Phys525:

Quantum Condensed Matter Physics: Quantum Criticality

Basics, Dynamics and Topological criticality

Episode Four:

Quantum effects on symmetry restoring and Mapping between Quantum and Classical Models

More on U(1) Symmetry breaking: Quantum effects

$$\hat{N} = \frac{1}{i}\frac{\partial}{\partial \varphi}, \quad [\varphi, \hat{N}] = i$$

$$\Psi_{N\text{-boson}} \to \Psi_{N\text{-boson}}\, e^{iN\varphi}$$

if $\varphi_0 \to \varphi_0 + \varphi$

With short range interactions,

$$H_{eff} = g\int \rho^2(r)\, d^d\vec{r} \sim \frac{\hat{N}^2}{2C}$$

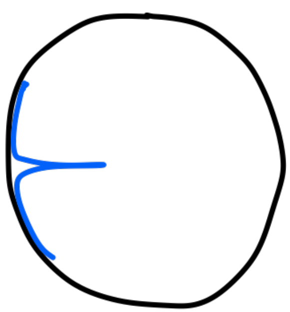
$$C = \frac{\Omega}{g}$$

Ω ← Vol.

g ← interaction

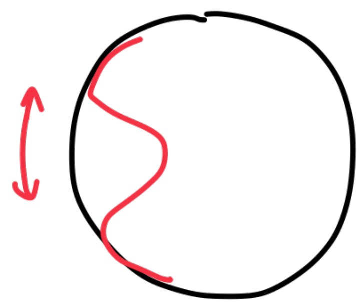
$t = 0$

$\langle \Delta\varphi^2 \rangle \sim \frac{1}{\alpha}$



S^1 ← internal space

$t \neq 0$



$\langle \Delta\varphi^2(t) \rangle \sim \frac{\alpha t^2}{C^2}$

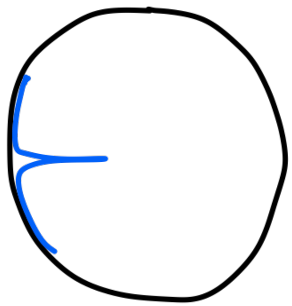
S^1

$(C \to \infty \text{ if } g \to 0)$

More on U(1) Symmetry breaking : Quantum effects

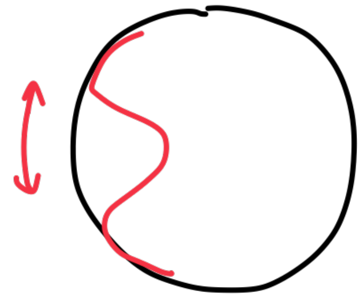
$t=0$

$\langle \Delta\varphi^2 \rangle \sim \frac{1}{\alpha}$



S^1 ← internal space

$t \neq 0$



S^1

$\langle \Delta\varphi^2(t) \rangle \sim \frac{d\, t^2}{c^2}$

$(c \to \infty$ if $g \to 0)$

time it takes for $\langle \Delta\varphi^2(t) \rangle \sim O(1)$,

$$t_{O(1)} \sim \frac{1}{\sqrt{\alpha}} \cdot c = \frac{1}{\sqrt{\alpha}} \cdot \frac{\Omega}{g} \longrightarrow \infty \text{ if } \Omega \to \infty.$$

← Vol.

