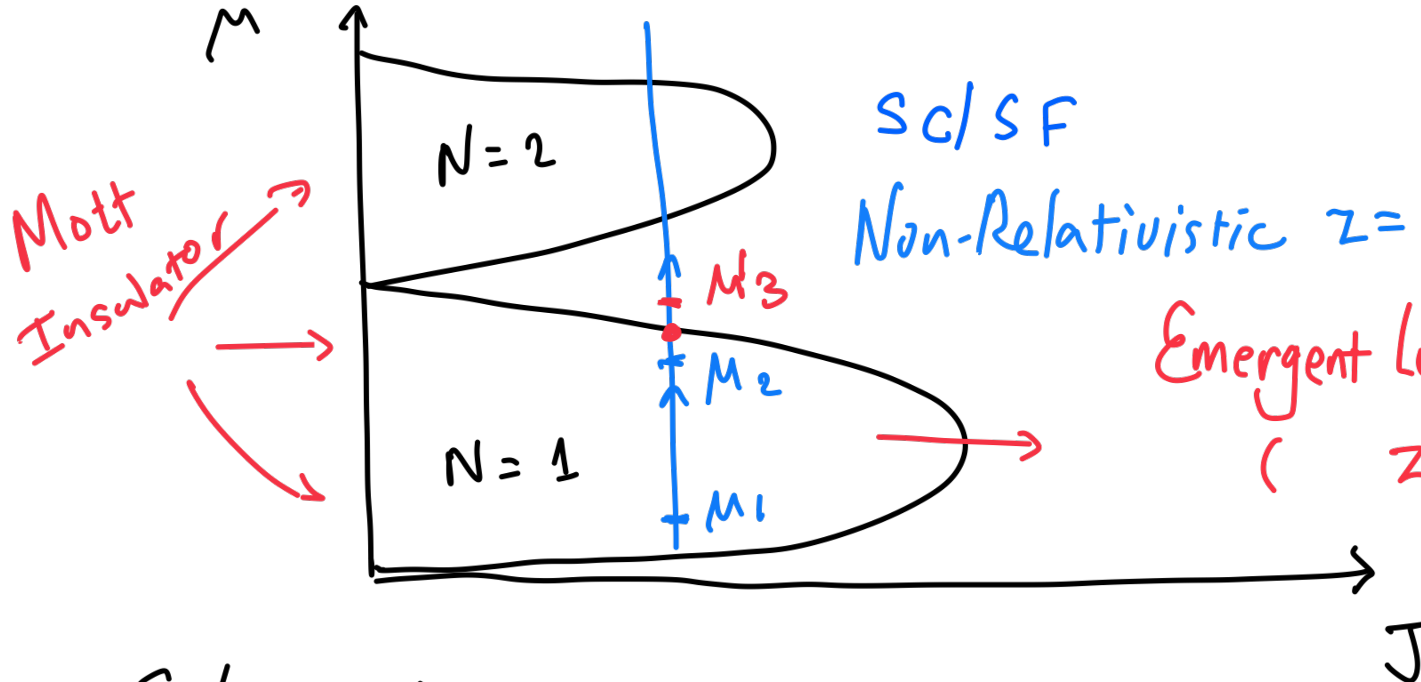
Quantum Model II : $[b_i, b_j] = 0$, $[b_i, b_j^\dagger] = \delta_{ij}$

$$\mathcal{H}_{BH} = \sum_i \frac{\hat{N}_i^2}{2c} - \mu \hat{N}_i + J \sum_{\langle ij \rangle} b_i^\dagger b_j + h.c.$$

$(\hat{N}_i = b_i^\dagger b_i)$

Bose-Hubbard Model





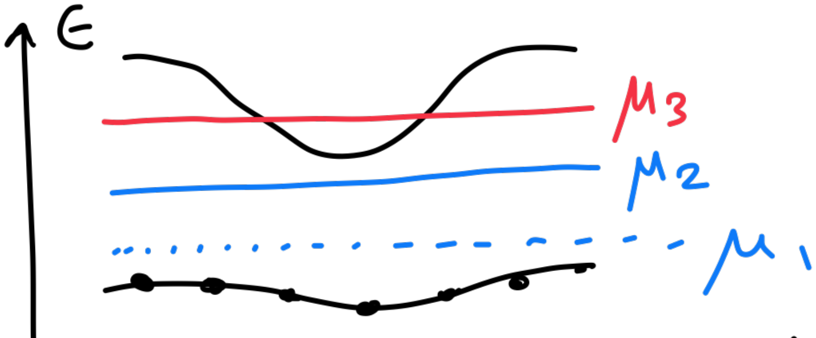
Sc/SF

Non-Relativistic $z=2$

Emergent Lorentz Symmetry

($z = 1$)

Schematics



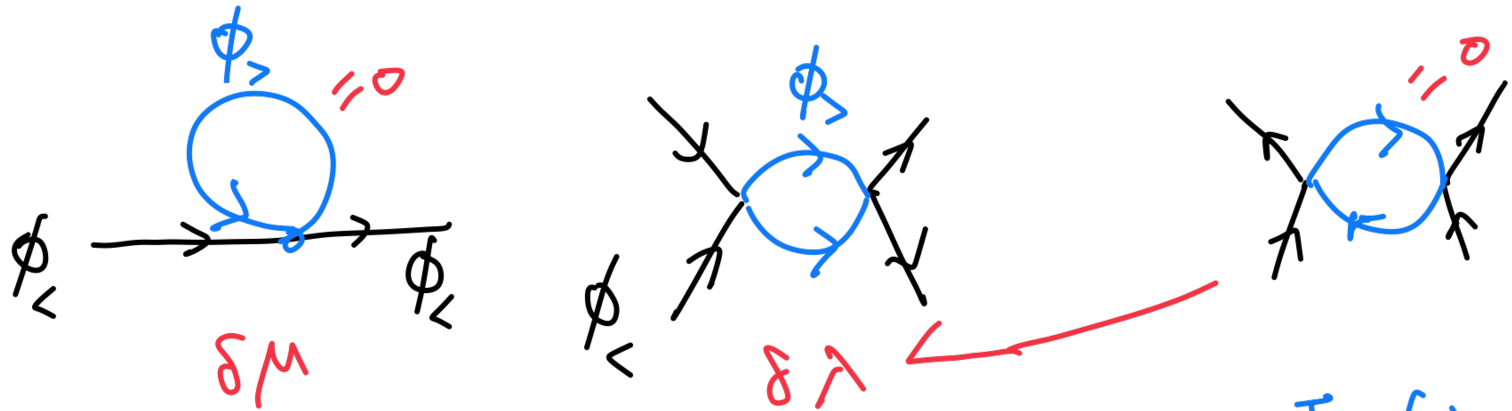
Lower Hubbard band

($N = 1$)

