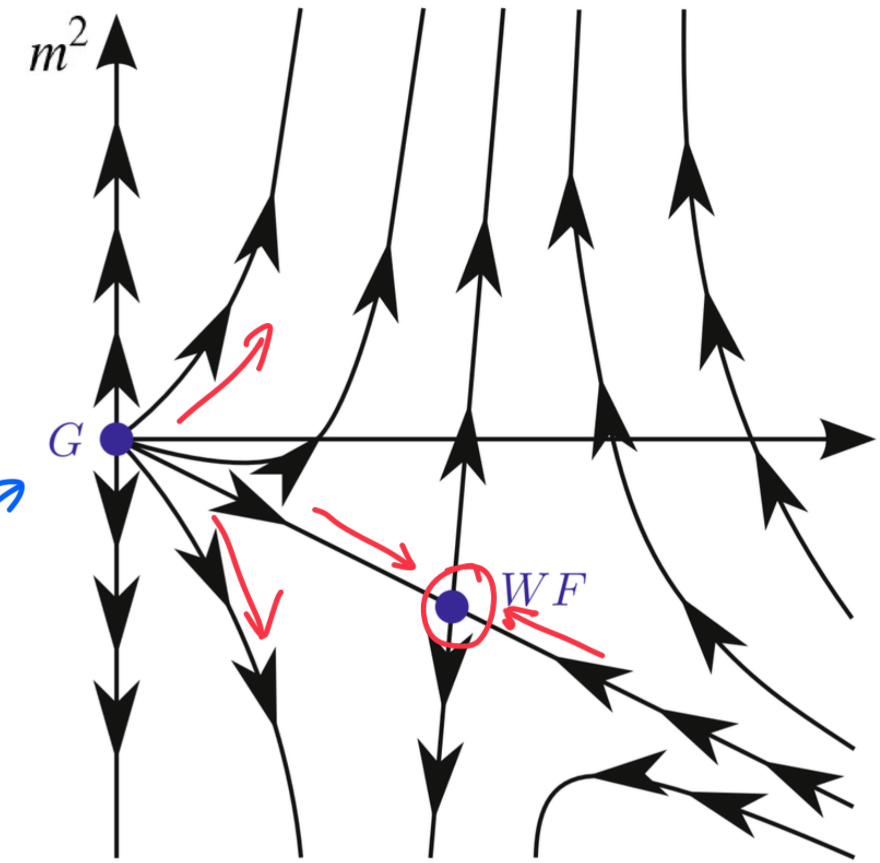
Phys525:
Quantum Condensed Matter Physics: Quantum Criticality
Basics, Dynamics and Topological criticality

Renormalization group equations and simple applications to BHM

RG flows (i.e. scale transformation) of a scalar model : Scale Symmetric Wilson-Fisher F.P. (d+1<4)

$d < 3$



mass

$\lambda = g$ interaction

$g\phi^4$

MF fixed pt

IR unstable

WF fixed pt ($d<3$)

