Phys525:
Quantum Condensed Matter Physics: Quantum Criticality
Basics, Dynamics and Topological criticality

Episode Nine:
introduction to scale symmetry at QCPs and emergent light-cone/horizon in dynamics

Slightly formally speaking, if $\quad a \to \lambda a$ $\quad$ (d+1) D space-time

A) $\vec{x} \longrightarrow \vec{y} = \dfrac{\vec{x}}{\lambda}$ or $\vec{x} = \lambda \vec{y}$ $\quad \in R^{d+1}$

$$\psi(\vec{x}) \longrightarrow \psi'(\vec{y}) = \lambda^{\eta} \psi(\vec{x} = \lambda \vec{y}),$$

B) $G'(\vec{y}', \vec{y}) = \langle \psi'(\vec{y}') \psi'(\vec{y}) \rangle$

$$= G(\vec{y}', \vec{y})$$

$$= \langle \psi(\vec{y}') \psi(\vec{y}) \rangle$$

then

The system has a scale symmetry

$$\psi(\vec{x}) \longrightarrow \psi'(\vec{y}) = \lambda^{\eta}\,\psi(\vec{x} = \lambda\vec{y})$$

$$G'(\vec{y}', \vec{y}) = \langle \psi'(\vec{y}')\, \psi'(\vec{y}) \rangle$$

$$= G(\vec{y}', \vec{y})$$

Example: A QCP ($\xi_c \to \infty$), $G(\vec{y}', \vec{y}) = \dfrac{1}{|\vec{y}' - \vec{y}|^{\beta}}$

$$G'(\vec{y}', \vec{y}) = \lambda^{2\eta} \langle \psi(\lambda\vec{y}')\, \psi(\lambda\vec{y}) \rangle$$

$$= \lambda^{2\eta} \frac{1}{\lambda^{\beta}\, |\vec{y}' - \vec{y}|^{\beta}} = G(\vec{y}', \vec{y})$$

if $2\eta = \beta$ ($\beta = d-1$ in MF)

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if $2\eta = \beta$ ($\beta = d-1$ in MF)

Counter Example: ξ_c = finite.

$$G'(\vec{y}', \vec{y}) = \frac{1}{|\vec{y}' - \vec{y}|^{\beta}} e^{-\frac{|\vec{y}' - \vec{y}|}{\underbrace{\xi_c \lambda^{-1}}_{\to \xi_c'}}}$$

$$\neq G(\vec{y}', \vec{y}; \xi_c)$$

What does Scale symmetry imply at QCPs ?

dynamical critical Exponent

Scale dimension of field "ψ"

Scale transformation :

$$\vec{r}\,' = \vec{r}\lambda, \quad \tau' = \tau\lambda^{z}, \quad \psi'(\vec{r}\,', \tau') = \lambda^{-\eta}\psi(\vec{r}, \tau)$$

(for MI-SF QCP at $J = J_c$, $z = 1$, $\eta = \frac{d-1}{2}$ MF fixed pt)

Scale invariance at QCPs

$$\boxed{\psi(\vec{r}, \tau) = \lambda^{\eta}\psi'(\vec{r}\lambda, \tau\lambda^{z})}$$

$$S(\{\psi(r, \tau)\}) \longrightarrow S'(\{\psi'(\vec{r}\,', \tau')\}) = S(\{\psi'(\vec{r}\,', \tau')\})$$

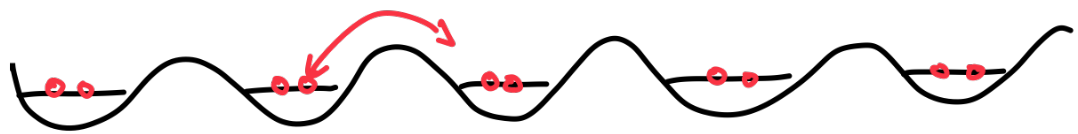
or $$\mathcal{H}(\{b^{+}_{\vec{r}}, b_{\vec{r}}\}) \longrightarrow \mathcal{H}'(\{b^{+\prime}_{\vec{r}\,'}, b'_{\vec{r}\,'}\}) = \lambda^{-z}\mathcal{H}(\{b^{+\prime}_{\vec{r}\,'}, b'_{\vec{r}\,'}\})$$