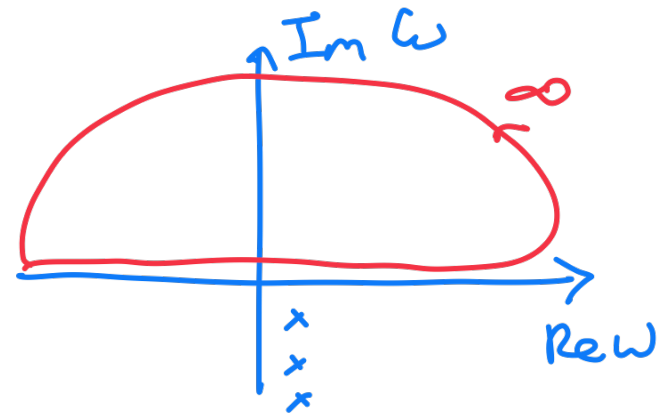
At $\mu = \mu_c$, particle-hole/charge conjugation C-Symmetry fully broken!

Non-Relativistic Model.

$$\mathcal{L} = \varphi^*\left(\partial_\tau - \frac{\nabla^2}{2}\right)\varphi + (\mu_c - \mu)\varphi^*\varphi + \lambda|\varphi|^4 + \dots$$



$$G(\omega, p) = \frac{1}{-i\omega + \varepsilon_p}$$



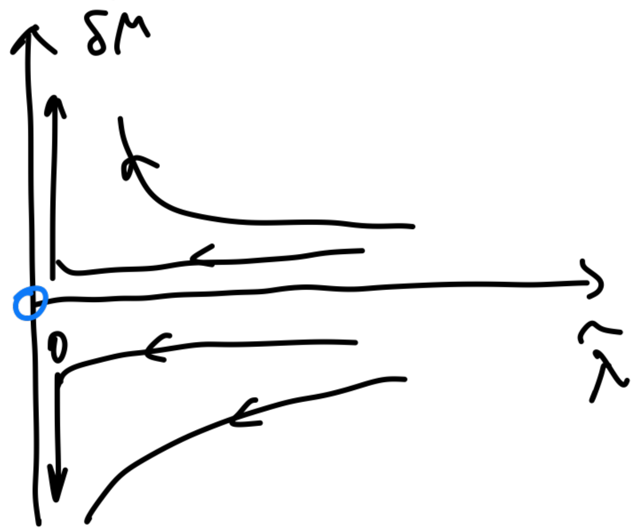
$$\langle vac|\, T\, b_p(t)\, b_p^\dagger(0)\, |vac\rangle = \theta(t)\dots + \theta(-t)\cdot 0$$

$$\mathcal{L} = \varphi^*\left(\partial_\tau - \frac{\nabla^2}{2}\right)\varphi + (\mu_c - \mu)\varphi^*\varphi + \lambda|\varphi|^4 + \dots$$

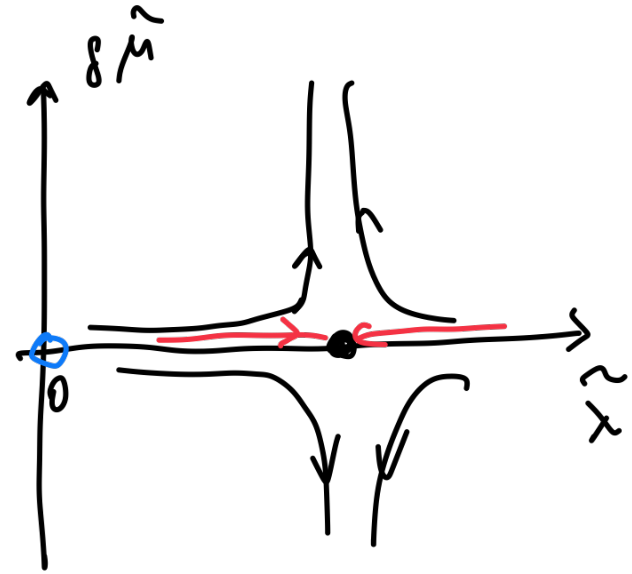
$$\frac{d\delta\tilde{\mu}}{dt} = -2\,\delta\tilde{\mu} + 0, \qquad \frac{d\tilde{\lambda}}{dt} = (d-2)\tilde{\lambda} + \tilde{\lambda}^2 C_d$$

$$\left(\delta\tilde{\mu} = (\mu_c - \mu)\Lambda^{-2}, \quad \tilde{\lambda} = \lambda\Lambda^{d-2}\right)$$

