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PHYSICS 206

# MECHANICS

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Course web site: **http://www.physics.ubc.ca/~birger/n206toc/index.html**
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# Contents

# 1 Introduction

## 1.1 About this course

The material in these notes was last presented in class January-April 2002. Included are also a number problems that mostly constitute assignments and exams from the winter sessions 1999–2002. The problems are very much part of the course, and without doing them you cannot expect to fully understand the material. There were also optional end of term projects and you can find a list of some of these near the end of these notes.

**AIMS**
Physics 206 at UBC covers material from Newtonian mechanics to Lagrangian and Hamiltonian dynamics. The most important goals are to

- Give needed background for later courses in quantum and statistical mechanics and electricity and magnetism.
- Make you familiar with the often formal and abstract approach of physics. Classical mechanics is a good place to do this– one can at the same time relate to familiar experiences.
- Introduce you to powerful problem solving techniques and allow practice of skills.
- Show applications to practical problems.

**TEXTS**
There was no assigned text for the course. A possible suitable text (of which I unfortunately was not aware when I gave the course) is the book by Florian Scheck[17]. Much of my lecture material was inspired by the "classic" text Landau and Lifshitz[11], and to a lesser extent by another "classic", Goldstein[8] (I had the first edition as an undergraduate a long time ago). In previous years Hand and Finch[9], and Cassiday and Fowles[7] have been assigned to the course. Cassiday and Fowles introduce Lagrangian Mechanics at a much later stage than I do, so you may prefer the other texts. They do require, more mathematical background than Cassiday and Fowles, so it may

still be useful. The book by Landau and Lifshitz is quite brief and extremely concise. Actually, everything that is needed for the course is there, and some students may prefer it as a text because of its elegance and clarity. Hand and Finch, and also Cassiday and Fowles, contain quite a bit of historical background and many more examples than it is possible to find room for in a one term course. For some students this extra material will be helpful in providing better understanding.

A common problem facing undergraduate physics students is that the necessary mathematics is often taught after it is required in the physics courses. The book by Riley, Hobson and Bence[19] may bridge the gap - it contains most of the math required for an undergraduate degree in physics and I recommend it as a general mathematical reference for physics students. A similar book is Arfken and Weber[2].

You are encouraged to use soft-ware such as Maple or Mathematica to solve the assignments! Both programs have excellent help menus. There are also many books available describing how to use both programs. For Maple I found Kofler[10] very helpful. You will find links to a number of Maple worksheets complementing the lecture material on the course web site. These have been produced using Maple 6 or Maple V release 5 operating under Windows 1998. Hopefully these worksheet will work in other environments too! You should feel free to download them from the course web site and edit them yourself for particular problems. The books by Enns and McGuire[4] and Lynch[12] are intended for a more advanced level than PHYS 206, but they contains many wonderful Maple examples that would be understandable to PHYS 206 students. The Enns and McGuire book is also available in a Mathematica edition. Sometimes working from examples is not enough, and you may need to learn more about the Maple language. I have found "The Maple V Programming guide" by Monagan *et al.*[14] a useful reference in this respect. In the case of Mathematica the standard reference is Wolfram[19]. Some useful web links can be found at the course web site.

## 1.2 Vector Algebra.

I will start by covering some mathematical preliminaries:

Much of mechanics is concerned with establishing the **position**, **velocity** and **acceleration** of particles as well as the forces acting on these. We represent these quantities by **vectors**. I will assume that you have had some exposure to vector algebra, but in order to establish notation (and to refresh your memory) I will quickly go through some of the rules.

