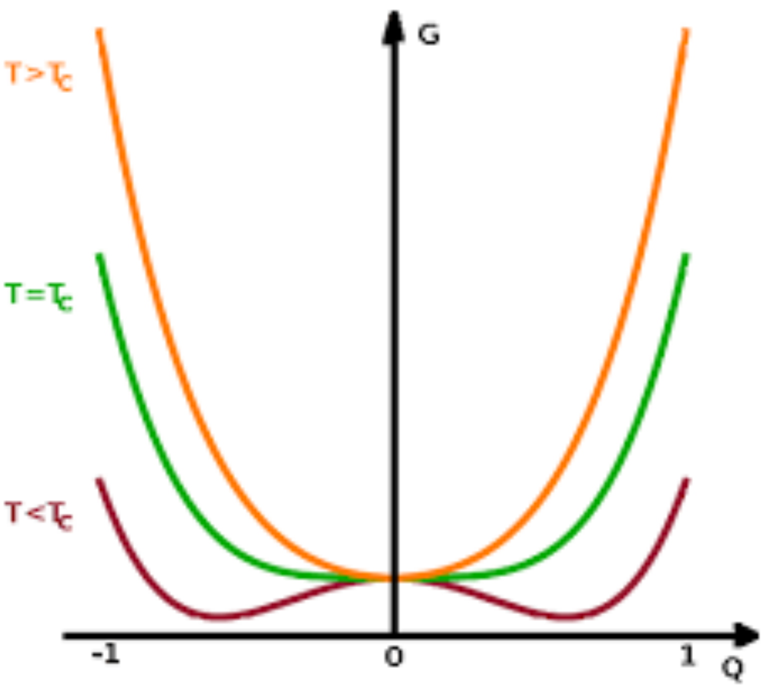
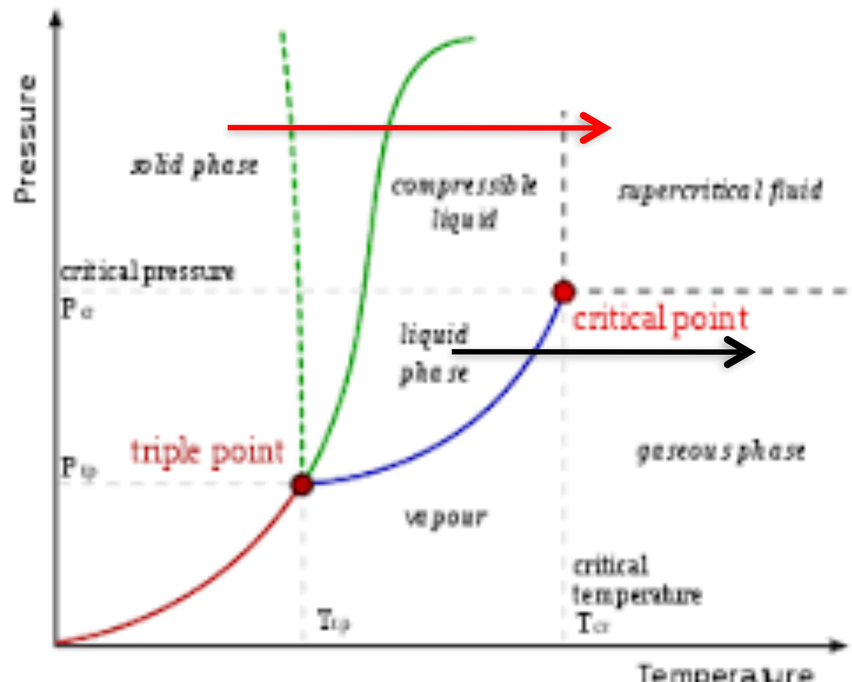
Phys525:
Quantum Condensed Matter Physics: Quantum Criticality
Basics, Dynamics and Topological criticality

Episode Two: Order-disorder Phase transitions:

The concept of "coarse graining" or Great Unification

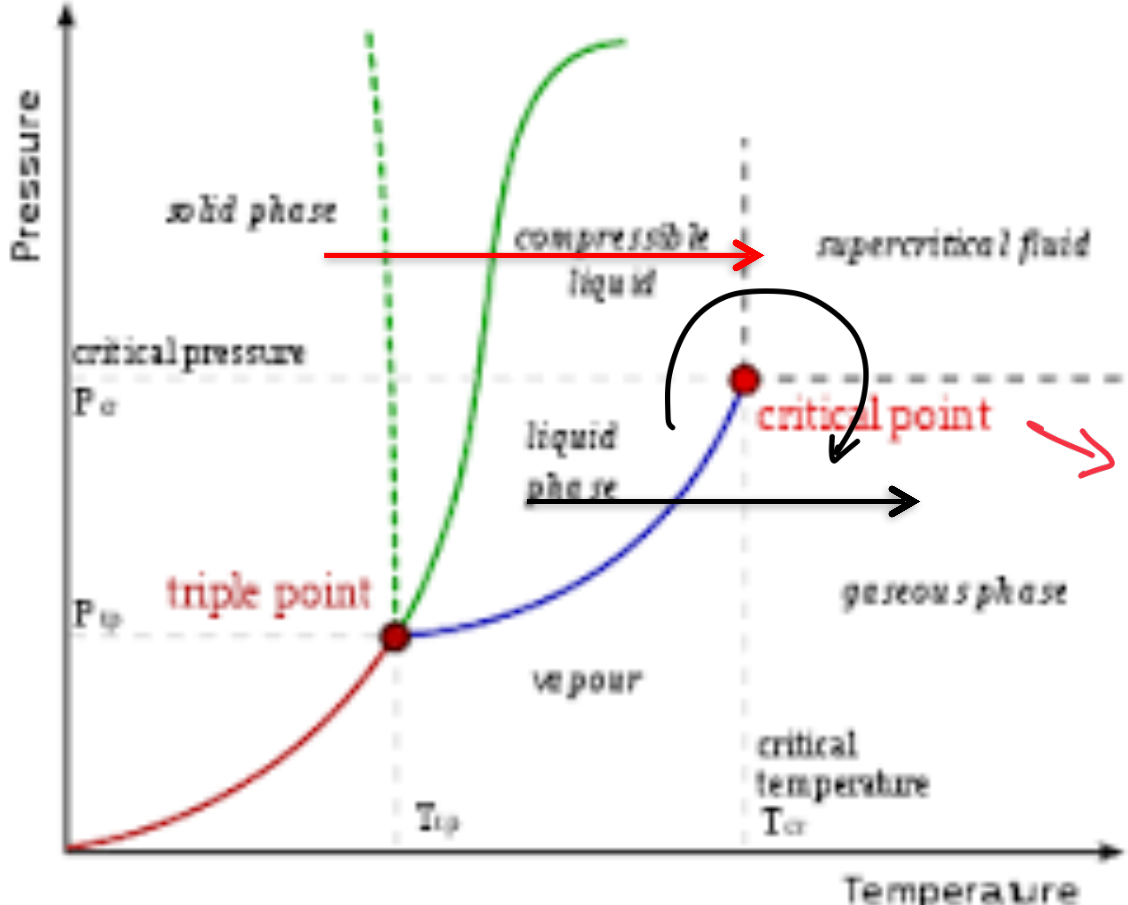
Landau paradigm for order-disorder Phase Transitions





There have to be phase transitions if ordering occurs.