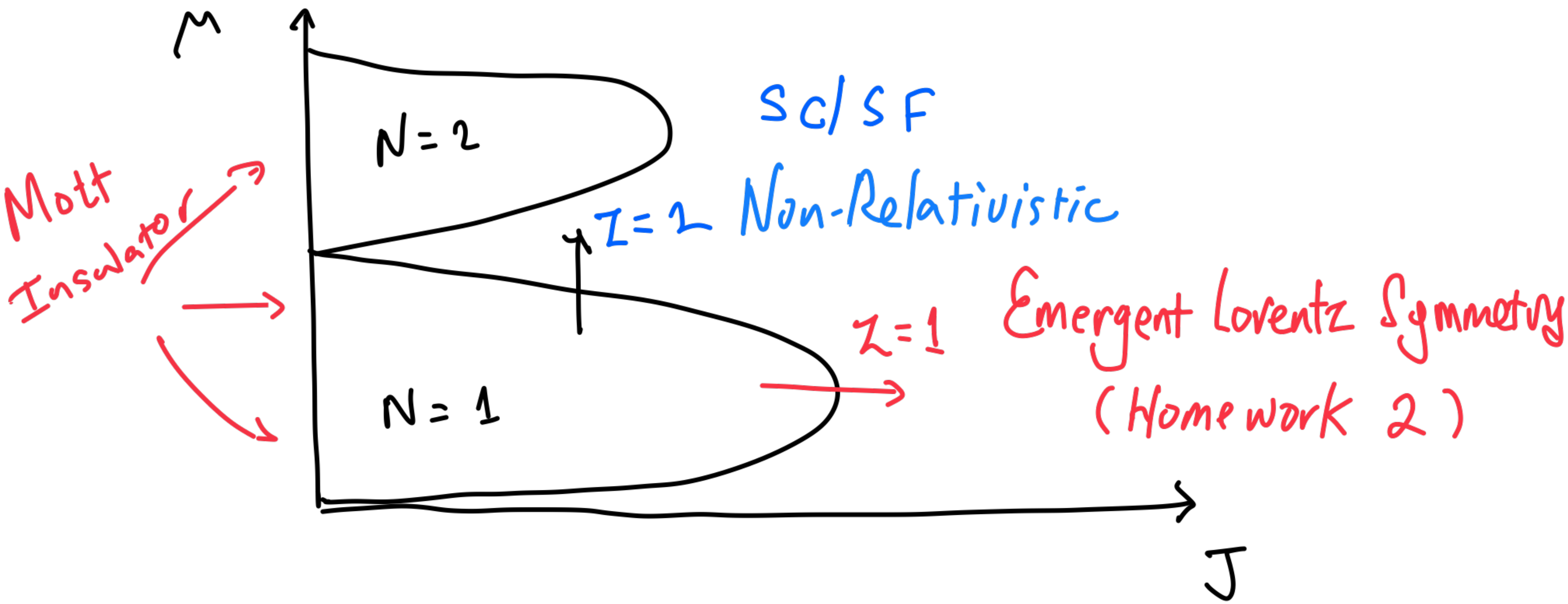
Quantum Model II : $[b_i, b_j]=0,\ [b_i, b_j^\dagger]=\delta_{ij}$

$$\mathcal{H}_{BH} = \sum_i \frac{\hat{N}_i^2}{2c} - \mu \hat{N}_i + J \sum_{\langle ij \rangle} b_i^\dagger b_j + h.c.$$

$$(\hat{N}_i = b_i^\dagger b_i)$$

Bose-Hubbard Model





$z=1$ $\qquad$ QCP $\qquad$ Example I.