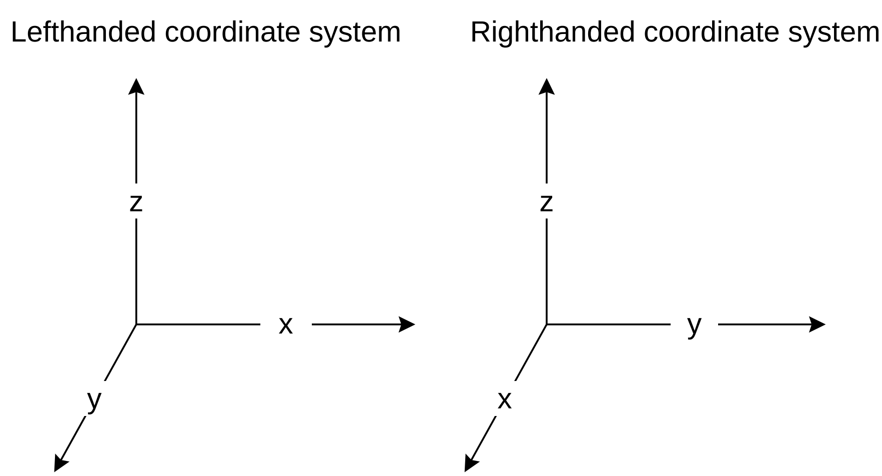
**Vectors** are objects that have a **magnitude** and **direction.** I will use the notation $\vec{A}$ to describe vectors (except in figures where I will write **A**).

To **specify** a vector we need a **coordinate system**.
The vector can then be expressed in terms of its **components**.

We will first specify formal rules of vector algebra in a 3-dimensional Cartesian system:

$$\vec{A} = (A_x, A_y, A_z)$$

Throughout this course we implicitly assume that this coordinate system is **right-handed**, i.e. the order of the components matters.




We **add** and **subtract** vectors by adding and subtracting the components

$$\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z)$$

$$\vec{A} - \vec{B} = (A_x - B_x, A_y - B_y, A_z - B_z)$$

**Scalars** are quantities which are independent of the coordinate system used, they can be specified by a single number in appropriate units.
When a **vector** is **multiplied** by a scalar the effect is to multiply all components by the same number

$$c\vec{A} = (cA_x, cA_y, cA_z)$$

Since they are different types, we cannot add scalars and vectors. So far the rules of vector algebra are much the same as ordinary algebra. The plot thickens when we come to vector multiplication! There are two kind of products:

DOT PRODUCTS

- The **scalar** or **dot** product of two vectors is defined by

$$\vec{A} \cdot \vec{B} = (A_x B_x + A_y B_y + A_z B_z) = \vec{B} \cdot \vec{A}$$

The result of the multiplication is a
**number** i.e. a **scalar**.

- The magnitude or length of a vector is

$$A = |\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

- The **null vector** has zero magnitude. The only way to achieve this to put all the components to zero. I will not bother to put a vector sign over the null vector
$$\vec{0} = (0, 0, 0) = 0$$

- **Unit vectors** have length 1. I will use the notation

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

Of special interest are the unit vectors in the three principal directions of a Cartesian coordinate system. They are often written

$$\hat{e}_x = (1,0,0);\ \hat{e}_y = (0,1,0);\ \hat{e}_z = (0,0,1)$$

Another common notation is

$$\hat{\imath} \equiv \hat{e}_x;\ \hat{\jmath} \equiv \hat{e}_y;\ \hat{k} \equiv \hat{e}_z$$

Clearly

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$$
$$\hat{\imath}\cdot\hat{\jmath} = \hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = 0$$

CROSS PRODUCT

The **vector** or **cross** product can be defined in terms of determinants

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x)$$
$$= \hat{\imath}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \hat{\jmath}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$
$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Remembering that a determinant changes sign when two rows (or columns) change place we find

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

Clearly

$$\vec{A}\times\vec{A} = 0$$

When evaluating cross products it is convenient to remember the cyclic relations

$$\hat{\imath}\times\hat{\jmath} = \hat{k};\ \hat{\jmath}\times\hat{k} = \hat{\imath};\ \hat{k}\times\hat{\imath} = \hat{\jmath}$$

which are equivalent to the familiar right hand rule.