Landau paradigm for order-disorder Phase Transitions



Related to Topological Quatum phase transition

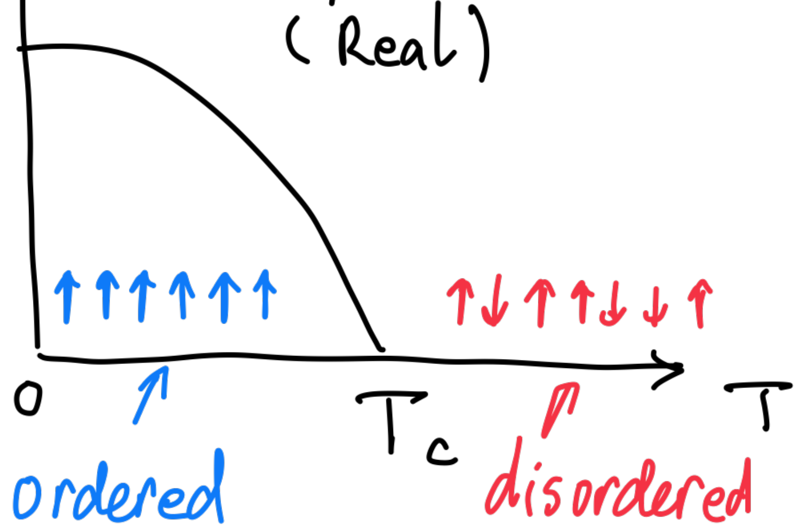
There have to be phase transitions if ordering occurs.

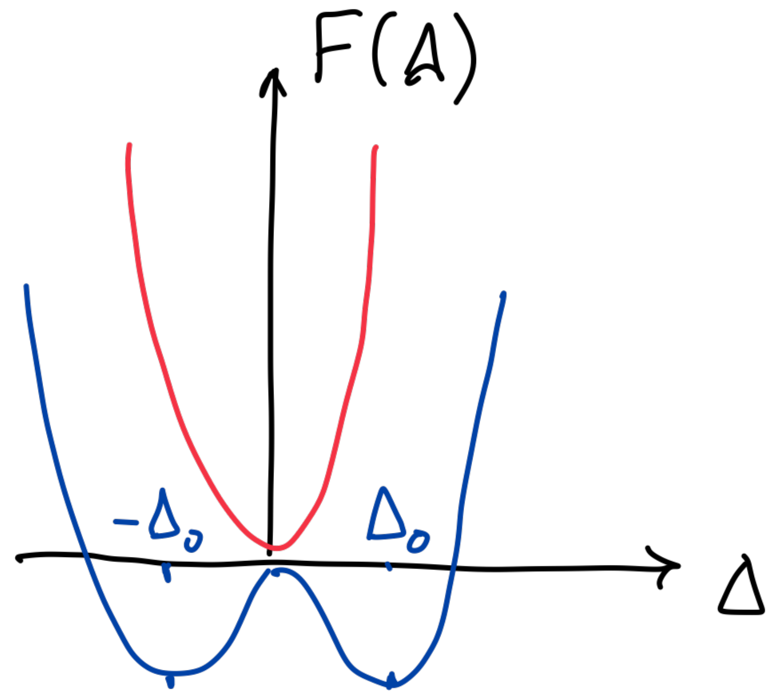
Example I: Ising Model

$$\beta H = -\frac{J}{T} \sum_{\langle ij \rangle} \sigma_{z_i} \cdot \sigma_{z_j}, \quad \left(\beta^{-1} = K_B T, \quad \sigma_z = \pm 1 \right)$$

$$\Delta_0(T) = \langle \sigma_{z_i} \rangle$$

Order parameter
(Real)

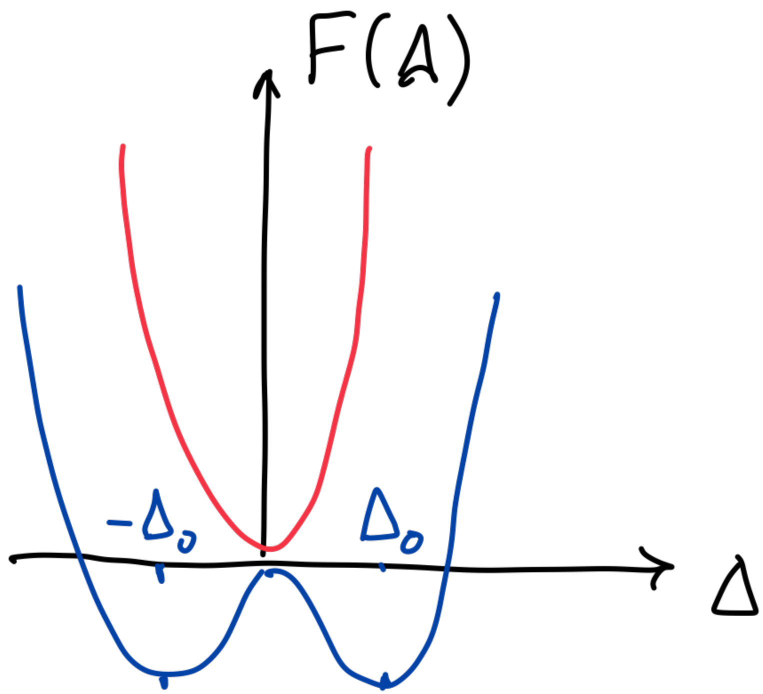




$$Z = \int D\Delta(\vec{r})\, e^{-\beta F(\Delta(\vec{r}))}$$

Ising symmetry / $\mathbb{Z}_2$ symmetry

$$\mathcal{H}(\{\sigma_{z_i}\}) = \mathcal{H}(\{-\sigma_{z_i}\}) = -J\sum_{\langle ij\rangle} \sigma_{z_i}\sigma_{z_j}$$