$c=1$, $\qquad J_c \qquad \longrightarrow J$

$$\mathcal{L} = |\partial_\tau \varphi|^2 + |\nabla \varphi|^2 + m^2 |\varphi|^2 + \underbrace{\cdots}_{\text{irrelavent } d>3}$$

$$\mathcal{L}_{QCP} = \mathcal{L}(m=0)$$

$$\tau \to \tau' = \tau\lambda \qquad x \to x' = x\lambda, \qquad \varphi \to \varphi'(x') = \lambda^{-\frac{d-1}{2}} \varphi(x,\tau)$$

$$\mathcal{L} = \{|\partial_{\tau'} \varphi'|^2 + |\nabla_{\vec{r}'} \varphi'|^2\} \lambda^{+\frac{d-1}{2}\times 2 + 2} + m^2 |\varphi'|^2 \lambda^{+\frac{(d-1)}{2}\times 2}$$

$$S = \int d^{(d)}x \, d\tau \, \mathcal{L} \longrightarrow S' = \int d^{(d)}x' d\tau'$$

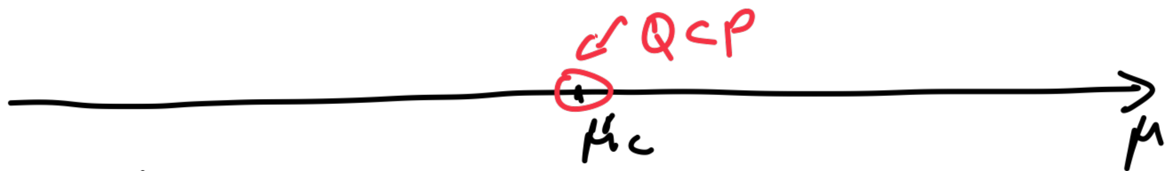
breaks scale symmetry

$$\{[|\partial_{\tau'} \varphi'|^2 + |\nabla_{\vec{r}'} \varphi'|^2] + \lambda^{-2} m^2 |\varphi'|^2\}$$

So that $\quad S \longrightarrow S' = S(\{\varphi'(\vec{r}',\tau')\})$ if $m=0$ or QCP!

Scale invariant.

$z=2$ Example II.



$$\mathcal{L} = \varphi^* \partial_\tau \varphi + |\nabla \varphi|^2 + \delta\mu\, |\varphi|^2 + \underbrace{\cdots}_{\text{irrelevant } d>2}$$

$$\mathcal{L}_{QCP} = \mathcal{L}(m=0)$$

$$\tau \to \tau' = \tau\lambda^2 \quad x \to x' = x\lambda, \quad \varphi \to \varphi'(x') = \lambda^{-\frac{d}{2}} \varphi(x,\tau)$$

$$\mathcal{L} = \{\varphi'^* \partial_{\tau'} \varphi' + |\nabla_{\vec{r}'} \varphi'|^2\} \lambda^{\frac{d}{2}\times 2 + 2} + \delta\mu\, |\varphi'|^2 \lambda^{\frac{d}{2}\times 2}$$

$$S = \int d^{(d)}x\, d\tau\, \mathcal{L} \longrightarrow S' = \int d^{(d)}x'\, d\tau' [\varphi'^* \partial_{\tau'} \varphi' + |\nabla_{\vec{r}'} \varphi'|^2] + \delta\mu\, \lambda^{-2} \cdots$$

So that $S \longrightarrow S' = S(\{\varphi'(\vec{r}', \tau')\})$ if $\delta\mu = 0$ or QCP!

Scale invariant.