SCALAR TRIPLE PRODUCT

Since $\vec{B}\times\vec{C}$ is a **vector**

$$\vec{A}\cdot(\vec{B}\times\vec{C})$$

is a scalar. From the determinantal expression for the cross product we find

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

We know that determinants changes sign when two rows or columns are interchanged.With two interchanges the determinant stays the same. Hence

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

There are also **vector triple products**.
It can be shown that

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

which is different from

$$(\vec{A} \times \vec{B}) \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{A}(\vec{B} \cdot \vec{C})$$

Often when solving problems it is useful to know how to apply symbol manipulation packages to problems. The worksheet WORKING WITH VECTORS IN MAPLE at
http://www.physics.ubc.ca/ birger/n206l1a.mws (or .html) demonstrates how to manipulate **vectors** using Maple.

SUMMARY
We have

- Worked out the basics of vector algebra in a Cartesian coordinate system.
- Described
  - vector addition and subtraction
  - dot product
  - vector product
  - scalar and vector triple products

**Example problems**

**Problem 1.2.1**
(Problem 1 of 1999 problem set 1)
Given the three vectors

$$\vec{a} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}};\ \vec{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}};\ \vec{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

Find

**a:** The angle between $\vec{a}$ and $\vec{b}$.

**b:** $\vec{a} \cdot (\vec{b} \times \vec{c})$

**c:** $\vec{a} \times (\vec{b} \times \vec{c})$

**d:** $(\vec{a} \times \vec{b}) \times \vec{c}$

**Problem 1.2.2**
(Problem 3 of 2000 problem set 1)
An electrical wire extends diagonally in a straight line down the north wall of a house. It makes a $30^o$ angle with the vertical. After it reaches the corner with the west wall it continues down along the west wall at a $30^o$ angle with the vertical. What is the angle between the two segments of the wire?

**Problem 1.2.3**
(Problem 4 of 2000 problem set 1)
The Lorentz force $\vec{F}$ on a charge $q$ moving with velocity $v$ in a magnetic field is

$$\vec{F} = q(\vec{v} \times \vec{B})$$

It is found that if (in appropriate units)

$$\vec{v} = \hat{\mathbf{i}} \quad \Rightarrow \quad \frac{\vec{F}}{q} = 2\hat{\mathbf{k}} - 4\hat{\mathbf{j}}$$

$$\vec{v} = \hat{\mathbf{j}} \quad \Rightarrow |F/q| = 5$$

What are the possible values of the magnetic induction $\vec{B}$ in these units.

**Problem 1.2.4**
(Problem 1 of problem set 1 2002)
The Lorentz force on a charge $q$ moving with velocity $\vec{v}$ in a magnetic field with induction $\vec{B}$ is

$$\vec{F} = q\vec{v} \times \vec{B}$$

It is found, in appropriate units, that if $\vec{v} = v\hat{\mathbf{i}}$ the force is given by

$$\frac{\vec{F}}{qv} = -3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

if $\vec{v} = v\hat{\mathbf{j}}$ the force is given by

$$\frac{\vec{F}}{qv} = 3\hat{\mathbf{i}} - \hat{\mathbf{k}}$$

and if $\vec{v} = v\hat{\mathbf{k}}$ the force is given by

$$\frac{\vec{F}}{qv} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

Find the induction $\vec{B}$

**Problem 1.2.5**
(Problem 1 of problem set 1 2001)
Identify the following surfaces:

**a:** $|\vec{r}| = 1$

**b:** $\vec{r} \cdot \hat{\mathbf{k}} = 1$

**c:** $|\vec{r} - (\vec{r} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}| = 1$

Here $\hat{\mathbf{k}}$ is a unit vector in the z-direction of a Cartesian coordinate system and $\vec{r}$ is a vector to a point on the surface your are asked to describe.