Free Boson Fixed pt



$d \geq 2$



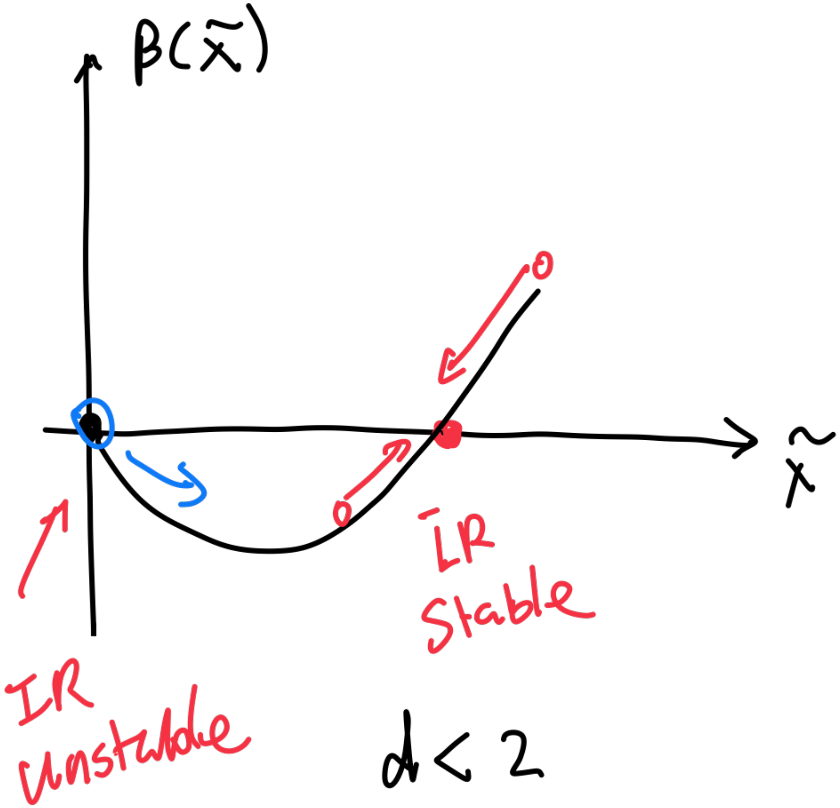
$d < 2$

$d<2$

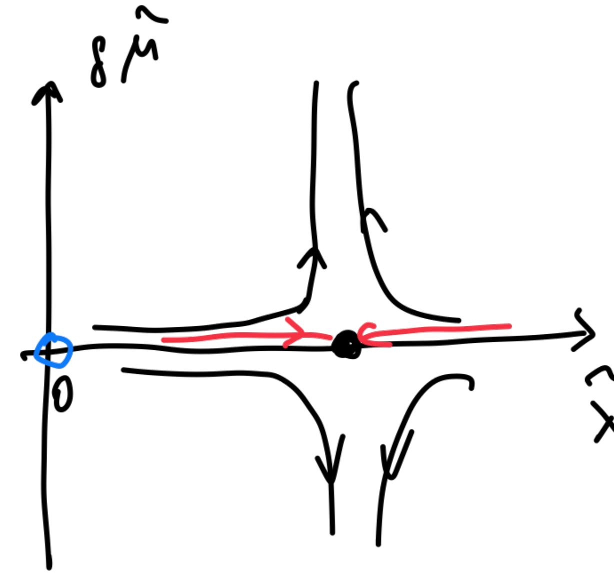
$$\mathcal{L} = \varphi^* \left(\partial_\tau - \frac{\nabla^2}{2} \right) \varphi + (\mu_c - \mu) \varphi^* \varphi + \lambda |\varphi|^4 + \dots$$

$$\frac{d\,\delta\tilde{\mu}}{dt} = -2\,\delta\tilde{\mu} + 0,$$

$$\frac{d\tilde{\lambda}}{dt} = \underbrace{(d-2)\tilde{\lambda} + \tilde{\lambda}^2 C_d}_{\beta(\tilde{\lambda})}$$



$d<2$



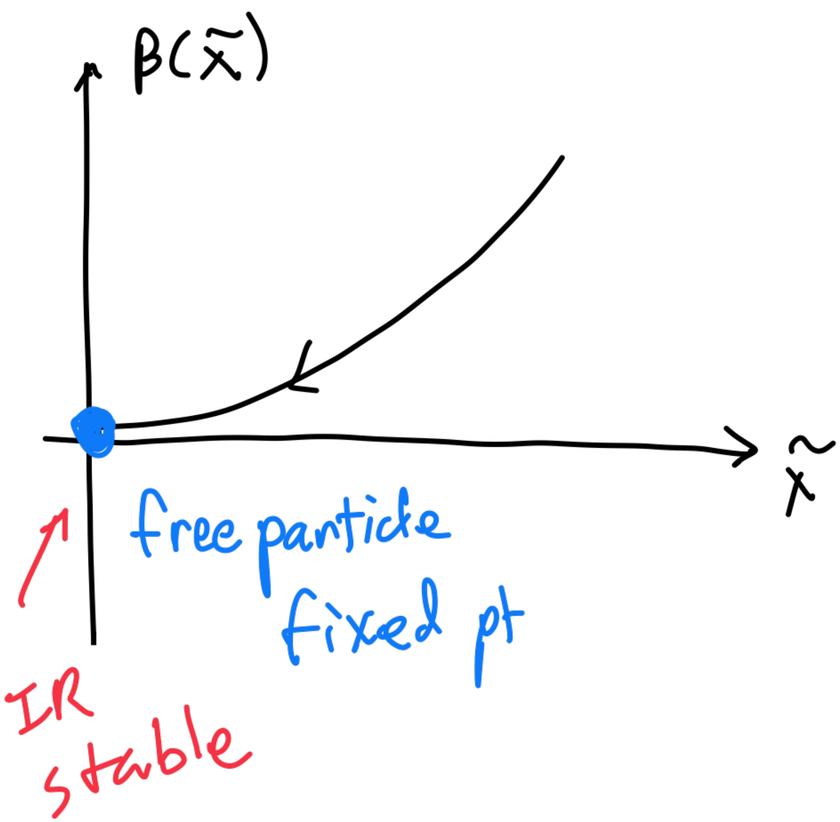
$d<2$

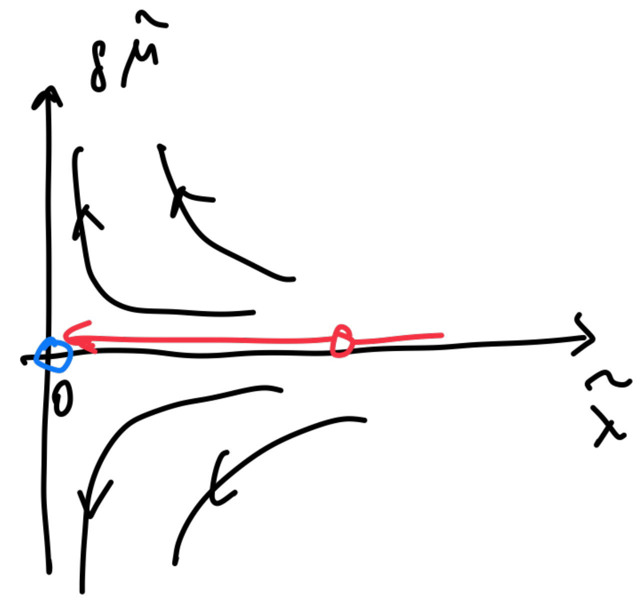
$d > 2$

$$\mathcal{L} = \varphi^*\left(\partial_\tau - \frac{\nabla^2}{2}\right)\varphi + (\mu_c - \mu)\varphi^*\varphi + \lambda|\varphi|^4 + \dots$$