↖ interaction

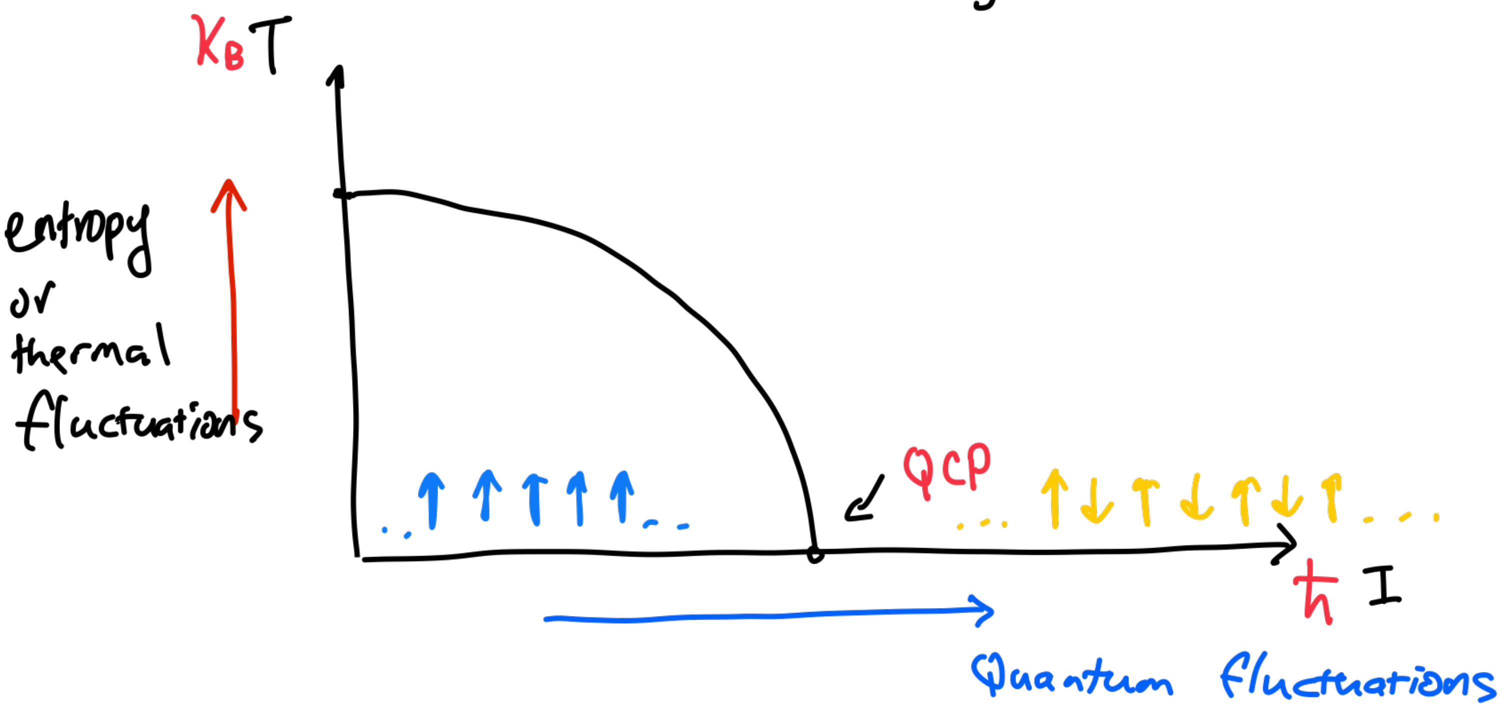
Spontaneous Symmetry Breaking Vs Symmetry Restoring

Towards Quantum Models

$$[S_{\alpha i}, S_{\beta j}] = i\hbar\, \epsilon_{\alpha\beta\gamma} S_{\gamma i}\, \delta_{ij}$$

$$\mathcal{H}_{Ising} = -J \sum_{\langle ij \rangle} S_{z i} \cdot S_{z j} + I \sum_{i} S_{x i}$$