Ground state breaks $\mathbb{Z}_2$ Symmetry Spontaneously !!
(2- fold degenerate)

The Concept of "<u>coarse graining</u>": an illustration

thermal physics

state : $\{\sigma_{z_i},\ i=1,2,\ldots\infty\}$

$$\cdots\ \uparrow\ \uparrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \cdots \longrightarrow Z = \sum_{\{\sigma_{z_i}\}} e^{-\beta H(\{\sigma_{z_i}\})}$$

$\beta \mathcal{H}(\{\sigma_{z_i}\})$

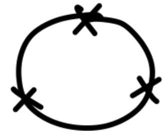
field : $\{\phi(\vec{r})\}$, $\phi(\vec{r}) = \langle \sigma_{z_i} \rangle_{\vec{r}}$

$$\cdots\ \underbrace{\uparrow\ \uparrow\ \uparrow\ \downarrow}_{\phi(\vec{r}_1)}\ \underbrace{\uparrow\ \downarrow\ \uparrow\ \downarrow}_{\phi(\vec{r}_2)}\ \cdots \longrightarrow Z = \sum_{\{\phi(\vec{r})\}} \underbrace{\sum_{\{\sigma_{z_i}\}\in\{\phi(\vec{r})\}} e^{-\beta H(\{\sigma_{z_i}\})}}$$

$\beta F(\{\phi(\vec{r})\})$

$$\simeq \sum_{\{\phi(\vec{r})\}} e^{-\beta F(\{\phi(\vec{r})\})}$$

3-Spin Grain



A Cartoon

$\{\sigma_{z_i}\}$	$\mathcal{H}$	$\longrightarrow$	$\phi = \langle \sigma_{z_i} \rangle$	$F(\phi) = U - TS$
$\uparrow\uparrow\uparrow$	$-3J$		$+1$	$-3J$
$\uparrow\uparrow\downarrow$	J			
$\uparrow\downarrow\uparrow$	J		$+\frac{1}{3}$	$J - T\ln 3$
$\downarrow\uparrow\uparrow$	J			
$\uparrow\downarrow\downarrow$	J			
$\downarrow\uparrow\downarrow$	J		$-\frac{1}{3}$	$J - T\ln 3$
$\downarrow\downarrow\uparrow$	J			
$\downarrow\downarrow\downarrow$	$-3J$		-1	$-3J$

$$Z = \sum_{\{\sigma_{z_i}\}} e^{-\beta H} \longrightarrow Z = \sum_{\{\phi(\vec{r})\}} e^{-\beta F(\phi)}$$

Proper field theory "Great Unifcation"!!

(Possible to derive it !)

4 important ingredients :

(A) Real fields (B) Same symmetry as Ising Model

(C) Continuous transition (D) Keep the most relevant terms

$$\beta H_{Ising} \longrightarrow \int d^d x \, f(\{\phi(x)\}), \quad \phi(x) = \phi^*(x)$$

$$f(\{\phi(x)\}) = (\nabla \phi(x))^2 + a\phi^2(x) + b\phi^4(x) + \cdots$$

terms $(\nabla^2 \phi(x))^2$, ϕ^6 are less relavent

terms $\phi^3(x)$, $\phi^5(x)$ are forbidden by (B).

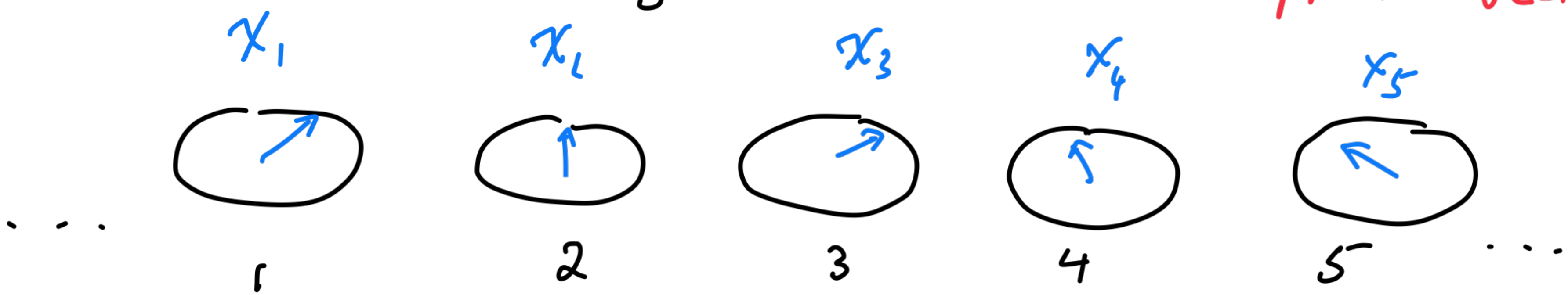
Example II: Superconductors / Superfluids

$$\beta \mathcal{H}_{Sc} = -\frac{J}{T} \sum_{\langle ij \rangle} \chi_i^* \cdot \chi_j + c.c.$$

$\chi_i^* \cdot \chi_i = 1$

$\chi_i = e^{i\varphi_i}$

planar vector



$$\Delta = \langle e^{i\varphi_i} \rangle$$

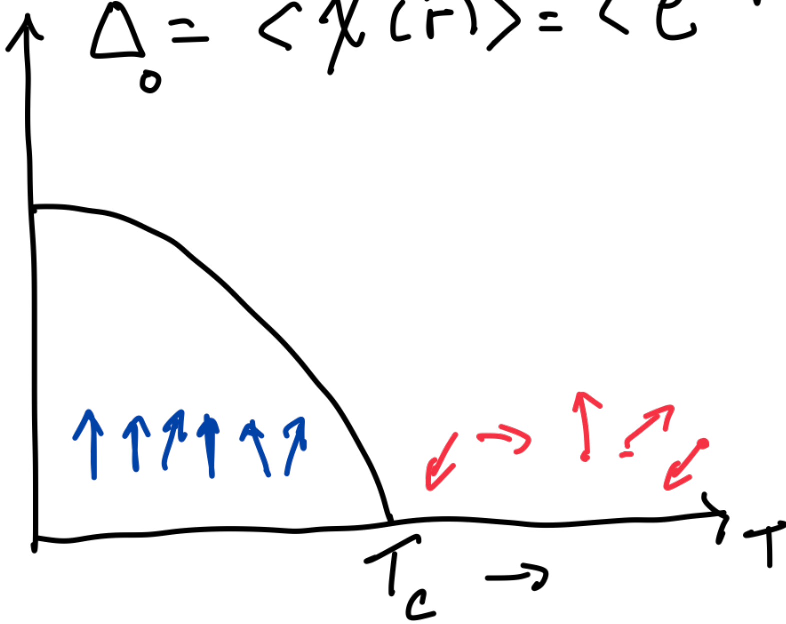
or $\beta \mathcal{H}_{SF} = -\frac{J}{T} \sum_{\langle ij \rangle} \hat{\chi}_i \cdot \hat{\chi}_j$, planar vector Rep.