- Towards time ordered Green's function **G**

$$G(r,\tau;0,0) = \langle g.s. | \mathcal{T}\, b_{\vec{r}}(\tau)\, b_0^{\dagger}(0) | g.s. \rangle$$

or

$$G(q,\tau;q,0) = \langle g.s. | \mathcal{T}\, b_q(\tau)\, b_q^{\dagger}(0) | g.s. \rangle$$

$$\overset{\tau>0}{=} \sum_{\vec{q}} \langle g.s. | \overbrace{b_{\vec{q}}(\tau)}^{e^{+H\tau} b_{\vec{q}}\, e^{-H\tau}} | \vec{q} \rangle \langle \vec{q} | b_{\vec{q}}^{\dagger}(0) | g.s. \rangle$$

$\vec{q}$ ← all Many-body eigen states with Momentum $\vec{q}$

$$= |\langle g.s. | b_{\vec{q}} | \vec{q}\rangle^{P} |\, \underbrace{e^{-E^{P}_{\vec{q}}\tau}}_{\text{"Particle-like M.B. States"}} + \underbrace{\cdots\cdots\cdots}_{\text{Continuum}}$$

"Multiparticles" M.B. States

Time ordered in Non-Relativistic Theory (free particles)

$|0\rangle = |vac\rangle$

$$\langle 0 | T\, b_{\vec r}(t)\, b_0^\dagger(0) | 0 \rangle \overset{t>0}{=} \sum_{\vec k} \langle 0 | \, b_{\vec r}(t) \underbrace{|\vec k\rangle\langle \vec k|}\, b_0^\dagger(0)|0\rangle$$

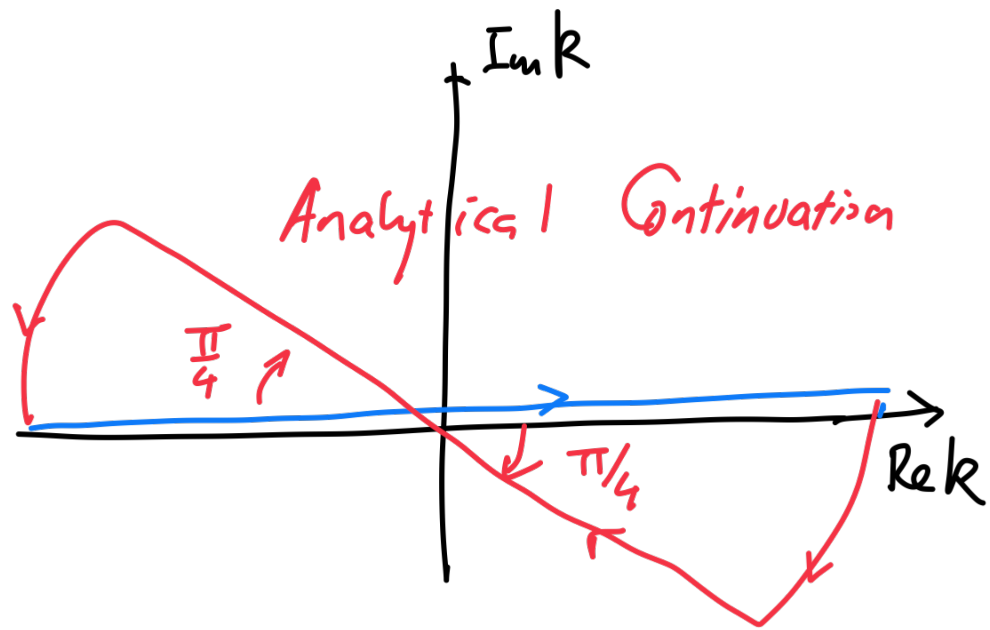
Single particle state

$(m=1)$

$$= \frac{1}{\Omega} \sum_{|\vec k|} e^{i\vec k\cdot(\vec r - 0)}\, e^{-i\varepsilon_k t}$$