$$[S_{z i}, H_{Ising}] = i\hbar\, I \sum_{i \in j} S_{y i}$$



Towards Quantum Models

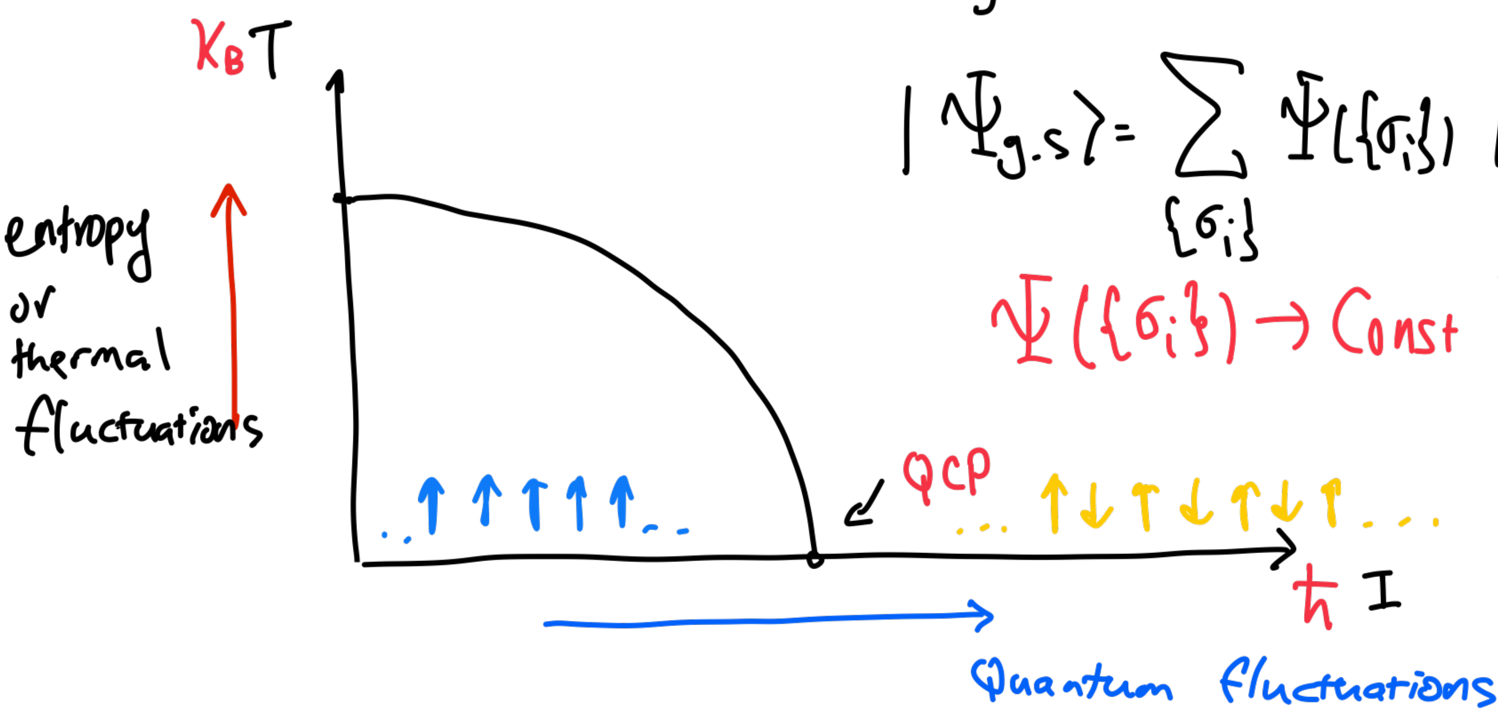
$$[S_{\alpha i}, S_{\beta j}] = i\hbar\, \epsilon_{\alpha\beta\gamma} S_{\gamma i}\, \delta_{ij}$$

$$\mathcal{H}_{Ising} = -J \sum_{\langle ij \rangle} S_{z_i} \cdot S_{z_j} - I \sum_i S_{x_i}$$

$$[S_{z_i}, H_{Ising}] = i\hbar I \sum_{i \in j} S_{y_i}$$

$$|\Psi_{g.s}\rangle = \sum_{\{\sigma_i\}} \Psi(\{\sigma_i\})\, |\sigma_1, \sigma_2, \ldots \sigma_N\rangle$$

$\Psi(\{\sigma_i\}) \to$ Const if $I \to \infty$



Quantum states of Ising Model

$$|\Psi_{g.s}\rangle = \sum_{\{\sigma_i\}} \Psi(\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_N) |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes |\sigma_3\rangle \ldots \otimes |\sigma_N\rangle$$