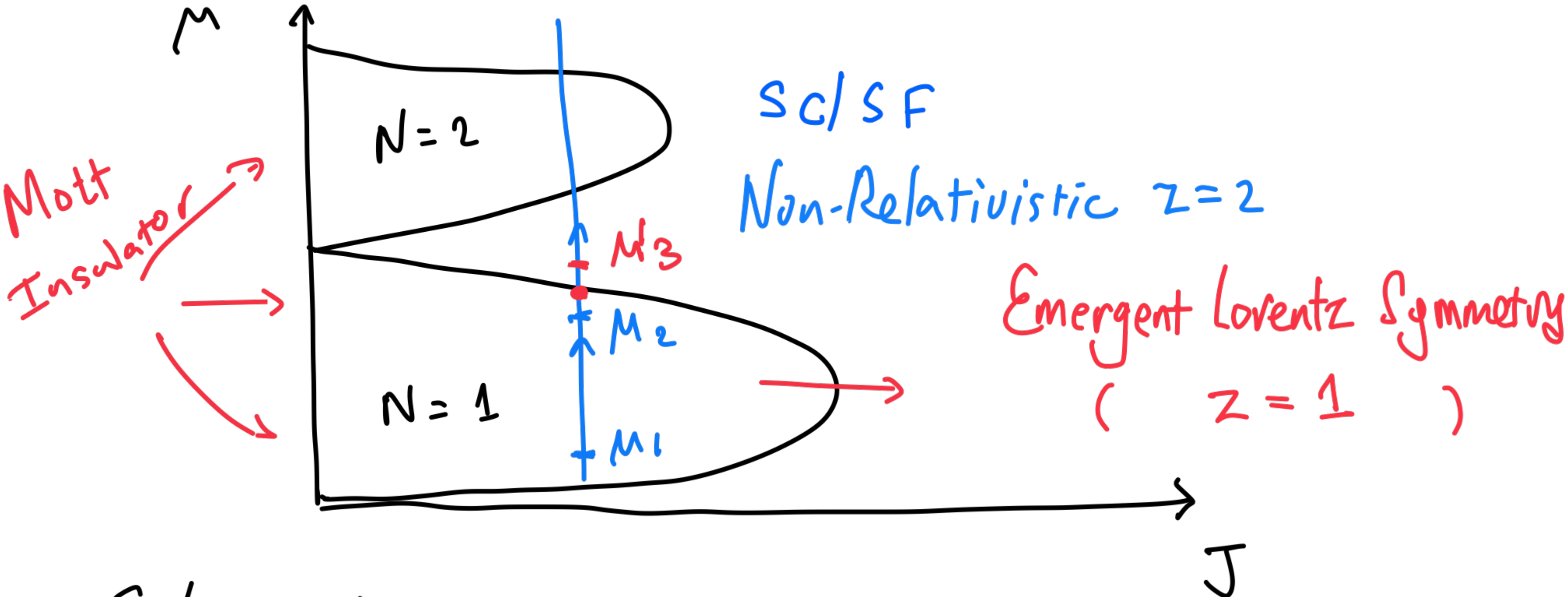
$$\frac{d\delta\tilde{\mu}}{dt} = -2\,\delta\tilde{\mu} + 0,$$

$$\frac{d\tilde{\lambda}}{dt} = \underbrace{(d-2)\tilde{\lambda} + \tilde{\lambda}^2 C_d}_{\beta(\tilde{\lambda})}$$