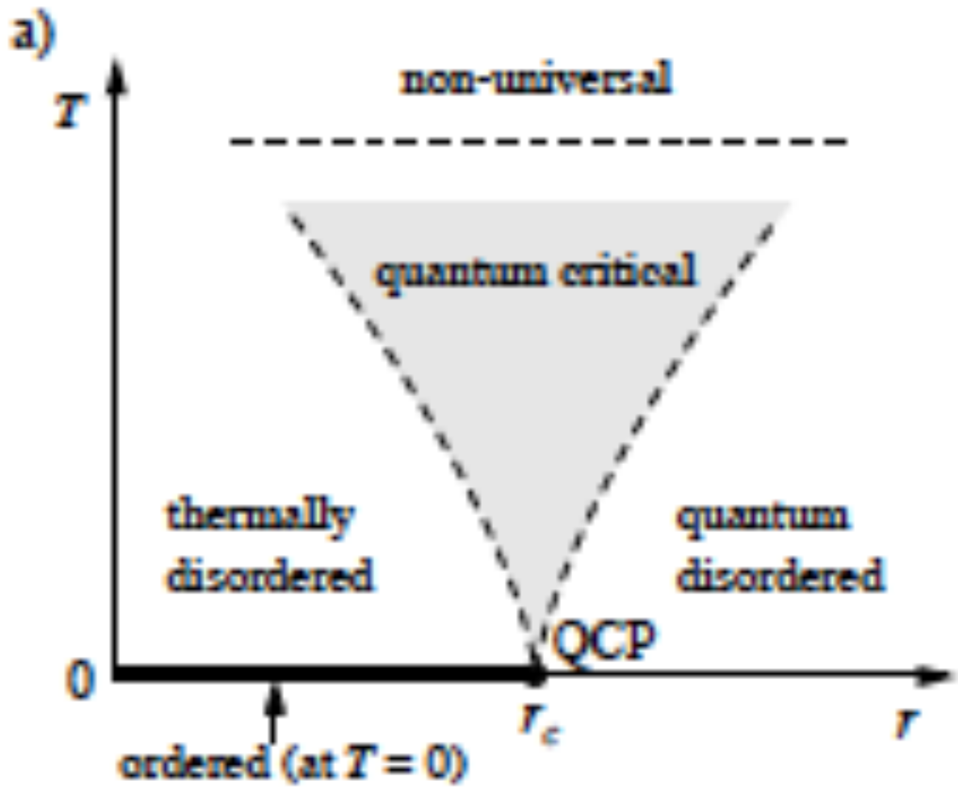
IR Stable

Scale invariance and Phase Transitions

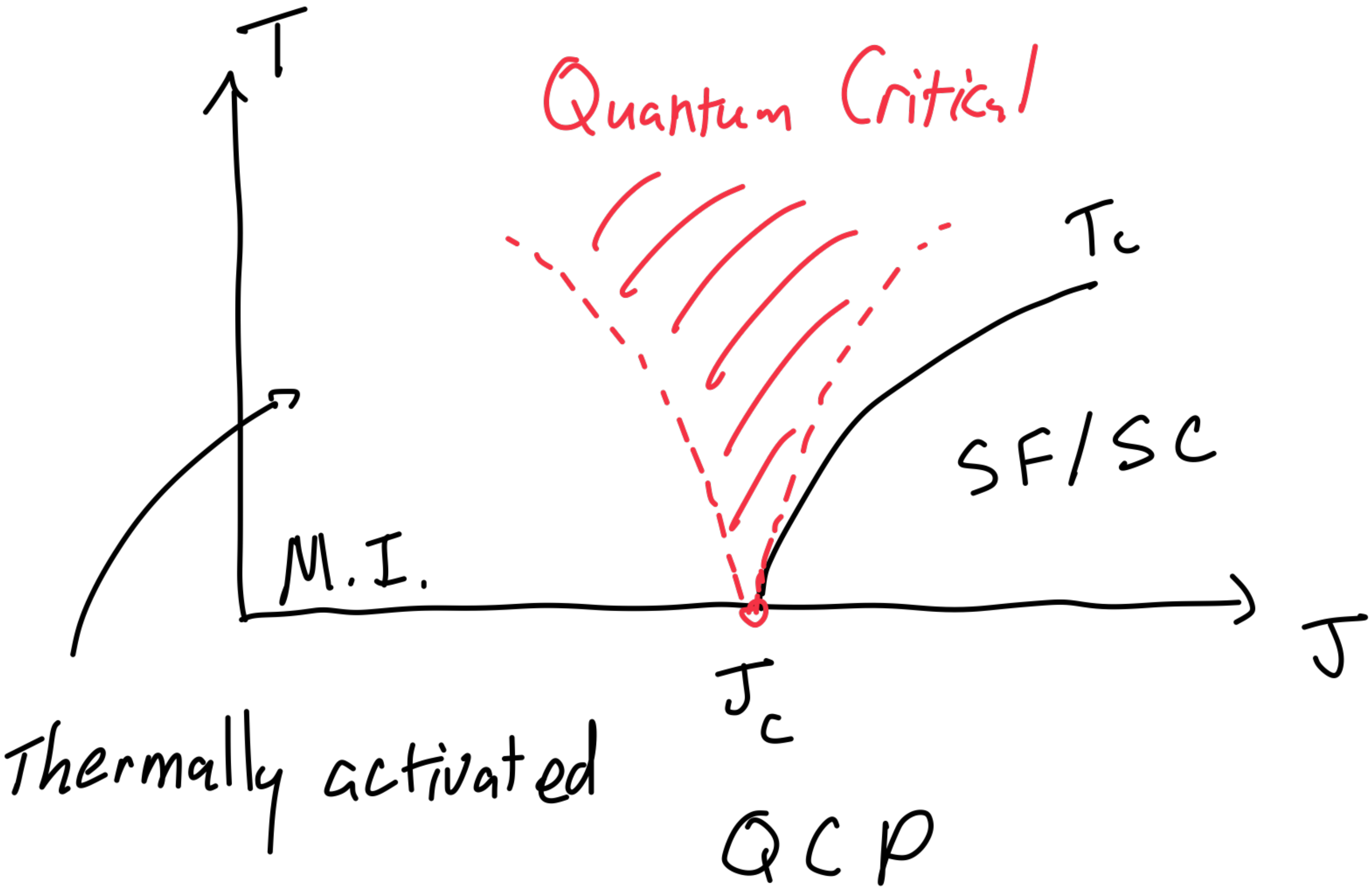
1) Effective QFT or EFT constructed out of symmetry and other general considerations.