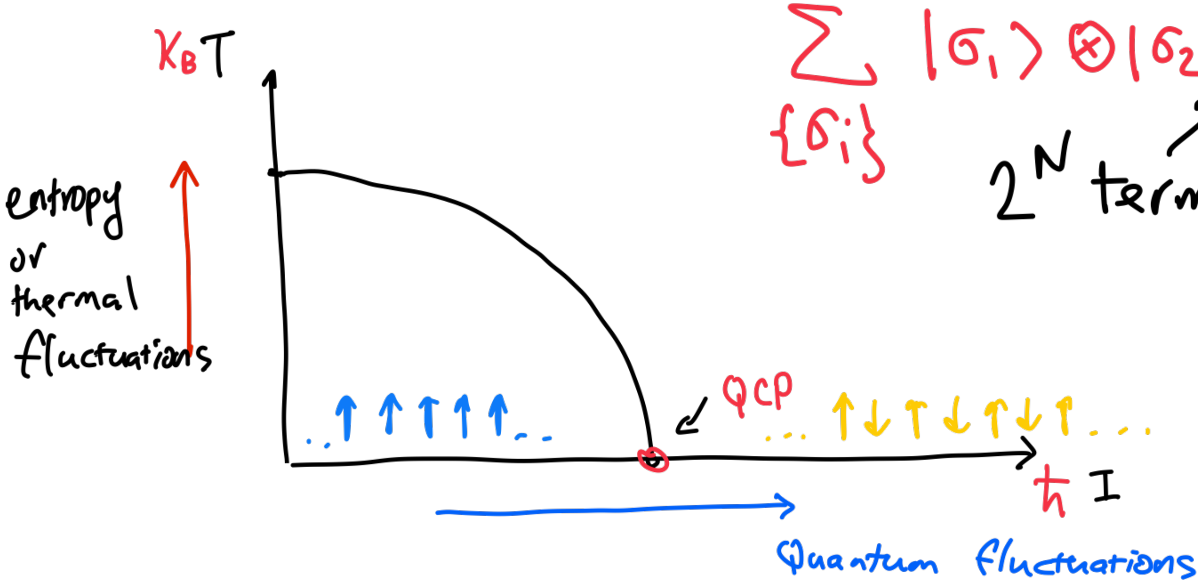
$\Gamma = 0$, $\quad \Psi(\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_N) = \underbrace{\delta_{\sigma_1,+1}\, \delta_{\sigma_2,+1}\, \delta_{\sigma_3,+1} \cdots \delta_{\sigma_N,+1}}_{\text{product state in terms of } |\sigma_i\rangle}$

$\Gamma = \infty$, $\quad \Psi(\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_N) = \text{Const}$

"Equal-amplitude spin state"

$$\sum_{\{\sigma_i\}} |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes |\sigma_3\rangle \ldots \otimes |\sigma_N\rangle$$

2^N terms



Some basic Approaches to many-body ground state

Imginary Time evolution

$$Z = \langle -\tfrac{T}{2} | \tfrac{T}{2} \rangle = \langle -\tfrac{T}{2} | e^{-HT} | -\tfrac{T}{2} \rangle$$

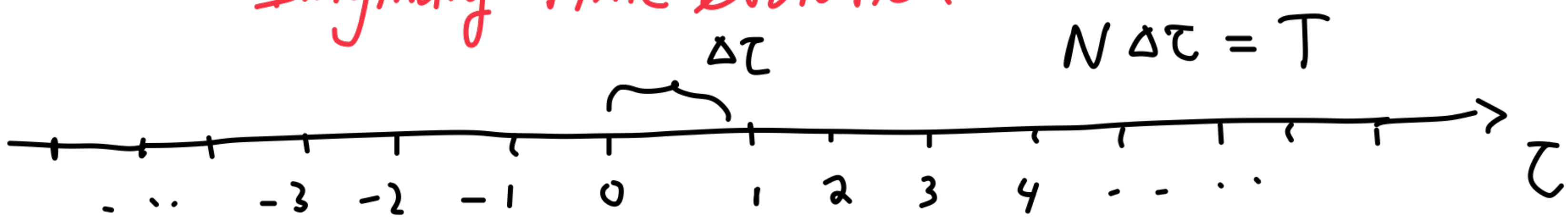
$$= \sum_n \langle -\tfrac{T}{2} | n \rangle \langle n | -\tfrac{T}{2} \rangle \, e^{-E_n \cdot T}$$

$$\xrightarrow{T \to \infty} |\langle -\tfrac{T}{2} | g.s. \rangle|^2 \, e^{-E_{g.s.} T}$$

$$E_{g.s.} = \lim_{T \to \infty} \frac{-1}{T} \ln Z(T)$$

Some basic Approaches to many-body ground state

Imaginary Time evolution



$$Z = \langle 0 | e^{-HT} | 0 \rangle = \langle 0 | e^{-H\Delta\tau} \cdots e^{-H\Delta\tau} | 0 \rangle$$

$$= \cdots \sum_{\{n_0\}} \sum_{\{n_1\}} \sum_{\{n_2\}} \cdots \langle n_0 | e^{-H\Delta\tau} | n_1 \rangle \langle n_1 | e^{-H\Delta\tau} | n_2 \rangle \cdots$$