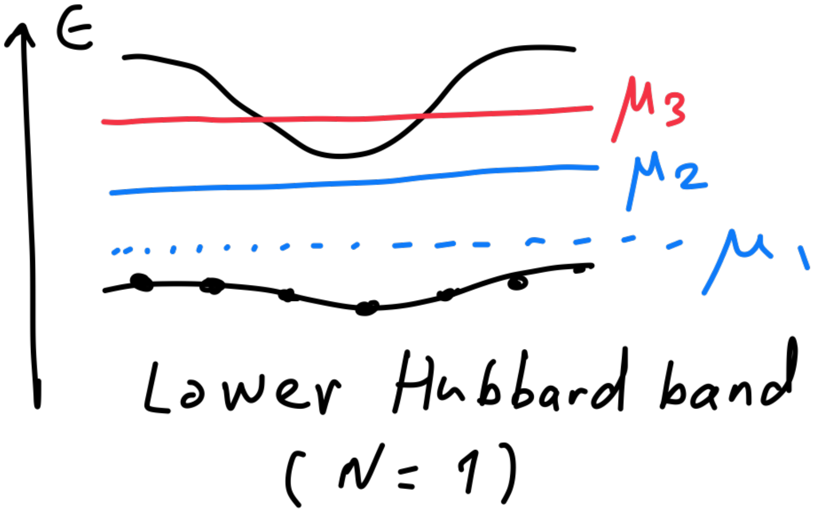




$d < 2$



Schematics



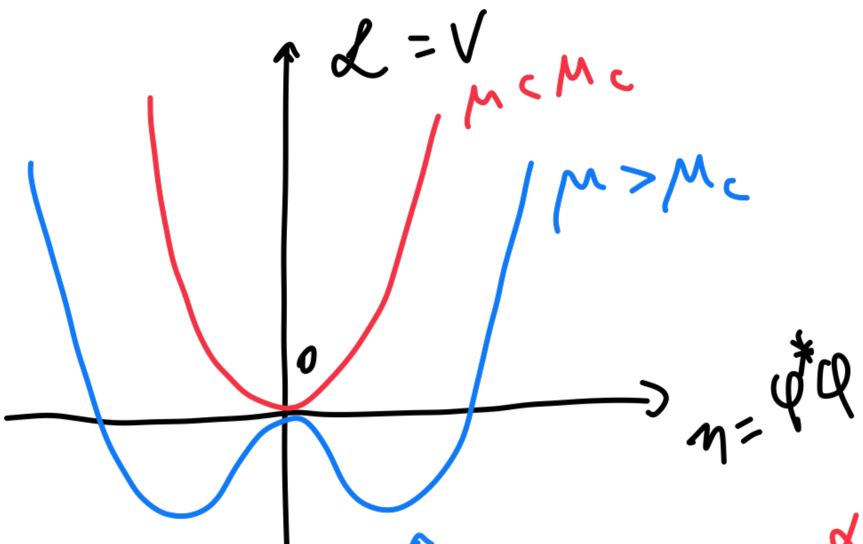
$n(\mu-\mu_c) = ?$

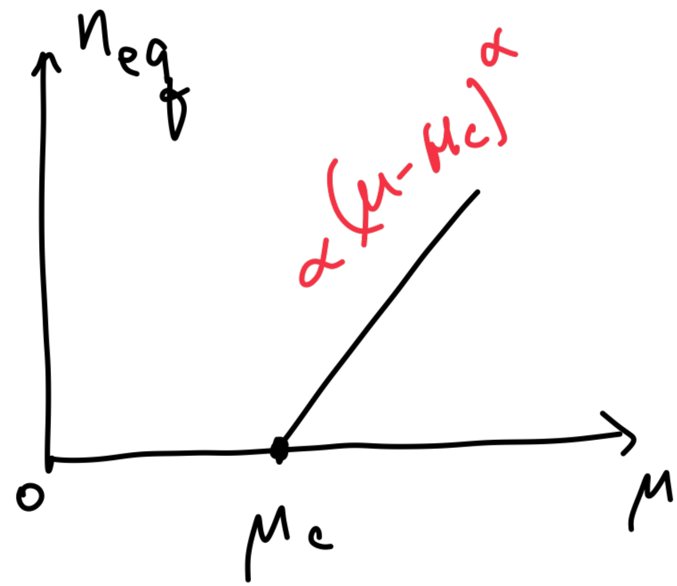
density

$$\mathcal{L} = \varphi^*\left(\partial_\tau - \frac{\nabla^2}{2}\right)\varphi + (\mu_c - \mu)\varphi^*\varphi + \lambda|\varphi|^4 + \dots$$

$d \geq 2$, free boson fixed pt / weakly interacting.

$$\mathcal{L} \simeq (\mu_c - \mu)\, n + \overset{>0}{\lambda}\, n^2,$$

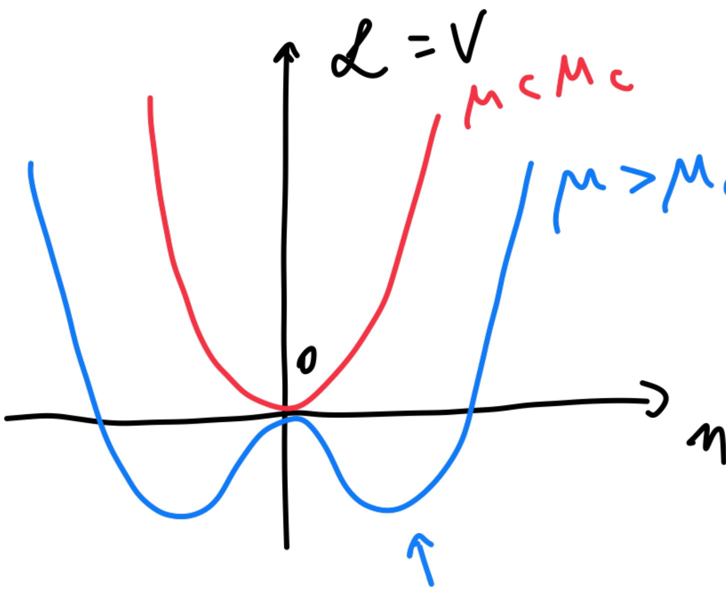


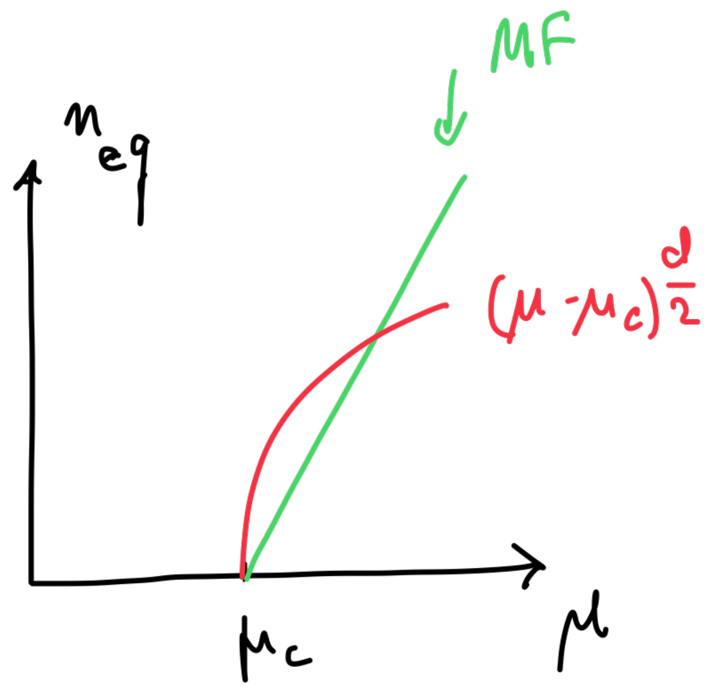


$$n_{eq} = \frac{(\mu - \mu_c)^\alpha}{2\lambda}, \quad \alpha = 1$$

$$\mathcal{L} = \varphi^* \left(\partial_\tau - \frac{\nabla^2}{2} \right) \varphi + (\mu_c - \mu)\, \varphi^* \varphi + \lambda |\varphi|^4 + \ldots$$

$d < 2$, strongly interacting fixed pt;



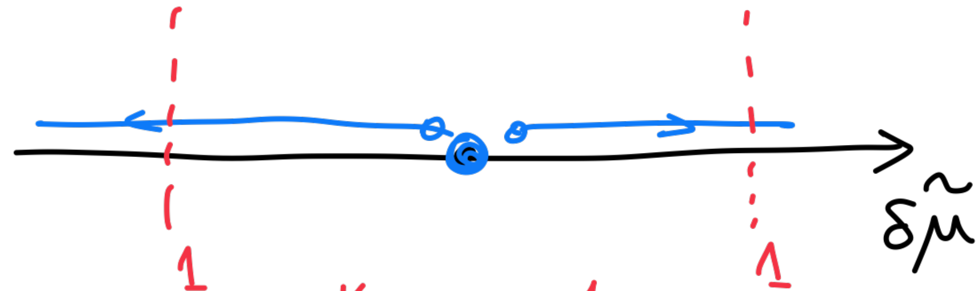


$n_{eq} \sim (\mu - \mu_c)^\alpha$, $\alpha = \frac{d}{2}$

($d=1$, $\alpha = \frac{1}{2}$ instead of 1)

Heuristically,

$$\frac{d\tilde{\delta\mu}}{dt} = -2\,\tilde{\delta\mu}$$



$$\tilde{\delta\mu}(L = \xi_c) \simeq 1, \qquad \xi_c = \delta\mu^{-1/2} = \frac{1}{(\mu-\mu_c)^{1/2}}$$

$$n_{eq} \sim \xi_c^{-d} \sim (\mu-\mu_c)^{d/2}$$ if and only if strong coupling fixed pt.