**Problem 1.2.6**
(Problem 2 of problem set 1 2001)
Let $\vec{A}$ be an arbitrary vector and $\hat{n}$ a unit vector in an arbitrary direction. Show that

$$\vec{A} = (\vec{A} \cdot \hat{n})\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}$$

**Problem 1.2.7**
(Problem 2 of problem set 1 2002)
Consider a triangle ABC, whose sides are the three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$. The angles opposite to the three sides are, respectively $\alpha$, $\beta$, and $\gamma$. Show that

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}$$

# 2 Newtonian Mechanics.

## 2.1 Momentum, work and kinetic energy.

**Newtonian mechanics** is the foundation on which classical mechanics is built. This course is mostly concerned with the **Lagrangian** and **Hamiltonian** reformulation of classical mechanics. We will find that the reformulated theory is better suited to handle problems more difficult than the ones you were exposed to in high school and first year physics. Nevertheless, to motivate and justify the more advanced treatment we need to go back to the "roots".

NEWTON'S LAWS

1 Every body continues in the state of rest or with uniform motion in a straight line, unless it is compelled by a force.

2 Acceleration is proportional to the applied force and takes place in the direction of the force

$$\vec{F} = m\vec{a}$$

3 The mutual forces between two objects are equal and oppositely directed. (For every action there is an equal and opposite reaction).

The first law defines an **inertial reference frame**. If there is an inertial frame, then any frame that moves relative to it with constant velocity is also an inertial frame.

**Acceleration** is the rate of change of the **velocity**.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}$$

where $\vec{a}, \vec{v}$, position $\vec{x}$, as well as the force $\vec{F}$ are all **vectors** in **space**. In Newtonian mechanics time is universal $\Rightarrow$ observers in different reference frames can agree on the time. Space and time are **given** and no attempt is made to justify these concepts.

CONSERVATION OF MOMENTUM.

The proportionality constant $m$ in $\vec{F} = m\vec{a}$ is the **inertial mass**. In Newtonian mechanics the inertial mass is independent of the velocity of a particle and the product

$$m\vec{v} = \vec{p}$$

is the **linear momentum** (or simply the **momentum**) of an object. The second law can then be restated

$$\vec{F} = \frac{d\vec{p}}{dt}$$

The third law can be interpreted to state that the rates of change of momentum are equal and opposite for two bodies influencing each other, so that the total momentum is unchanged. This is just the law of **conservation of momentum**.

WORK is a form of energy given by:
**Force** $\times$ **displacement** $\equiv$ **work**.

Let us first consider motion in one dimension.

Suppose a body moves a distance $dx$ in a time interval $dt$, starting out at $x_i$ and ending up at $x_f$ under the influence of the force $F$. The initial velocity is $v_i$ and the final $v_f$. The work done on a particle is

$$W = \int_{x_i}^{x_f} F dx = m \int_{x_i}^{x_f} \frac{dv}{dt} dx$$

$$= m \int_{x_i}^{x_f} \frac{dx}{dt}\frac{dv}{dx} dx = m \int_{v_i}^{v_f} v dv$$

$$= \frac{1}{2} m (v_f^2 - v_i^2)$$

$\frac{1}{2}mv^2$=kinetic energy

Work done=change in kinetic energy.

The argument can be extended to more than one dimension
We express the trajectory $\vec{r}(t)$, velocity $\vec{v}(t)$ and Force $\vec{F}$ , on component form:

$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\vec{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

$$\vec{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, respectively, are unit vectors in the $x-, y-$ and $z-$ directions.
The work done on a particle is now

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

$$= m\int_{x_i}^{x_f} \frac{dv_x}{dt} dx + m\int_{y_i}^{y_f} \frac{dv_y}{dt} dy + m\int_{z_i}^{z_f} \frac{dv_z}{dt} dz$$

$$= \frac{m}{2}(v_x^2 + v_y^2 + v_z^2)|_i^f = \frac{mv_f^2}{2} - \frac{mv_i^2}{2}$$