1) Critical point is identified as a scale invariant QFT (can be CFT if z=1, 2) or a fixed point Hamiltonian under scale transformation.

3) Microscopic information (which appears in EFT) wiped out in the course of renormalization leading to universalities.



Standard paradigm for QPT/QCP
("T" can also be a general variable "X")



T
Quantum Critical
T_c
SF/SC
M.I.
J
J_c
Thermally activated
QCP

Scale transformation via RGE

$$\frac{d\tilde{m}}{dt} = \beta_m(\tilde{m}, \tilde{g}), \quad \frac{d\tilde{g}}{dt} = \beta_g(\tilde{m}, \tilde{g}) \quad \ldots$$

Two approaches pursuited

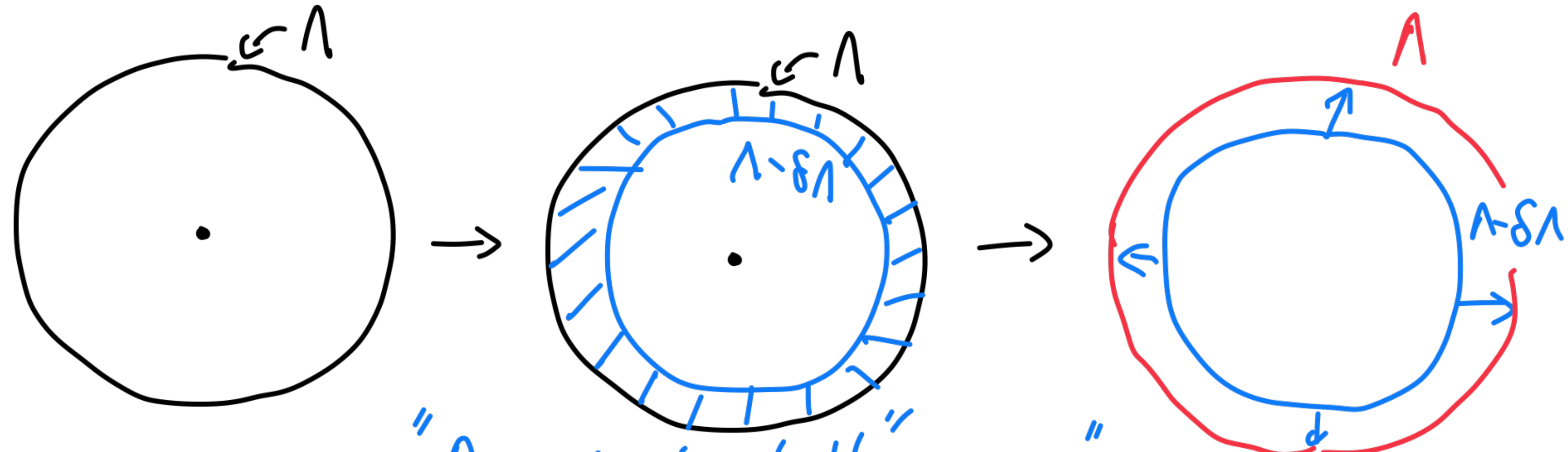
(A) Real Space RGE (for specific lattices)