vol

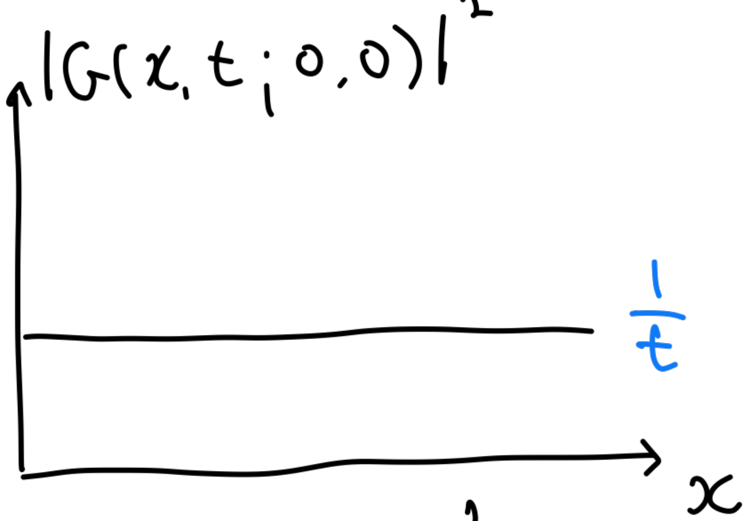
$$\underset{1D}{\hat=} \frac{\hbar^{-\frac{1}{2}}}{\sqrt{2\pi t i}}\, e^{i\frac{x^2}{2t}} \cdot \frac{1}{\hbar^{1/2}}$$