**Work done on particle = change in kinetic energy!**

CONSERVATIVE FORCES: ONE DIMENSIONAL CASE

Many forces in nature depend on position only (not explicitly on time or velocity). In one dimension we can always express a position dependent force as derivative of a **potential energy** $V(x)$.

$$F(x) = -\frac{dV}{dx}$$

$$V(x) = -\int^{x'} F(x')dx'$$

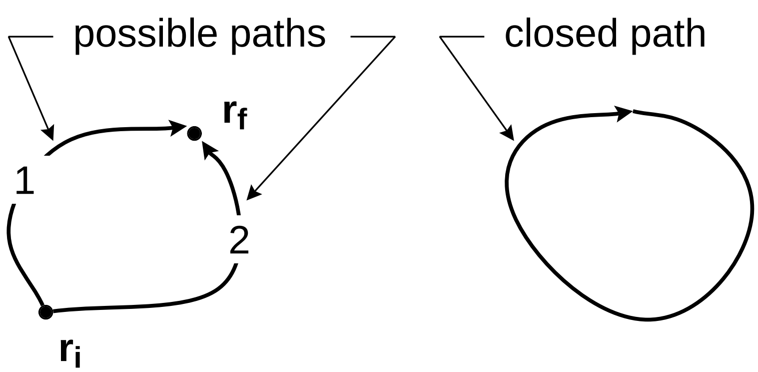
Only potential energy **differences** matter for the dynamics, the absolute value of the potential energy is not meaningful in classical mechanics. We indicated this by omitting the lower limit of integration above.

Often we **choose** the potential energy to be zero at some convenient reference point. (usually at infinity or the origin).

CONSERVATIVE FORCES: THREE DIMENSIONAL CASE

In higher dimension the concept of conservative forces is a bit more complicated.

A **position dependent force** is **conservative** if its line integral between two points is **independent of the path**.




We can then then define the **potential energy** as

$$V(\vec{r}_f) - V(\vec{r}_i) = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

$$= -\int_{x_i}^{x_f} F_x dx - \int_{y_i}^{y_f} F_y dx - \int_{z_i}^{z_f} F_z dx$$

If the line integral along any closed path is zero, the line integral must be independent of the path taken between the initial and final states! So, another way of defining a position dependent conservative force is that the line integral

$$\oint \vec{F} \cdot d\vec{r} = 0$$

for all possible closed paths.

THE GRADIENT OPERATOR

If the path infinitesimally short:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$
$$= -F_x dx - F_y dx - F_z dx$$

Define the operator $\nabla$ , or **nabla** by

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

We have

$$\vec{F} = -\nabla V$$

When $\nabla$ operates on a scalar such as $V$ it is called the **gradient**. The gradient of $-V$ is the **vector** $\vec{F}$.

Not all position dependent forces $\vec{F}(\vec{r})$ can be written as the gradient of a potential. The line integral along any closed path must be zero! We will in section 2.4 show how this can be verified in general using Stokes' theorem.

EXAMPLE
The force

$$\vec{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$$

is **not** conservative. To see this let us compute the line integral counterclockwise around a circle of radius $r$ surrounding the origin

$$x = r\ cos\theta;\ y = r\sin\theta;$$

$$dx = -r\sin\theta d\theta;\ dy = r\cos\theta d\theta$$

$$\oint \vec{F}\cdot d\vec{r} = -r^2\int_0^{2\pi}(\sin^2\theta + \cos^2\theta)d\theta = -2\pi r^2 \neq 0$$

EXAMPLE
The force

$$\vec{F} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$$

**is** conservative. We can show this by noting that it is the negative gradient of the potential

$$V(\vec{r}) = -xy$$

$$-\nabla V = (\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z})xy = \hat{\mathbf{i}}y + \hat{\mathbf{j}}x$$

q.e.d.

CONSERVATION OF ENERGY

Suppose $\vec{F}$ is a conservative force. Then

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = V_i - V_f$$

and we find

$$\frac{1}{2}mv_f^2 + V_f = \frac{1}{2}mv_i^2 + V_i = E_{tot} = const$$

which is the **law of conservation of energy.** You will recall from first year that the law of conservation of energy often can be used to simplify problems.