(B) QFT based approach (General)

(B1) Moment Shell Approach (Wilsonian)

(B2) Callan-Symanzik Green's function (General)

Momentum Shell Approach (d+1) dimension



"Remove the shell"
$[\Lambda - \delta\Lambda, \Lambda]$

"Reboost"

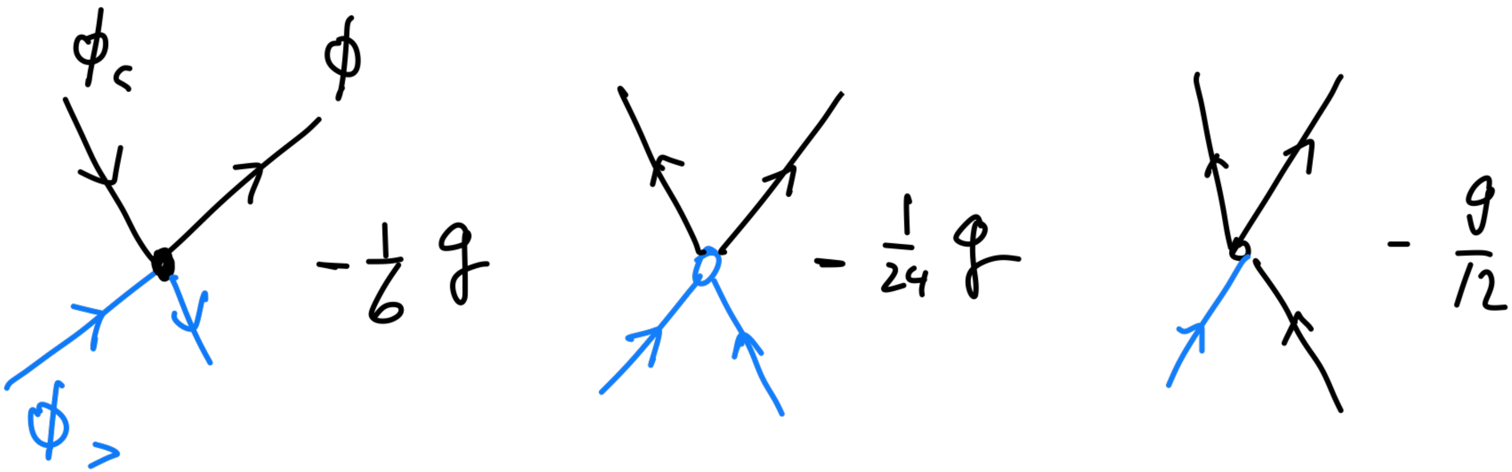
$$\phi = \phi_> + \phi_<$$

$$\mathcal{L}(\{\phi\}) = \mathcal{L}_<(\{\phi_<\}) + \mathcal{L}_>(\{\phi_>\}) + \mathcal{L}_{><}(\{\phi_>\}, \{\phi_<\})$$

$$\mathcal{L}_< \simeq |\partial_\tau \phi_<|^2 + |\nabla \phi_<|^2 + \frac{m^2}{2!}|\phi_<|^2 + \frac{g}{4!}|\phi_<|^4 \dots$$

$$\mathcal{L}_> \simeq |\partial_\tau \phi_>|^2 + |\nabla \phi_>|^2 + \frac{m^2}{2!}|\phi_>|^2 + \frac{g}{4!}|\phi_>|^4 \dots$$

$$\mathcal{L}_{><} = \frac{g}{24}\,\phi_>^{*2}\,\phi_<^2 + \frac{g}{6}(\phi_>^*\phi_>)(\phi_<^*\phi_<) + \frac{g}{12}\,\phi_>^*\phi_<\,\phi_<^*\phi_< + \text{c.c.}$$



$$Z = \int D\phi_<\, e^{-\int \mathcal{L}_<} \underbrace{\int D\phi_>\, e^{-\int \mathcal{L}_> + \mathcal{L}_{><}}}_{e^{-\int \delta \mathcal{L}_<}}$$