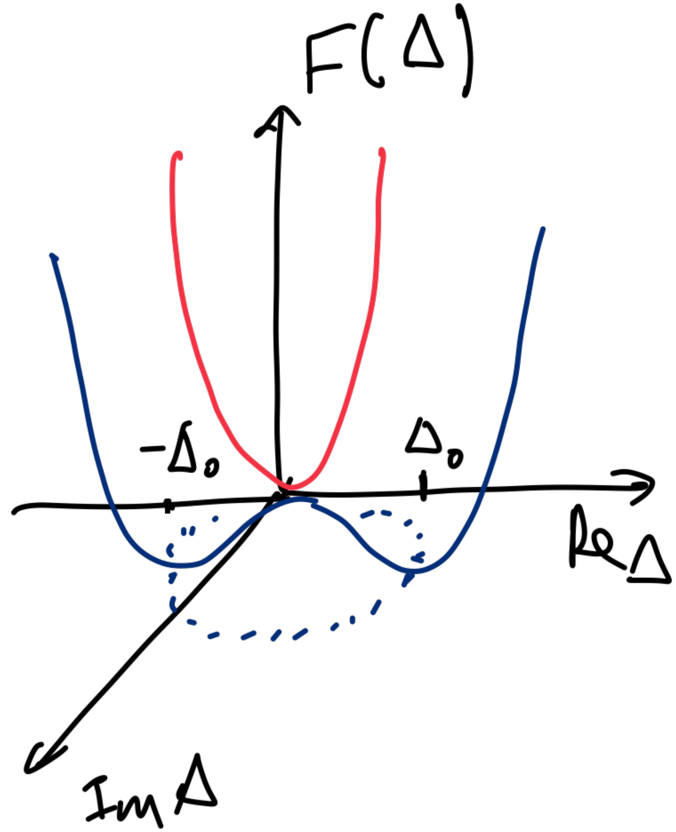
$\hat{\chi} = (\mathrm{Re}\,\chi, \mathrm{Im}\,\chi)$

Example II: Superconductors / Superfluids

$$\beta \mathcal{H}_{sc} = -\frac{J}{T} \sum_{\langle ij \rangle} \chi_i^* \cdot \chi_j + c.c.$$

$$\Delta_0 = \langle \chi(\vec{r}) \rangle = \langle e^{i\varphi} \rangle$$

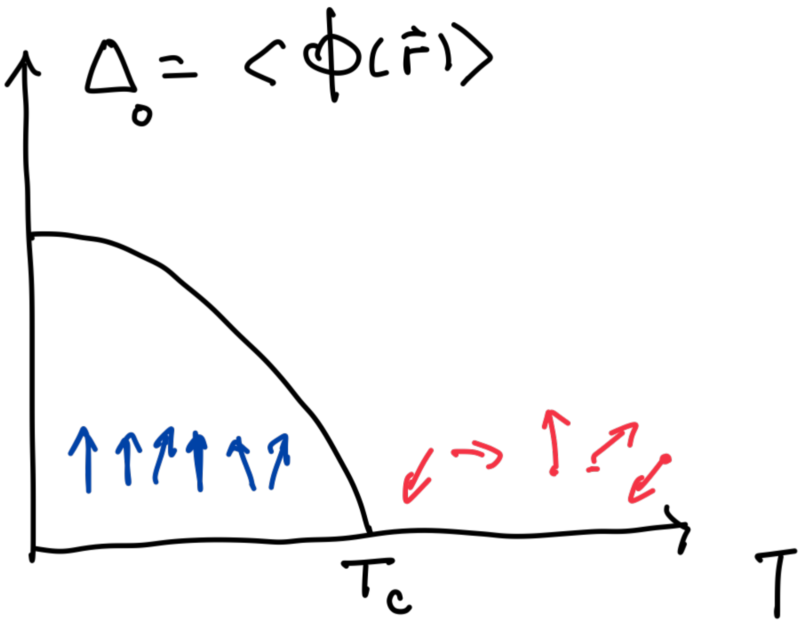


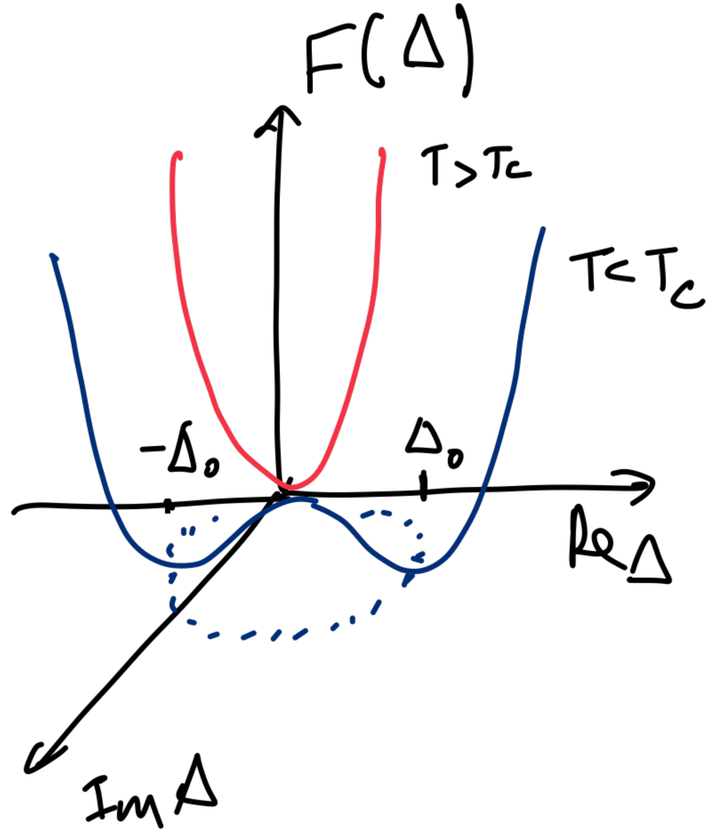


Proper field theory description (derived from BCS)

$$\beta f_{sc} = \int d^dx \; \frac{1}{2m} |\nabla \phi|^2 + a \, |\phi|^2 + b \, |\phi|^4 + \cdots$$

(ϕ : Complex field)