Complete Set

For Ising Model $\displaystyle\sum_{\{n_0\}} \rightarrow \sum_{\substack{\{\sigma_i\} \\ i=1\cdots N}}$, $|n_1\rangle = \underbrace{|\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \cdots |\sigma_N\rangle}$

2^N states $\leftarrow |\{\sigma_i\}, i=1\cdots N\rangle$

A technical Remark: $H = H_1 + H_2$, $[H_1, H_2] \neq 0$

$$\langle n_1 | e^{-H\Delta\tau} | n_2 \rangle \overset{\Delta\tau \to 0}{\cong} \langle n_1 | 1 - (H_1\Delta\tau + H_2\Delta\tau) | n_2 \rangle + O(\Delta\tau^2)$$

$$\cong \langle n_1 | e^{-H_1\Delta\tau} \, e^{-H_2\Delta\tau} | n_2 \rangle$$

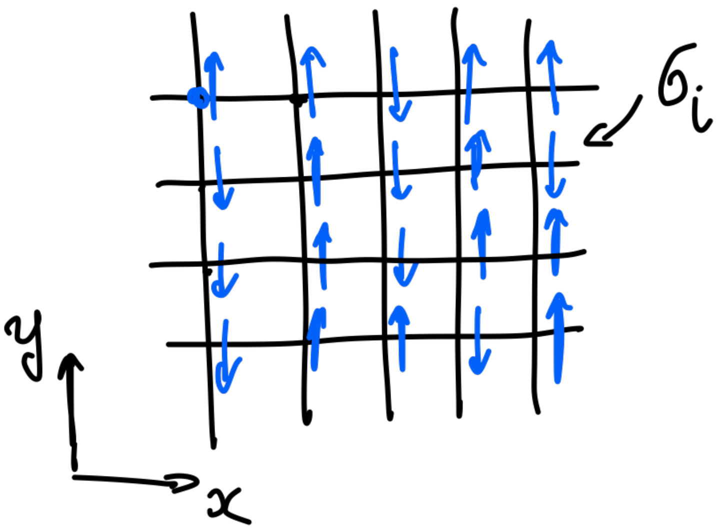
Sometimes

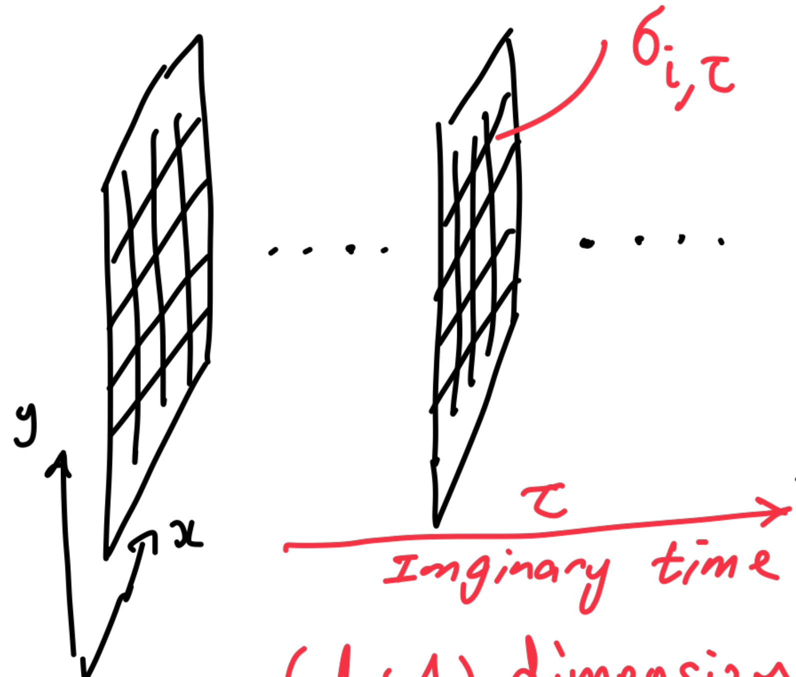
$$\cong \sum_{\{M\}} \langle n_1 | e^{-H_1\Delta\tau} | M \rangle \langle M | e^{-H_2\Delta\tau} | n_2 \rangle$$

This is needed for Superconducting problem.