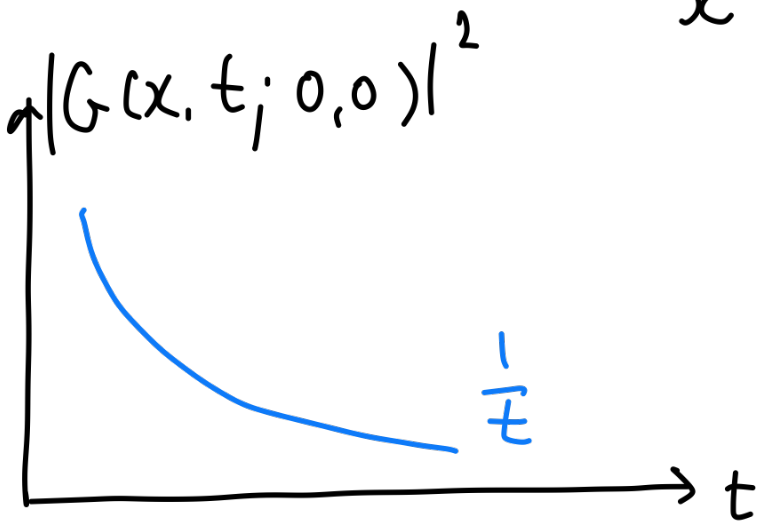


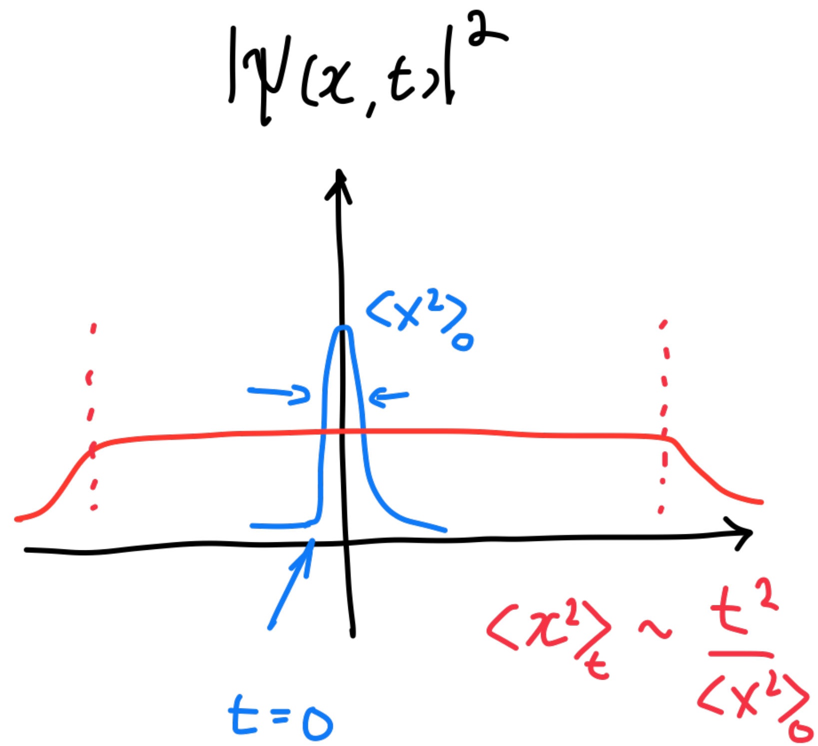
QM 100

O $\quad$ D

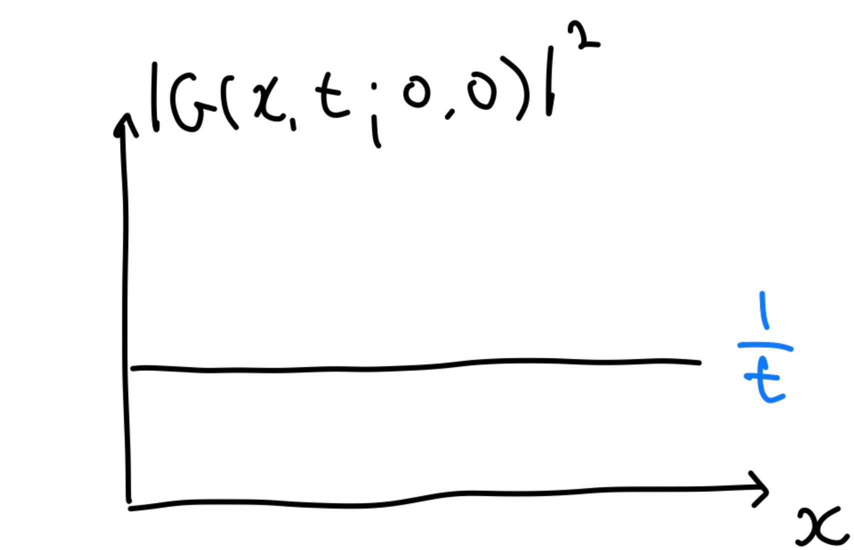
$(0,0)$ $\quad$ (x,t)