In addition to the conservative case discussed above, energy is also conserved if the force $\vec{F}$ is perpendicular to the direction of motion $\hat{v}$ (as in the case of the magnetic induction acting on a charge). In that case the force does no work and does not affect the energy.

A SPECIAL CASE: SIMPLE HARMONIC MOTION

You will recall from first year physics that the potential energy of a harmonic oscillator with displacement $x$ can be written

$$V(x) = \frac{1}{2}kx^2$$

where

$$k = \left.\frac{d^2V(x)}{dx^2}\right|_{x=0}$$

is the **spring constant**. In the more general case where the potential energy has a **minimum** at some value $x = a$, the second derivative of the potential about this point will be the spring constant for **small oscillations** about $a$.

MICROSCOPIC AND MACROSCOPIC MODELS
In microscopic models energy is typically conserved.

In the everyday (macroscopic) world there are **dissipative** forces such as **drag** and **friction** and energy is typically not conserved unless one wants to get involved in a microscopic description of the motion of individual atoms.

SUMMARY
We began our discussion of mechanics:

- by introducing Newton's Laws
- first law postulates existence of inertial systems
- second law defines inertial mass as ratio between force and acceleration
- third law is equivalent to law of conservation of momentum
- we defined conservative forces and conservation of energy
- We also discussed
  - work
  - kinetic energy
  - potential energy

**Example problems**

**Problem 2.1.1**
(Problem 2 of 1999 problem set 2)

**a:** Find the potential energy $V(x)$ as a function of displacement $x$ for a particle subject to the force

$$F_x = F_0 e^{-cx}$$

**b:** Find the velocity $v = \dot{x}$ as a function of displacement $x$ for a particle subject to the force in **a:** of mass $M$ which starts out with $v = 0$ at $x = 0$.

**c:** Plot the displacement and velocity as a function of time between $t = 0$ and $t = 10$ when $M = F_0 = c = 1$. The expressions for $v(t)$ and $x(t)$ should be evaluated numerically for the plot.

**Problem 2.1.2**
(Problem 3 of 1999 problem set 4)
A gun can fire a shell with speed $v_0$ in any direction. Show that the gun can reach any target between the ground and the surface

$$g^2\rho^2 = v_0^4 - 2gv_0^2 z$$

where $z$ is the vertical direction, $\rho = \sqrt{x^2 + y^2}$, $g$ is the acceleration of gravity, air resistance can be neglected, and the earth is flat.

**Problem 2.1.3**
(Problem 3 of 2001 problem set 1)
A pebble is thrown from the top of a hill which slopes downwards uniformly with angle $\phi = 60^o$. At what angle $\theta$ from the horizontal should the pebble be thrown to have the greatest range?

**Problem 2.1.4**
(Problem 4 of 2001 problem set 1)
A car can accelerate uniformly at the rate $2m\ s^{-2}$, it can brake with a maximum deceleration of $6m\ s^{-2}$. What is the minimum time to travel 500m, starting and ending the trip at rest?

**Problem 2.1.5**
(Problem 3 of 2002 problem set 2)
The potential energy of a molecule is

$$U(r) = \frac{a}{r^{12}} - \frac{b}{r^6}$$

where $r$ is the separation between the atoms and $a$ and $b$ are constants.
**a:** Plot the potential as a function of x if in appropriate units $a = b = 1$

**b:** For what value of $r$ is the potential a minimum?
**c:** How much energy does it take to separate the two atoms?
**d:** What is the "spring constant" associated with small amplitude oscillation?