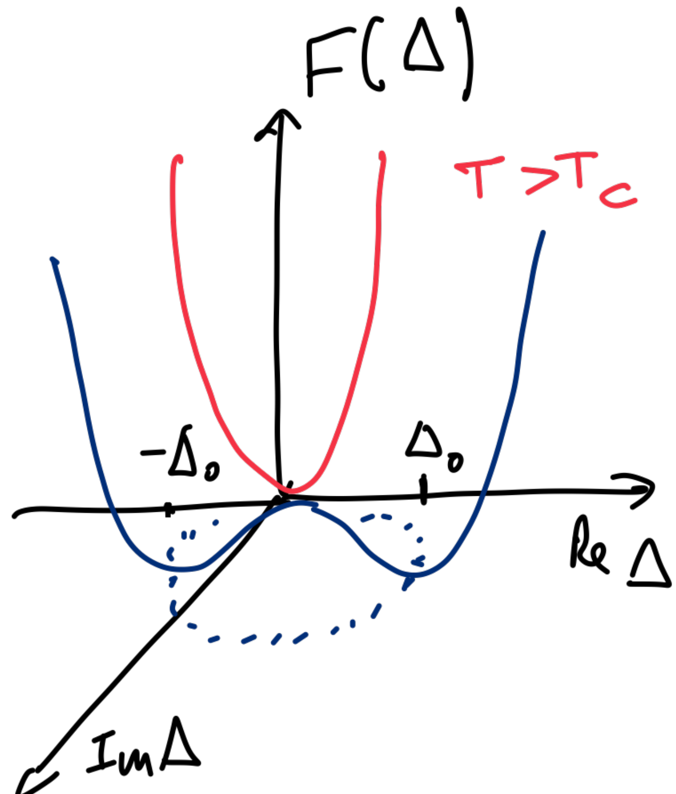




$a \sim T - T_c$

U(1) Gauge symmetry. $\mathcal{H}(\{\chi(\vec{r})\}) = \mathcal{H}(\{\chi(\vec{r})e^{i\varphi}\})$

"spontaneously broken"



Ground state manifold = S^1 (unit circle)

breaks the Gauge symmetry !!

Are cubic terms always forbidden? "$\phi|\phi|^2$, $\phi^*|\phi|^2$"

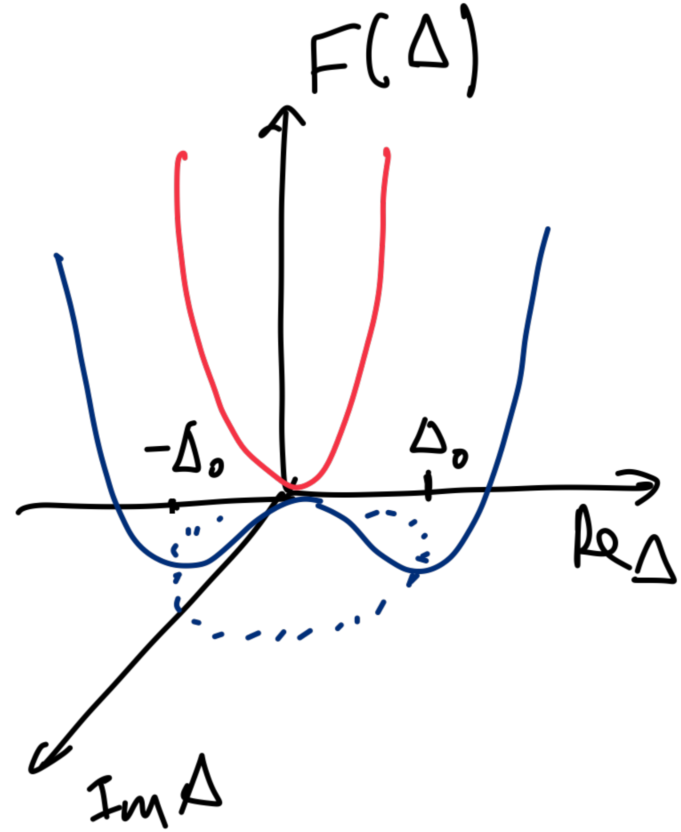
$$\beta F_{sc} = \int d^d x \; \frac{1}{2m}|\nabla\phi|^2 - \mu|\phi|^2 + \lambda|\phi|^4 + \cdots$$

(ϕ : Complex field)

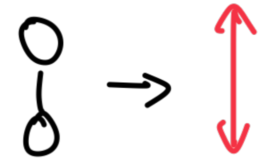
"$\phi|\phi|^2$, $\phi^*|\phi|^2$
are forbidden because of
symmetries"

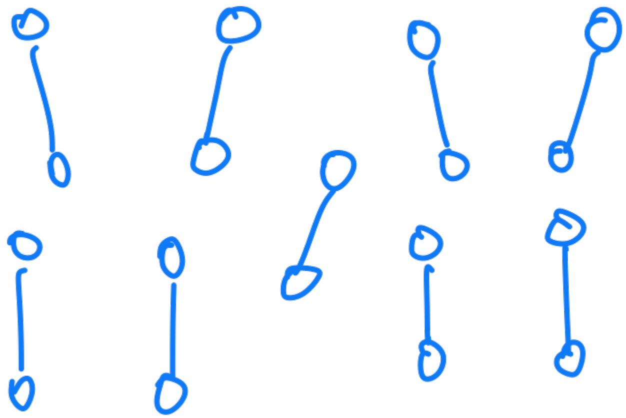
U(1) gauge symmetry

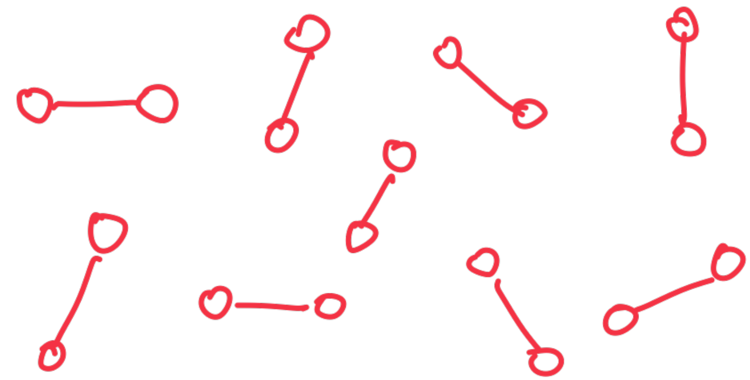
or $\mathbb{Z}_2$ Symmetry in Ising Model.