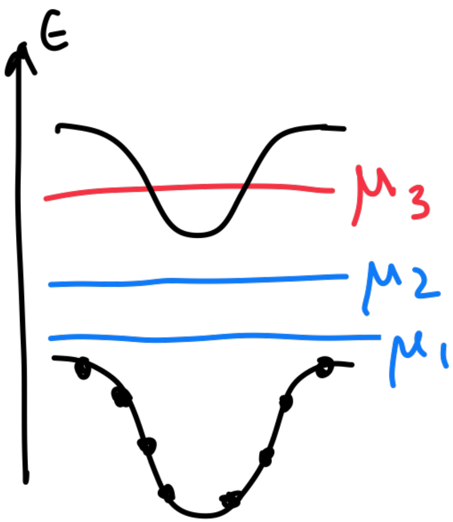
(This not valid for weak coupling fixed pt.)

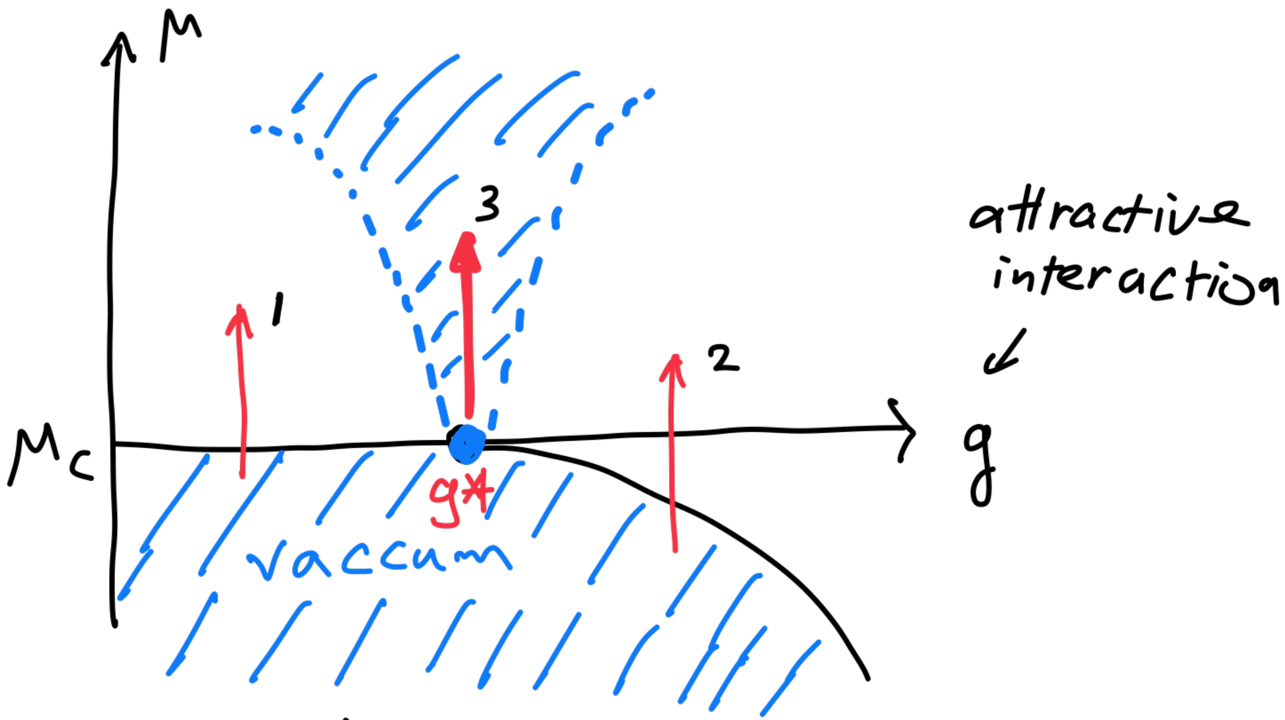
$$1D, \quad n_{eq} \sim (\mu-\mu_c)^{1/2}$$

Consistent with CFT and Bethe-Ansatz solutions.

Related $z=2$

General fermionic Model (3d)





1 : free - fermion Universality class

2 : free - boson Universality class

3 : SO(2, 1) Conformal field Theory Class

Applications in quantum magnetism, quantum gases and ion trap

Magnetically stabilized nematic order: Three-dimensional bipartite optical lattices
F. Zhou, M. Snoek, J. Wiemer, and I. Affleck, Phys. Rev. B 70, 184434 (2004)

Magnetically stabilized nematic order. II. Critical states and algebraically ordered nematic spin liquids in one-dimensional optical lattices, Hui Zhai and Fei Zhou, Phys. Rev. B 72, 014422 (2005)

Universal Bose gases near resonance: A rigorous solution,Shao-Jian Jiang, Wu-Ming Liu, Gordon W. Semenoff, and Fei Zhou, Phys. Rev. A 89, 033614(2014).

Long-lived universal resonant Bose gases, Shao-Jian Jiang, Jeff Maki, and Fei Zhou
Phys. Rev. A 93, 043605 (2016).

Tricritical Physics in Two-Dimensional p-Wave Superfluids, Fan Yang, Shao-Jian Jiang, and Fei Zhou, Phys. Rev. Lett. 124, 225701 (2020).

Achieving continuously tunable critical exponents for long-range spin systems simulated with trapped ions, Fan Yang, Shao-Jian Jiang, and Fei Zhou, Phys. Rev. A 99, 012119(2019).

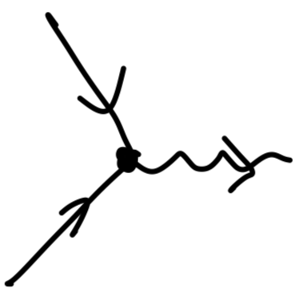
$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_b + \mathcal{L}_{bf}$$

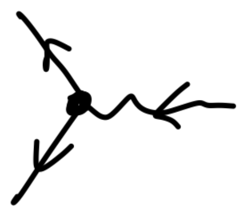
$$\mathcal{L}_f = \chi^*_\alpha \partial_\tau \chi_\alpha - \chi^*_\alpha \frac{\nabla^2}{2} \chi_\alpha + \mu_f \chi^*_\alpha \chi_\alpha \qquad \alpha = \uparrow, \downarrow$$