**Problem 2.1.6**
(Problem 1 of 2000 midterm)
The potential energy of a particle moving in the $x-y$ plane is

$$\mathcal{U}(x,y) = \frac{k}{2}(x^2 - y^2)$$

while the kinetic energy is

$$\mathcal{T} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

**a:** Write down the equations of motion.
**b:** The particle starts from rest from the point $x_0, y_0$. Find the subsequent motion.
**c:** Describe the motion qualitatively assuming that $x_0 \neq 0, y_0 \neq 0$.

**Problem 2.1.7**
(Problem 1 of 1999 final)
Three particles $A, B, C$ each have mass $m$ and they move in the $x, y$ plane under the influence of forces between them. These forces satisfy Newton's third law. To ascertain if there are any additional, external, forces acting the coordinates of the particles are measured at times $t = 0, t = 1, t = 2$ (in appropriate units) The results are

| *Time* | $A$ | $B$ | $C$ |
|---|---|---|---|
| 0 | $(1,1)$ | $(2,2)$ | $(3,3)$ |
| 1 | $(1,0)$ | $(0,1)$ | $(3,3)$ |
| 2 | $(0,1)$ | $(1,2)$ | $(2,0)$ |

Is there a net external force acting on the system of particles?

## 2.2 Friction

LAST TIME

We began our discussion of mechanics:

- by introducing Newton's Laws:
  - first law postulates existence of inertial systems
  - second law defines inertial mass as ratio between force and acceleration
  - third law is equivalent to law of conservation of momentum
- we defined conservative forces and conservation of energy
- we also discussed
  - work
  - kinetic energy
  - potential energy

TODAY
Consider **dissipative** systems $\Rightarrow$ energy not conserved.

We distinguish two types of dissipative forces:

**Drag:** force between moving body and surrounding fluid (air,water...)

**Friction:** tangential motion-resisting force between solid bodies. We will discuss friction first.

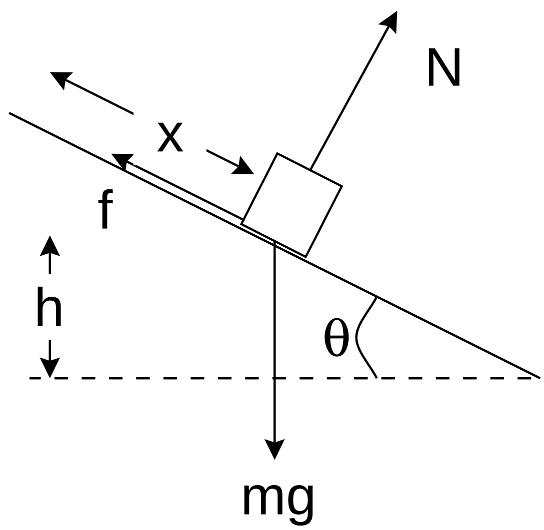
CAUSES OF FRICTION

- **Adhesion**. If two solids are pressed together, bonds are formed between surface atoms on opposite sides and they need to be broken as surfaces slide past each other
- **Asperities**. For rough surfaces to slide past over each other, one surface must be lifted over high spots of the other.
- **Ploughing and plastic deformation**. A hard material may scratch softer material. Asperities may deform in ways which do not recover after surfaces slide past each other.

- **Electrostatic effects** (in insulators).

These effects are too complex to compute theoretically (at least by me). Need **phenomenological** description!

STATIC FRICTION




Two surfaces are pressed together with a normal force $N$. If the surfaces are at rest, and we apply tangential force $F_{||}$, they will remain at rest if

$$F_{||} < \mu_s N$$

i.e. $F_{||}$ will be balanced by an opposing **frictional force**. $\mu_s =$ **coefficient of static friction**. Consider a block resting on an incline plane as in the figure. If the angle of the plane exceeds the**static angle of repose** $\theta_s = \tan^{-1}\mu_s$ the block will start sliding down the plane.

KINETIC FRICTION

If $F_{||} > \mu_s N$ sliding starts. The opposing frictional force is written

$$f = \mu_k N$$

$\mu_k$ is the **coefficient of kinetic friction**. If a block slides down a plane with angle of inclination less than $\theta_k = \tan^{-1}\mu_k =$ **kinetic angle of repose**, the friction cannot overcome the force of gravity and the block will decelerate. If $\theta > \theta_k$ the block will accelerate.