Nematic liquid Crystal


"director"

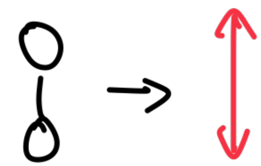

orientation order


orientation disorder

Nematic Liquid Crystal:
breaks Rotation Symmetry but not translational.

(believe to exist in HT_c systems !)

Nematic liquid Crystal



"director"

$$\Delta_{\alpha\beta} = \langle n_\alpha n_\beta \rangle - \frac{1}{3}\delta_{\alpha\beta} T_r \langle n_\alpha n_\beta \rangle , \quad T_r \Delta_{\alpha\beta} = 0$$

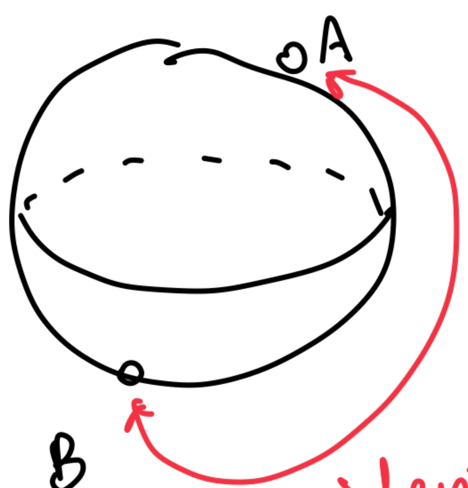
$\alpha = x, y, z$

<u>Real Symmetric, traceless tensor</u>

manifold $\quad S^2/\mathbb{Z}_2$

cubic term allowed!

$$F(\Delta) = a T_r \Delta^2 + \underbrace{b T_r \Delta^3}_{} + c T_r \Delta^4 + \ldots$$



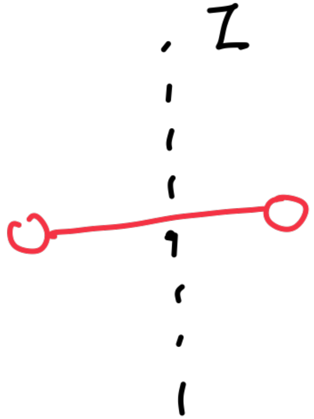
$$\Delta \rightarrow \Delta$$

if $\hat{n} \rightarrow -\hat{n}$.

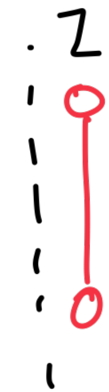
$$\Delta_1 = \begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ & \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix} \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

$$\Delta_2 = \begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ & \begin{bmatrix} -1 & & \\ & -1 & \\ & & 2 \end{bmatrix} \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

$$= -\Delta_1$$



biaxial



uniaxial

Important feature for classical Order-disorder transition

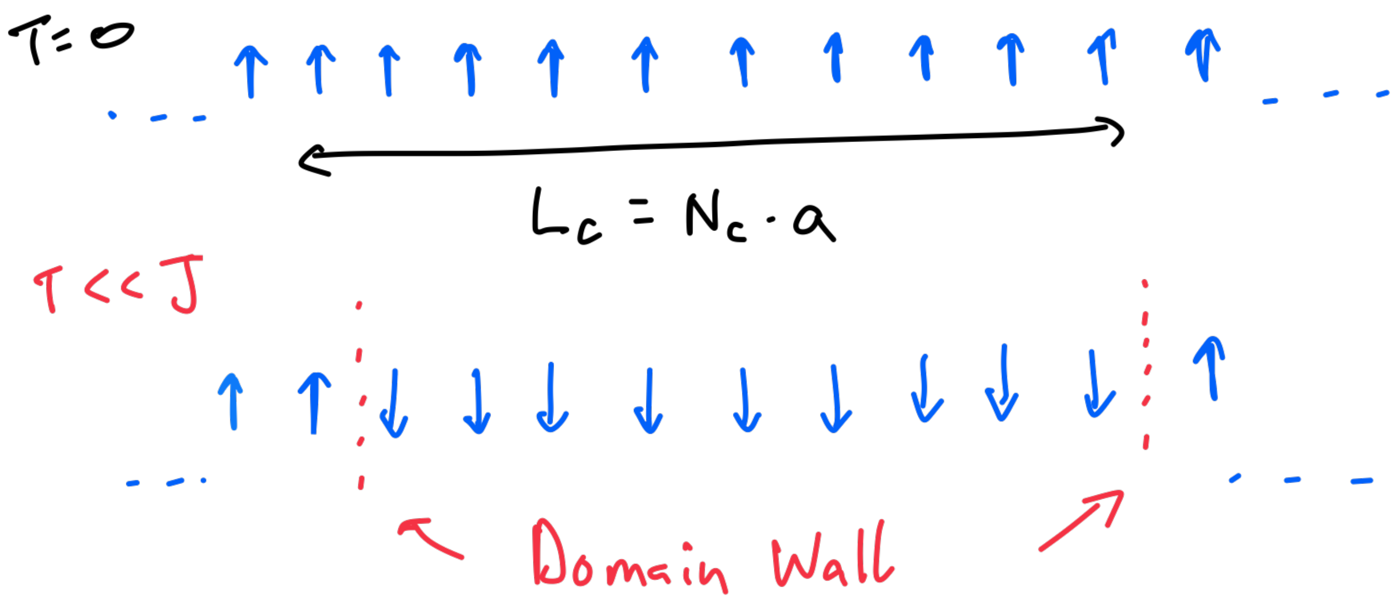
A: The operator defines the order parameter communtes with $\mathcal{H}$.

$$[\sigma_{z_i}, \mathcal{H}_{Ising}] = [\chi_i, \mathcal{H}_{Ising}] = 0$$

B: At lower temperature is always ordered.