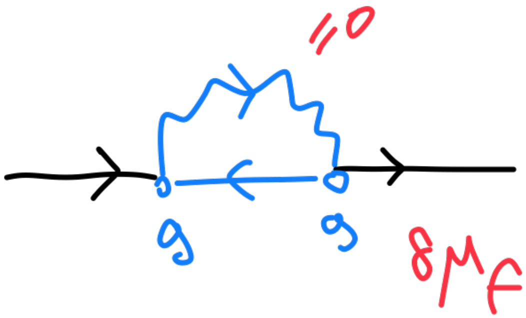
$$\mathcal{L}_b = \phi^* \partial_\tau \phi - \phi^* \frac{\nabla^2}{4} \phi + (\epsilon_b \underbrace{- 2\mu_f}_{-\mu_b}) \phi^* \phi$$

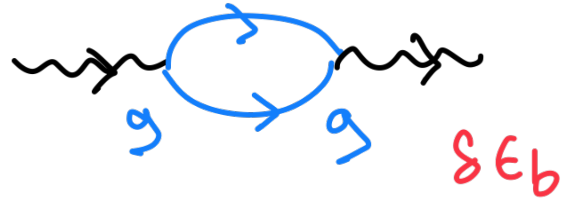
$$\mathcal{L}_{bF} = g\, \chi^*_\alpha\, \sigma_{y,\alpha\beta}\, \chi^*_\beta\, \phi + c.c.$$

Anti-Symmetric

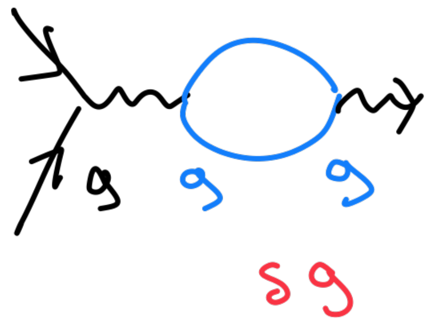




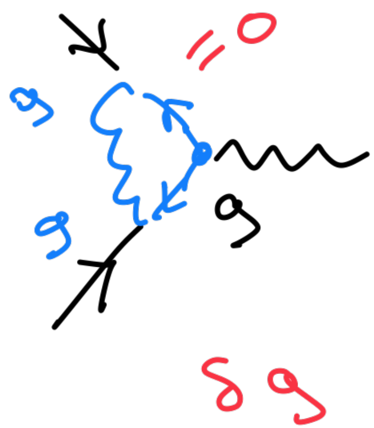

$=0$
g
g
δM_F


g
g
$\delta \epsilon_b$


g


g
g
g
δg


$=0$
g
g
g
δg


$=0$
g
g
g
δg

RGE

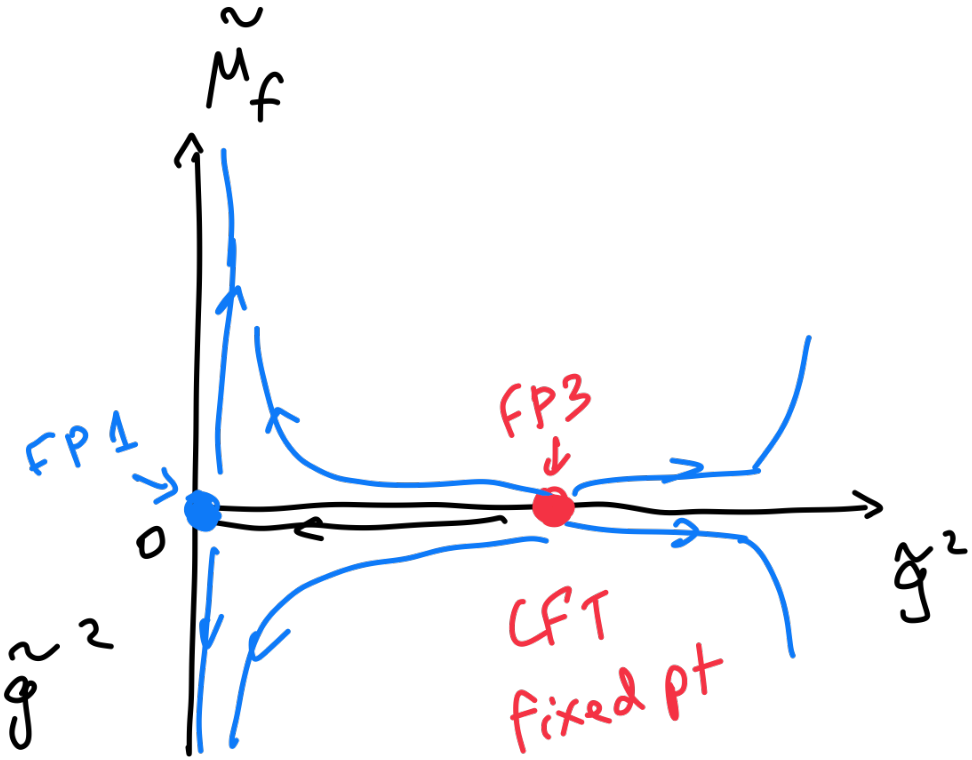
$$\frac{d\tilde{\mu}_f}{dt} = -2\,\tilde{\mu}_f$$

$$\frac{d\tilde{g}^2}{dt} = (d-4)\,\tilde{g}^2 + \tilde{g}^4$$

$$\frac{d\tilde{\mu}_b}{dt} = -2\,\tilde{\varepsilon}_b + 2\,\tilde{\mu}_f + \tilde{g}^2$$

$$\frac{dZ_b^{-1}}{dt} = -\,\tilde{g}^2$$

$$\left(\frac{d\tilde{\varepsilon}_b}{dt} = -2\,\tilde{\varepsilon}_b + \tilde{g}^2\right)$$