Towards Quantum Models   Why QCP ?

$$\mathcal{H}_{Ising} = -J \sum_{\langle ij \rangle} S_{z_i} \cdot S_{z_j} - I \sum_i S_{x_i} \quad (d\text{-dimension})$$

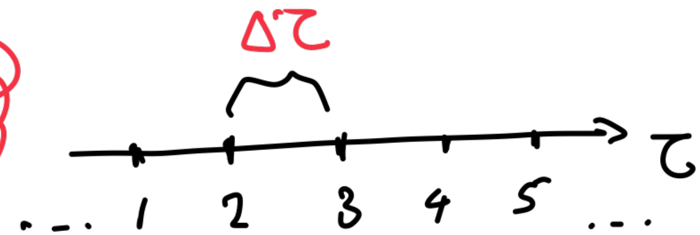




(d+1) dimension

$$\beta \mathcal{H}_{Ising} = -\tilde{J}(J\Delta\tau) \sum_{\langle ij \rangle} \sigma_{i,\tau}\, \sigma_{j,\tau} - \tilde{K}(I\Delta\tau) \sum_{\langle \tau\tau' \rangle} \sigma_{i,\tau} \cdot \sigma_{i,\tau'}$$

The physics shall not depend on "$\Delta\tau$": discretization procedure



$$I = gJ, \qquad \tilde{K}(I\Delta\tau) = \tilde{K}(g, J\Delta\tau); \quad \tilde{J} = J\Delta\tau$$

Set $\tilde{K} = \tilde{J}$ to deform into <u>isotropic</u> Model

$$\tilde{K}(g, J\Delta\tau) = \tilde{J}(J\Delta\tau) \longrightarrow J\Delta\tau = f(g)$$

$$\tilde{K} = \tilde{J} = f(g), \quad \text{Ising Model} \longrightarrow f(g) = 1$$

$g = g_c$

transition

QCP at $g = g_c$.

$$\beta\mathcal{H}_{Ising} = -\tilde{J}(J\Delta\tau)\sum_{\langle ij\rangle}\sigma_{i,\tau}\sigma_{j,\tau} - \tilde{K}(I\Delta\tau)\sum_{\langle\tau\tau'\rangle}\sigma_{i,\tau}\cdot\sigma_{i,\tau'}$$

One Spin System $\qquad H = h\sigma_x$

$$\langle 0| e^{-HT} |0\rangle = \sum_{\cdots\,\sigma_0=\pm 1}\ \sum_{\sigma_1=\pm 1}\ \sum_{\sigma_2=\pm 1} \cdots \langle\sigma_0| e^{-h\sigma_x\Delta\tau}|\sigma_1\rangle\langle\sigma_1| e^{-h\sigma_x\Delta\tau}|\sigma_2\rangle \cdots$$

$$\langle\sigma_1| e^{-h\sigma_x\Delta\tau}|\sigma_2\rangle = \cosh h\Delta\tau + \sigma_x \sinh h\Delta\tau$$

transition Amplitude

$$= X \cdot \underbrace{\begin{bmatrix} e^{\tilde K} & e^{-\tilde K} \\ e^{-\tilde K} & e^{\tilde K} \end{bmatrix}}$$

||

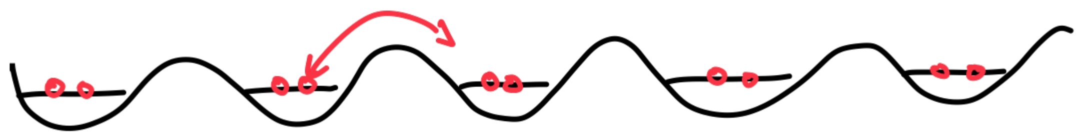
Boltzmann Weight $\rightarrow$ $\quad e^{\tilde K (h\Delta\tau)\sigma_1\sigma_2}$