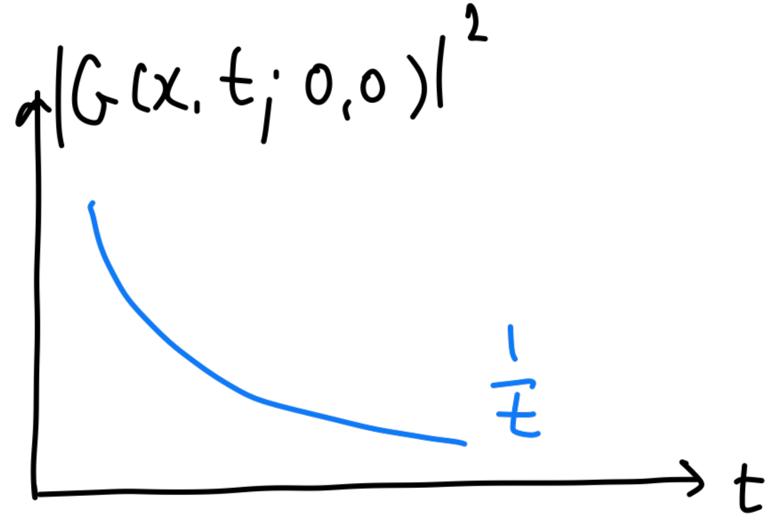




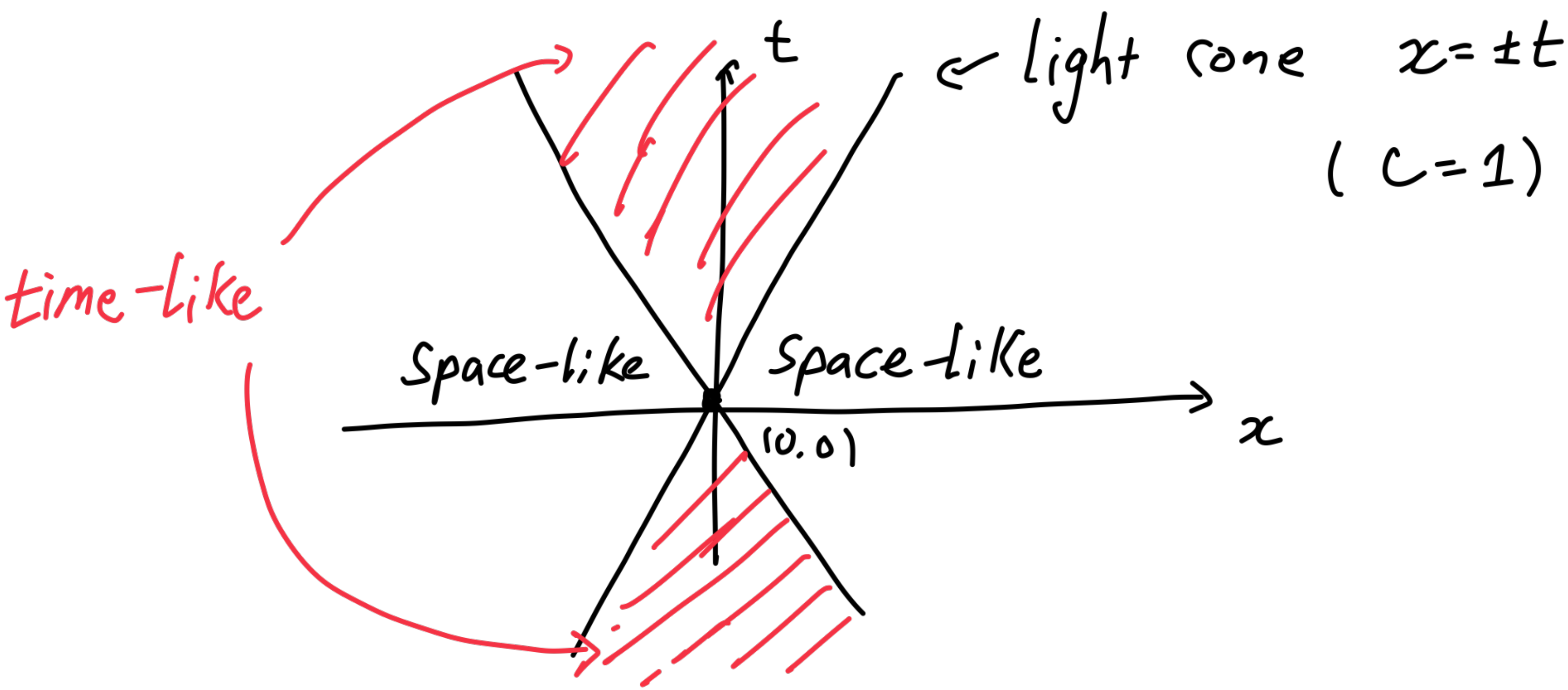


$$|\psi(x,t)|^2 \xrightarrow{\langle x^2\rangle_0 \to 0} |G(x,t;0,0)|^2$$





What about $Z=1$ Relativistic QCP?



Generally, Non-Relativistic Theory $z=2$
has $SO(2,1)$ conformal symmetry.

$$\langle vac | T\, \hat{O}_{\vec{r}}(t)\, \hat{O}_0(0) | vac \rangle \sim \frac{1}{t^{2\Delta_0}}\, e^{i\frac{r^2}{2t}}$$

Δ_0 : Scaling dimension of Operator $\hat{O}_0$
"prime"

($\Delta_0 = \frac{1}{2}$ in 1D for $\hat{O}_0 = b_0^{+}$)