fixed pt 1: $\tilde{\mu}_f = \tilde{g}^2 = 0,\ Z_b = 1$

fixed pt 3: $\tilde{\mu}_f = \tilde{\mu}_b = Z_b = 0$

$\tilde{g}^2 = 4-d$

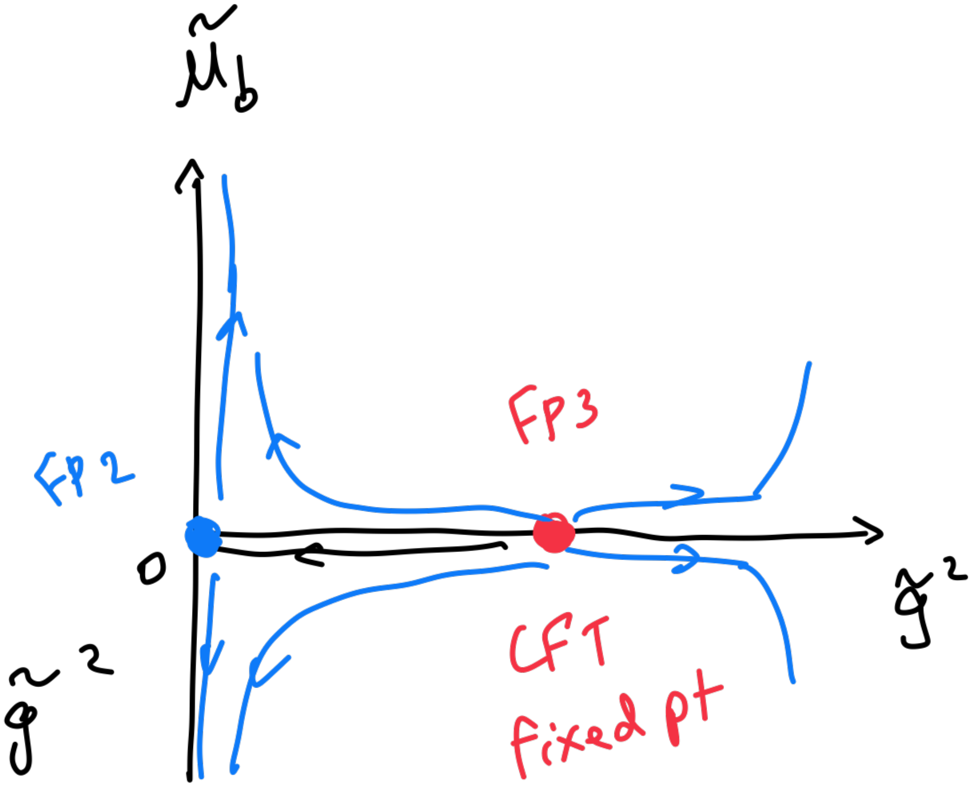
RGE

$$\frac{d\tilde{\mu}_f}{dt} = -2\,\tilde{\mu}_f$$

$$\frac{d\tilde{g}^2}{dt} = (d-4)\,\tilde{g}^2 + \tilde{g}^4$$

$$\frac{d\tilde{\mu}_b}{dt} = -2\,\tilde{\varepsilon}_b + 4\,\tilde{\mu}_f + \tilde{g}^2$$

$$\frac{dZ_b^{-1}}{dt} = -\,\tilde{g}^2$$



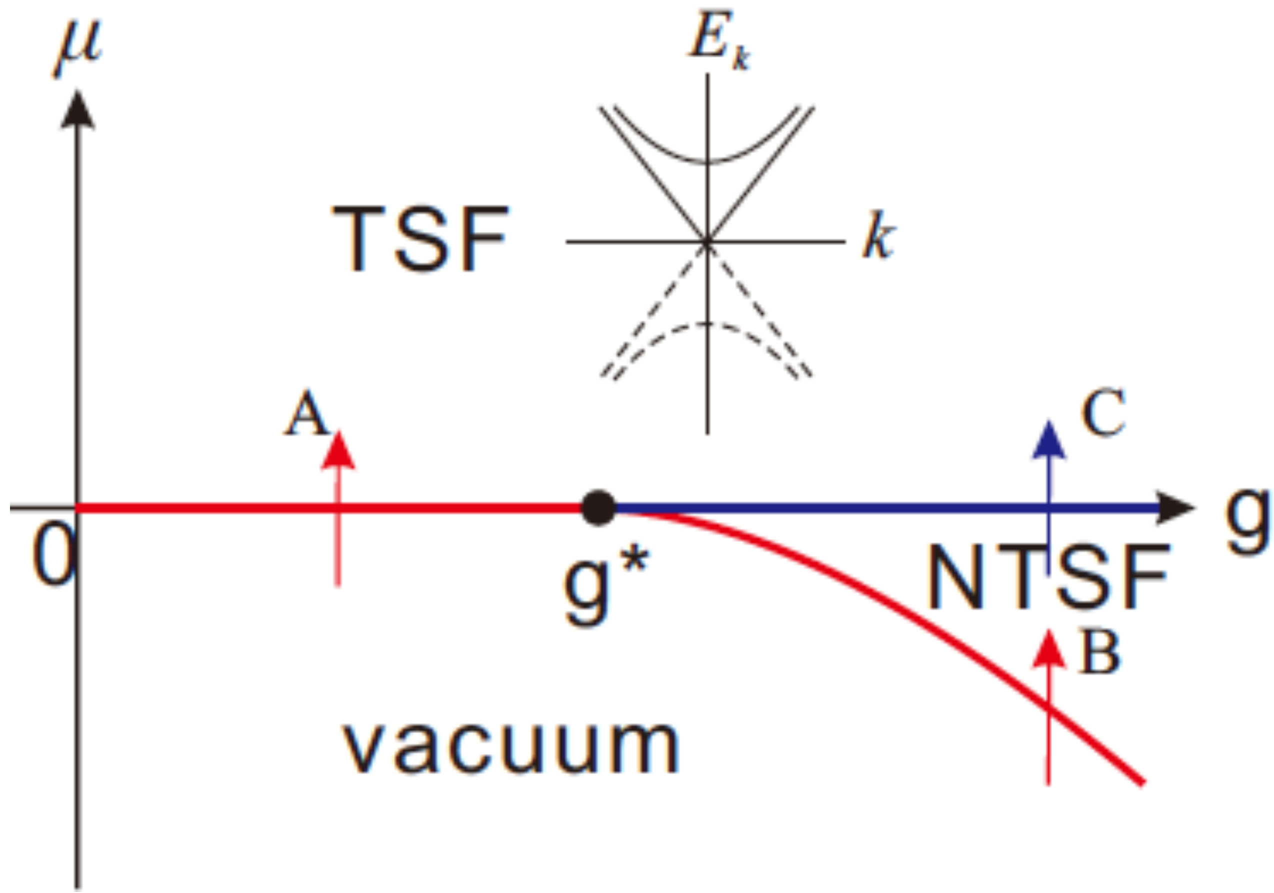
fixed pt 2: $\tilde{\mu}_b = \tilde{g}^2 = 0$

$(\; 2\tilde{\mu}_f - \tilde{\varepsilon}_b = 0 \;)$

fixed pt 3: $\tilde{\mu}_f = \tilde{\mu}_b = Z_b = 0$

$\tilde{g}^2 = 4-d$

Example: Phase diagram of p+ip spinless SF



A: Free Fermion; B: Wilson-Fisher/Free boson; C: Majorana class
g*: QCP of SO(2,1) CFT.

Yang, Jiang and FZ, 2019