$$Z = \int D\phi_< \, e^{-\int \mathcal{L}_<} \left\{ \int D\phi_> \, e^{-\int \mathcal{L}_> + \mathcal{L}_{><}} \right\} \to e^{-\int \delta\mathcal{L}_<}$$

$$e^{-\int \delta\mathcal{L}_<} = \frac{\int D\phi_> \, e^{-\int \mathcal{L}_>} \, e^{-\int \mathcal{L}_{><}}}{\int D\phi_> \, e^{-\int \mathcal{L}_>}} \cdot \int D\phi_> \, e^{-\int \mathcal{L}_>}$$

$$= \left\langle e^{-\int \mathcal{L}_{><}} \right\rangle_{Z_>} \cdot Z_>$$

$$e^{-\int \delta\mathcal{L}_<} = \left\langle e^{-\int \mathcal{L}_{><}} \right\rangle_{Z_>} \cdots\cdots$$

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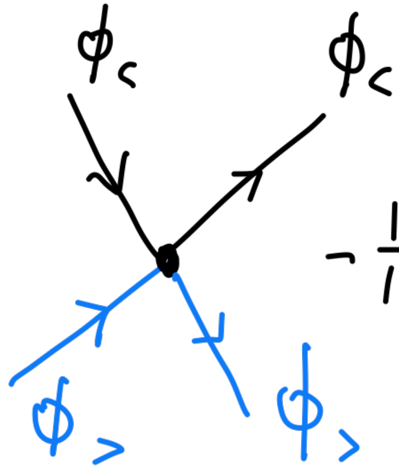
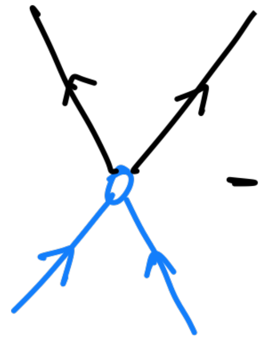
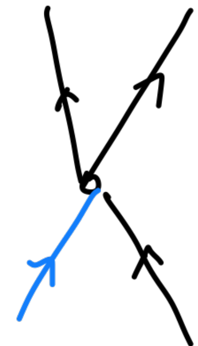


$-\frac{1}{12} g$



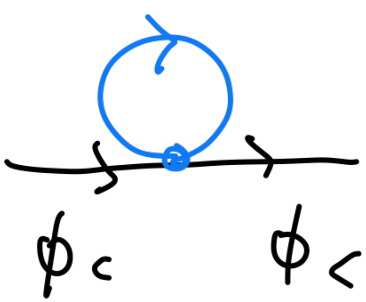
$-\frac{1}{24} g$



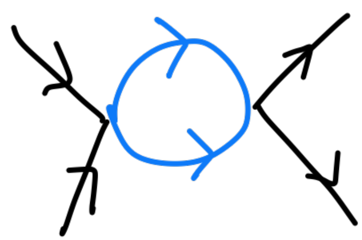
$-\frac{g}{12}$

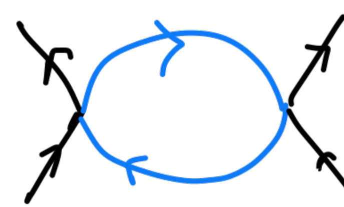
$\langle \phi^*_>(k)\phi_>(k)\rangle_{Z_>} = G(\omega, k)$, $\quad \langle \phi_>(k)\phi_>(k)\rangle_{Z_>} = 0$

$\langle\ \rangle_{Z_>}$:



δm^2



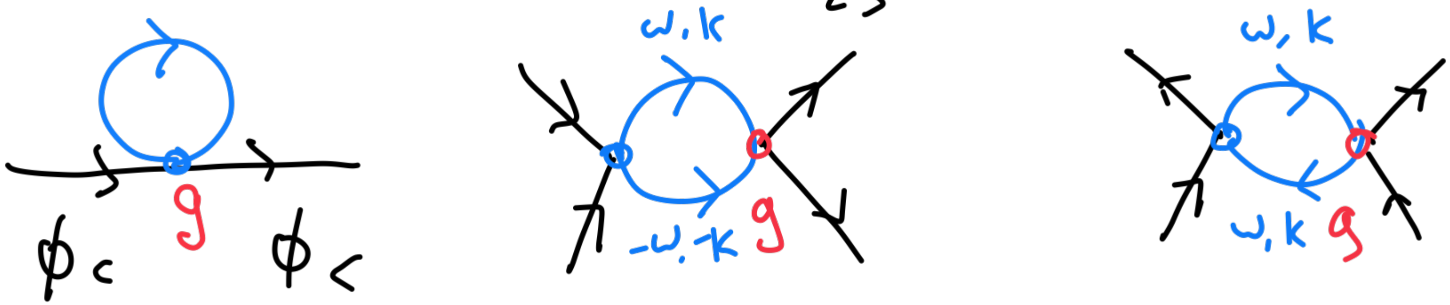


δg

$G = \frac{1}{\omega^2 + p^2 + m^2}$

$$e^{-\delta \mathcal{L}_<} = \left\langle e^{-\int \mathcal{L}_{><}} \right\rangle_{Z_>} \cdots\cdots$$

$$G(\omega, k) = \frac{1}{\omega^2 + k^2 + m^2}$$

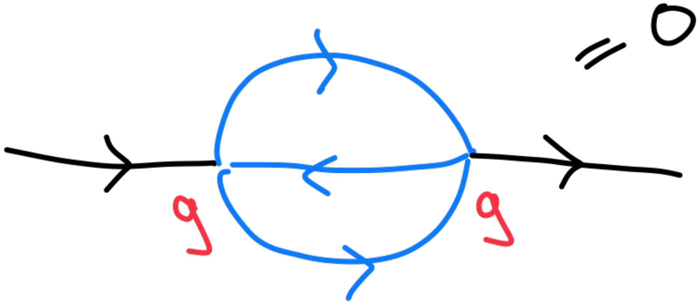


$$\delta m^2 = (-1)\left(-\frac{g}{6}\right) \int_{\Lambda - \delta\Lambda}^{\Lambda} G(\omega, k) \frac{d\omega}{2\pi} \frac{d^{(d)}k}{(2\pi)^d} + \cdots$$

$$\delta g = (-1)\, g^2 \cdot C \cdot \int_{\Lambda - \delta\Lambda}^{\Lambda} G(\omega, k)\, G(-\omega, -k) \frac{d\omega}{2\pi} \frac{d^{(d)}k}{(2\pi)^d} + \cdots$$

perturbative RGE

$O(g^2)$ Limitations of Momentum shell approach



$= 0$

Vanishes in the Momentum shell approach,

but contributes to "Z" or anomalous dimension of the "φ" field ; $\varphi(\vec{x}', t') = e^{-t\eta} \varphi(\vec{x}' e^{t}, t' e^{t})$

(η depends on "Z".)

Can be treated in Callan - Symanzik approach.

(See General Reference II)

$$\delta m^2 = (-1)\left(-\frac{g}{6}\right)\int_{\Lambda-\delta\Lambda}^{\Lambda} G(\omega,k)\,\frac{d\omega}{2\pi}\frac{d^{(d)}k}{(2\pi)^d}+\cdots$$

$$\delta g = (-1)\; g^2 \cdot C \cdot \int_{\Lambda-\delta\Lambda}^{\Lambda} G(\omega,k)\,G(-\omega,-k)\,\frac{d\omega}{2\pi}\frac{d^{(d)}k}{(2\pi)^d}+\cdots$$

$$\begin{cases} \dfrac{d\tilde{m}^2}{dt} = -2\tilde{m}^2 - a_d\tilde{g} + b_d\tilde{m}^2\tilde{g}, & \tilde{m}^2 = m^2\Lambda^{-2} \\[2ex] \dfrac{d\tilde{g}}{dt} = (d-3)\tilde{g} + c_d\tilde{g}^2, & \tilde{g} = g\Lambda^{d-3}\ (>0) \end{cases}$$

("$Z=1$" - No field Renormalization.)

$$\begin{cases} \dfrac{d\tilde{m}^2}{dt} = -2\tilde{m}^2 - a_d\tilde{g} + b_d\tilde{m}^2\tilde{g}, & \tilde{m}^2 = m^2\Lambda^{-2} \\ \dfrac{d\tilde{g}}{dt} = (d-3)\tilde{g} + c_d\tilde{g}^2, & \tilde{g} = g\Lambda^{d-3} \ (>0) \end{cases}$$

$d>3$: $\tilde{g}^* = 0 = \tilde{m}^*$, MF fixed point.