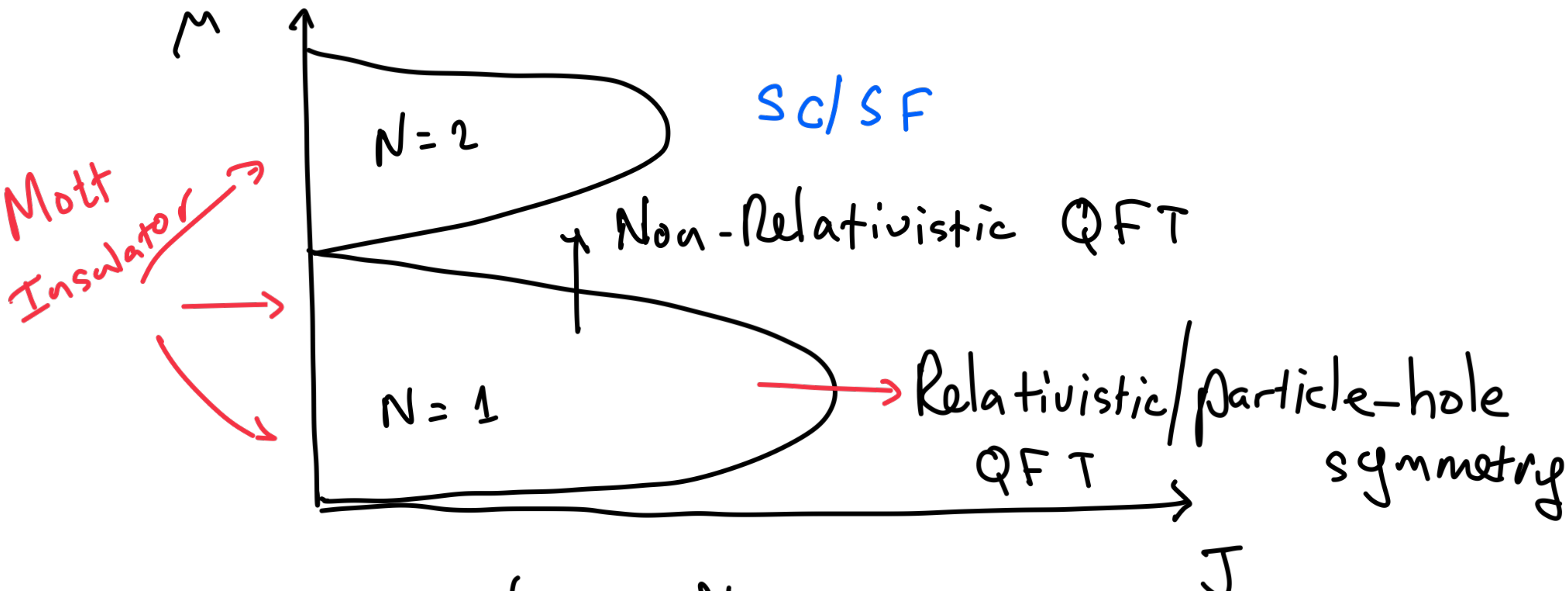
Quantum Model II: $[b_i, b_j] = 0, \quad [b_i, b_j^\dagger] = \delta_{ij}$

$$\mathcal{H}_{BH} = \sum_i \frac{\hat{N}_i^2}{2c} - \mu \hat{N}_i + J \sum_{\langle ij \rangle} b_i^\dagger b_j + h.c.$$

$$(\hat{N}_i = b_i^\dagger b_i)$$

Bose-Hubbard Model





T = 0 phase diagram

Order-Disorder quantum phase transitions

1) “Order” usually involves condensation of bosonic fields

Ising model: Real scalar field;
Superfluids: Complex scalar field;
Nematic order: Real “Director” field;
Ferromagnetic order: Complex vector field
……

2) So transitions are usually described by Bosonic quantum field theories.