C. phase transition driven by the entropy effect.

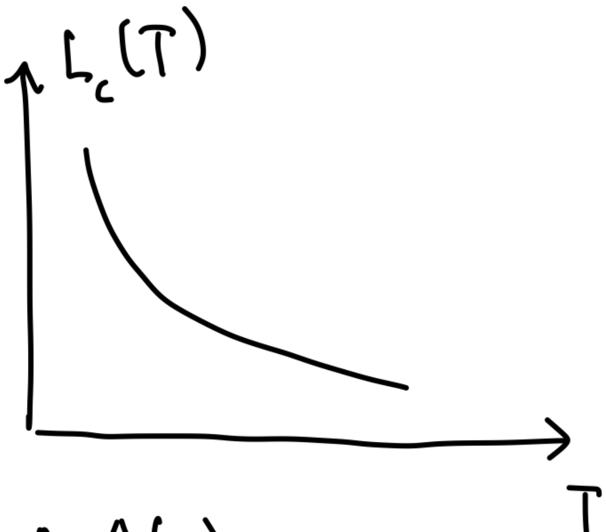
Entropy effect : 1D Ising Model $\mathcal{H}_{Ising} = -J \sum_{\langle i j \rangle} \sigma_{z_i} \cdot \sigma_{z_j}$



$$\underbrace{P_{DW}}_{\text{Probability}}(L_c) \sim e^{-\beta F_{DW}}, \quad F_{DW} = \underset{U}{4J} - \underset{T}{T}\, \underset{S}{\ln \frac{N_c(N_c-1)}{2}}$$

$$L_c \sim e^{\frac{2J}{T}} \quad \text{when} \quad P_{DW}(L_c) \sim O(1)$$

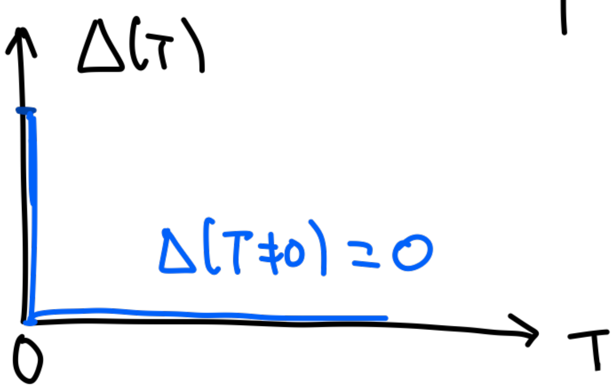
typical distance between DWs



$L_c(T) \to \infty$ when $T \to 0$

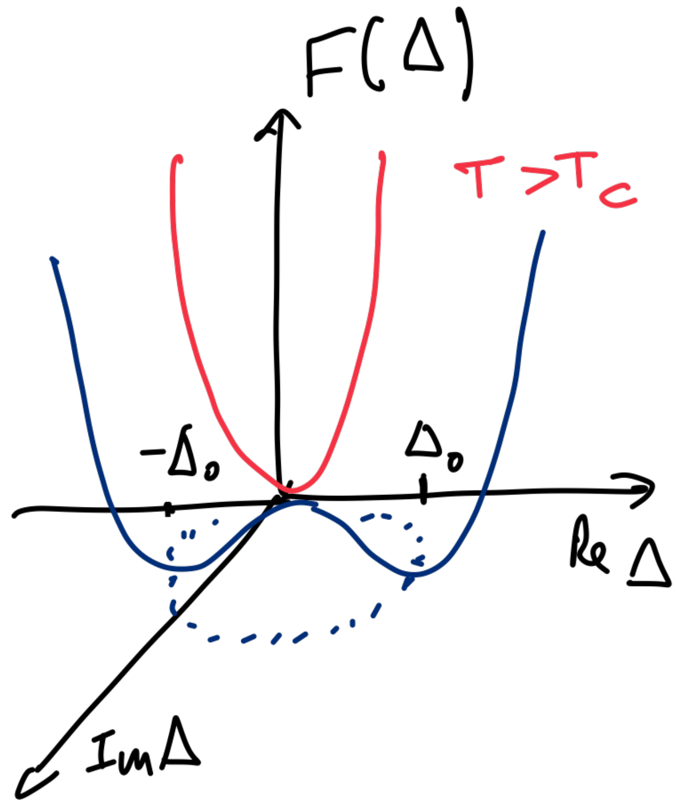
"A Dilute Gas of Domain Walls"

= Ising Model Ground state at $T \neq 0$



U(1) Gauge symmetry. $H(\{\chi(\vec{r})\}) = H(\{\chi(\vec{r})e^{i\varphi}\})$

"spontaneously broken"



Ground state manifold = S^1 (unit circle)

More on U(1) Symmetry breaking : Quantum physics

How long it takes to Restore U(1) symmetry for a system of "N" particles, quantum mechanically?

$$\psi_{N\text{-body}} \to \psi_{N\text{-body}} e^{iN\varphi}$$

if $\varphi_0 \to \varphi_0 + \varphi$

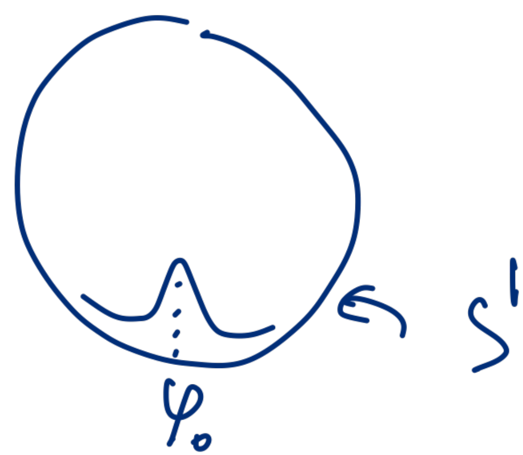
$$N = \frac{1}{i}\frac{\partial}{\partial\varphi} \qquad [\varphi, N] = i$$

For short range interactions, $\mathcal{H}_{eff} = g\int \rho^2(r)\, d^d r \sim \frac{N^2}{2C}$

$$C = \frac{\Omega}{g}$$

$\Omega \leftarrow$ Vol.