$d<3$: $\tilde{g}^* = \dfrac{3-d}{c_d} = \dfrac{\epsilon}{c_d}$, $\quad 3-d=\epsilon$

dimension Regularization

$$\tilde{m}^2 = -\frac{a_d}{2}\cdot\frac{\epsilon}{c_d}$$

(HW Set III)

$$
\begin{cases}
\dfrac{d\tilde{m}^2}{dt} = -2\tilde{m}^2 - a_d\,\tilde{g} + b_d\,\tilde{m}^2\tilde{g}, \\[2ex]
\dfrac{d\tilde{g}}{dt} = (d-3)\,\tilde{g} + c_d\,\tilde{g}^2,
\end{cases}
$$

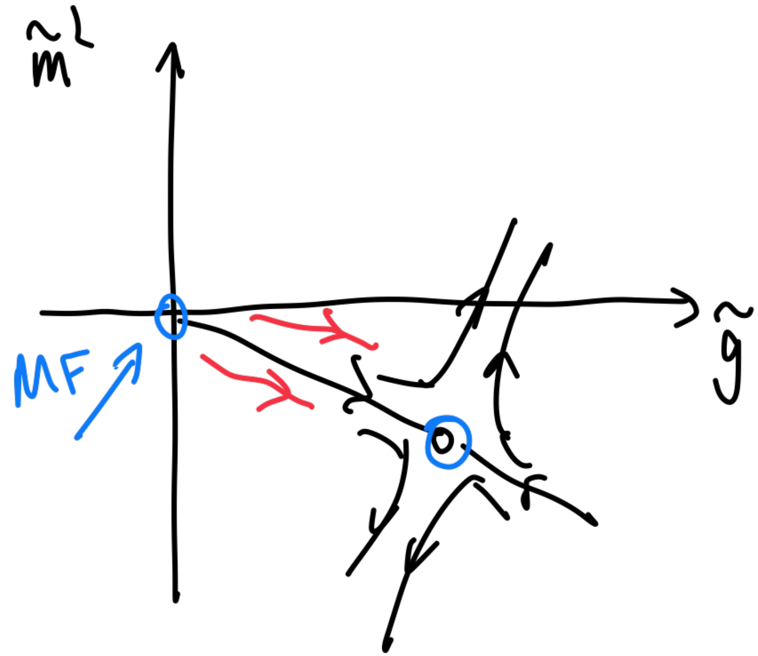
$d < 3$

$-\tilde{m}^{2*} \sim \tilde{g}^* \sim \epsilon$

Near WF fixed pt

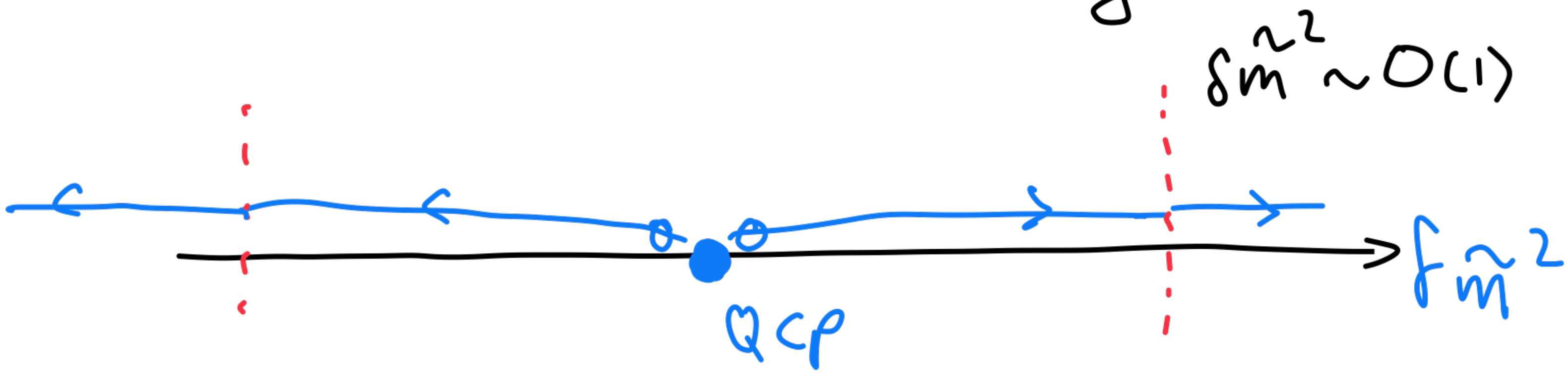
$$
\frac{d}{dt}\begin{bmatrix} \delta\tilde{m}^2 \\ \delta\tilde{g} \end{bmatrix} = \begin{bmatrix} -2+e_d\epsilon, & -a_d \\ 0 & \epsilon \end{bmatrix}\begin{bmatrix} \delta\tilde{m}^2 \\ \delta\tilde{g} \end{bmatrix}
$$