$g \leftarrow$ interaction

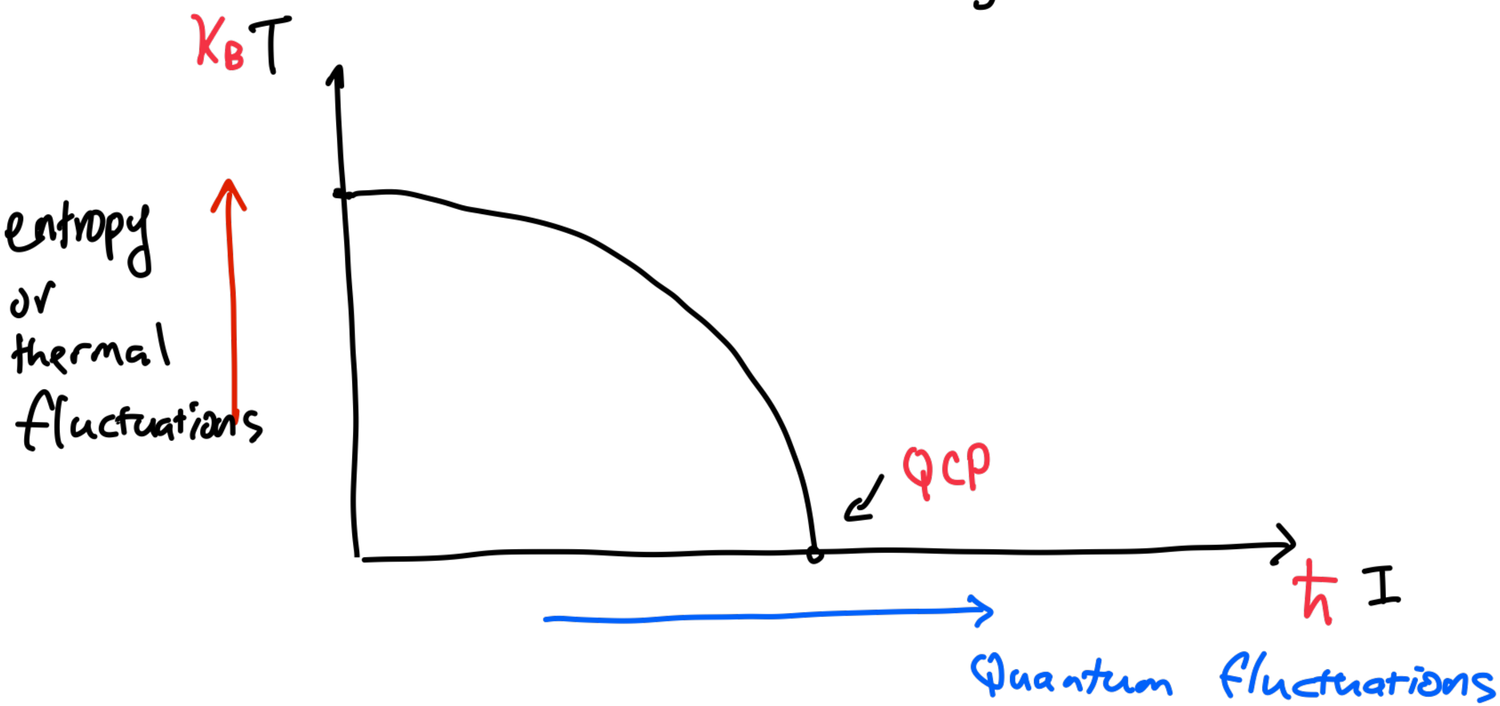


Towards Quantum Models

$$[S_{\alpha i}, S_{\beta j}] = i\hbar \epsilon_{\alpha\beta\gamma} S_{\gamma i} \delta_{ij}$$

$$\mathcal{H}_{Ising} = J \sum_{\langle ij \rangle} S_{z_i} \cdot S_{z_j} + I \sum_i S_{x_i}$$

$$[S_{z_i}, H_{Ising}] = i\hbar I \sum_{i \in j} S_{y_i}$$

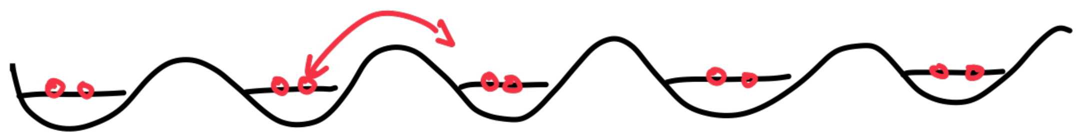


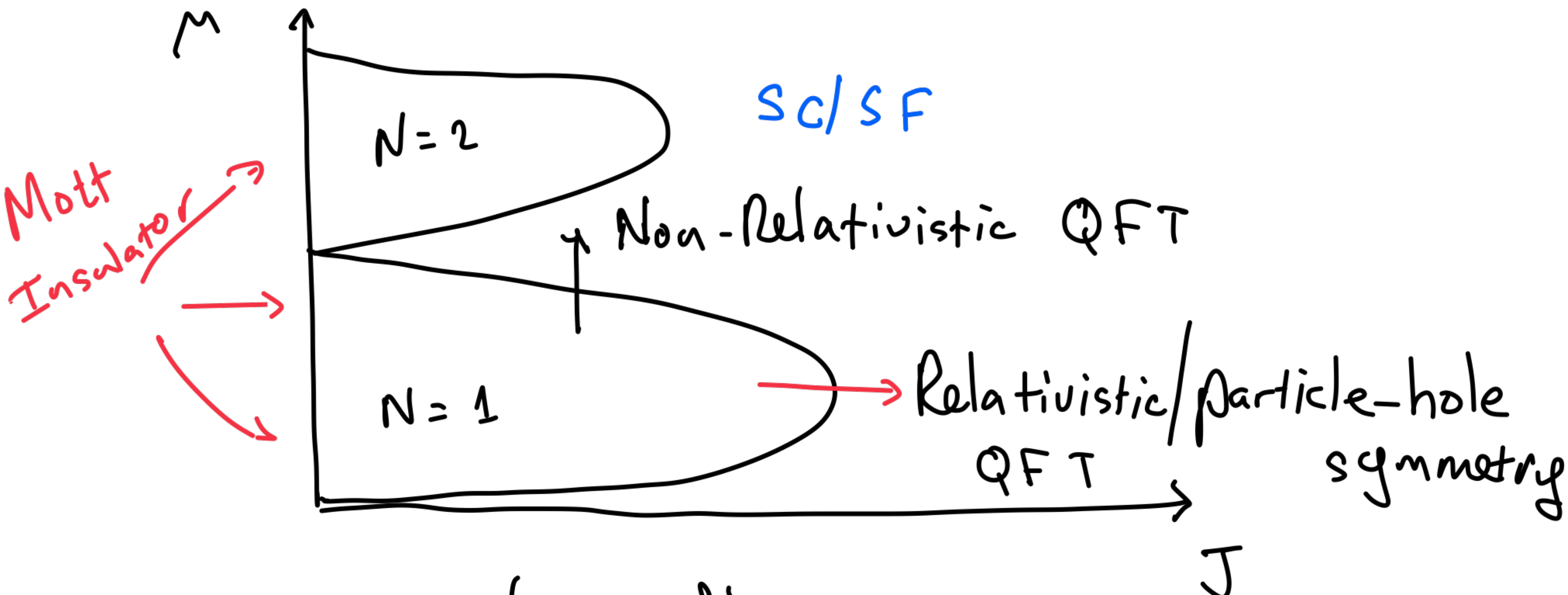
Quantum Model II : $[b_i, b_j] = 0, \quad [b_i, b_j^\dagger] = \delta_{ij}$

$$\mathcal{H}_{BH} = \sum_i \frac{\hat{N}_i^2}{2c} - \mu \hat{N}_i + J \sum_{\langle ij \rangle} b_i^\dagger b_j + h.c.$$

$$(\hat{N}_i = b_i^\dagger b_i)$$

Bose-Hubbard Model





$T=0$ phase diagram