$(a_d > 0,\ b_d > 0,\ c_d > 0)$



WF fixed pt

RGE Vs Correlation length



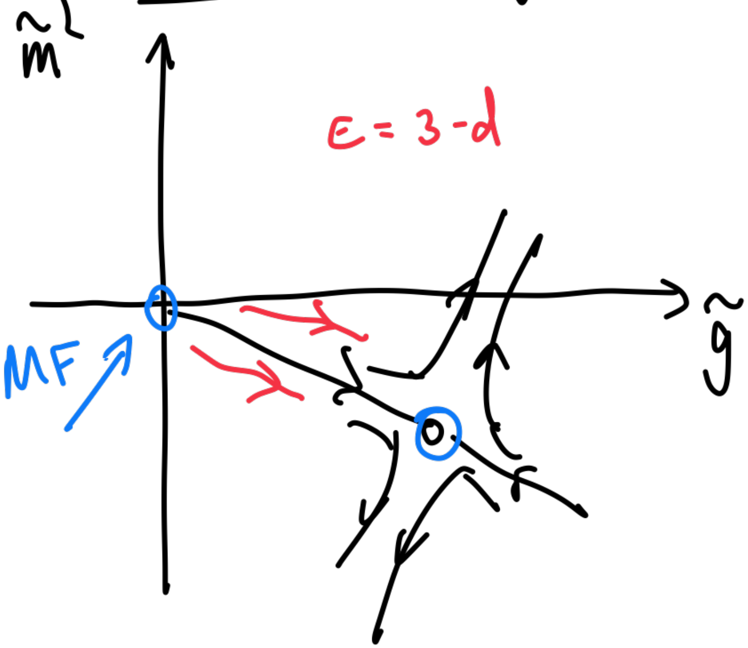
$\delta\tilde{m}^2 = 0 \longrightarrow$ QCP

$\delta\tilde{m}^2 \ll 1 \longrightarrow$ Quantum Critical Regime – QCR

$\delta\tilde{m}^2 \sim 1 \longrightarrow$ No longer QCR

$$\delta\tilde{m}^2(L = \xi_c) \simeq 1$$

Correlation length



$$\frac{\delta\tilde{m}^2(L)}{\delta\tilde{m}(\Lambda_{UV})} \sim L^{(2-e_d\epsilon)}.$$

$$\xi \, |J-J_c|$$

$$\xi_c \sim |\delta\tilde{m}^2|^{-1/2} \leftarrow \text{MF}$$

$|\delta\tilde{m}^2| \sim |J-J_c|$ for BHM

Near WF fixed pt

$$\frac{d}{d\ell}\begin{bmatrix}\delta\tilde{m}^2\\ \delta\tilde{g}\end{bmatrix} = \begin{bmatrix}-2+e_d\epsilon, & -a_d\\ 0 & \epsilon\end{bmatrix}\begin{bmatrix}\delta\tilde{m}^2\\ \delta\tilde{g}\end{bmatrix}$$

$$\xi_c \sim |J-J_c|^{-\frac{1}{2-e_d\epsilon}}$$

$$\epsilon \to 0, \quad \xi_c \sim |J-J_c|^{-\frac{1}{2}}$$