Generally $\mu_k < \mu_s$. When

$$\theta_k < \theta < \theta_s$$

**both** the state of rest and accelerated motion along inclined plane are possible. Which motion is selected depends on initial conditions!

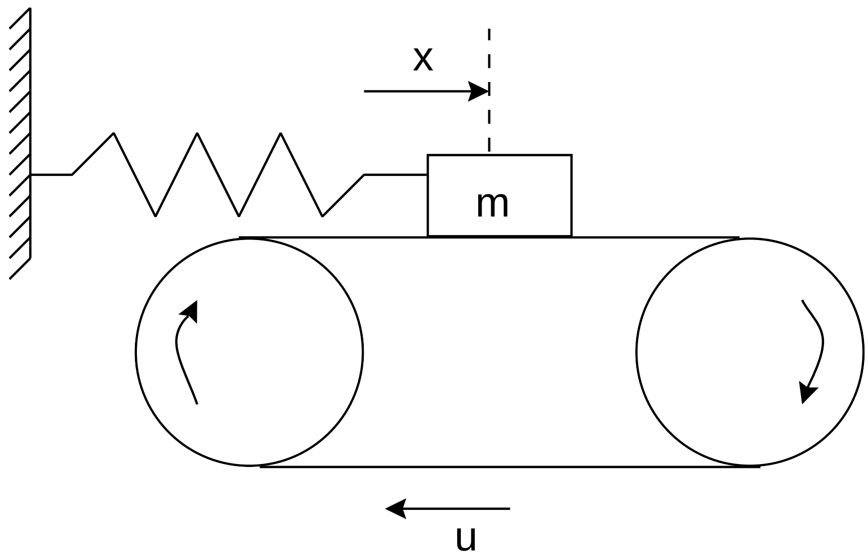
This is one reason why avalanches can be so treacherous. A peaceful looking snow slope can suddenly be set in motion by a "weak" cause such the sound of a plane flying by.

"LAWS" OF FRICTION

- $\mu_k$ and $\mu_s$ are, to a good approximation, independent of $N$, surface area and surface roughness. However, the force of friction is sensitive to surface contamination (e.g. lubrication).
- $\mu_k$ is typically weakly dependent on speed with a minimum at a nonzero speed.
- We will see that we sometimes have an instability if

  $$\frac{d\mu_k}{dv} < 0$$

  The corresponding situation occur in some electronic circuits and is then called **negative resistance**.
- $\mu_s$ typically increases slowly with time.
- Often a good approximation to assume that $\mu_k$ independent of speed.
- The study of the effects of friction on moving machine parts (and of methods, such as lubrication, of obviating them) is called **tribology**.




STICK AND SLIP

As an example consider a slider block of mass $m$ that lies on top of a conveyor belt.

The block is attached to a fixed support by a friction-less spring with spring constant $k$.

The extension of the spring is $x$ and the speed of the conveyor belt is $u$.

The coefficients of static and kinetic friction are $\mu_s$ and $\mu_k$, respectively. We neglect the velocity dependence of $\mu_k$.

THE PHASE PLANE

Suppose we start with the block at rest with respect to the belt at $x = 0$. The block will first follow the belt until the spring force exceeds the static friction at which point it will start slipping.

The subsequent equation of motion is

$$m\frac{d^2x}{dt^2} + kx = mg\mu_k$$

It is instructive to transform this equation into two first order equations

$$\frac{dx}{dt} = v$$

$$m\frac{dv}{dt} = -kx + mg\mu_k$$

The **trajectories** of the blocks are then **curves** in the $v - x$ plane. We call this plane the **phase plane**.

We can eliminate time from the coupled first order equations. Multiply the second equation by $v$. The left hand side becomes

$$mv\frac{dv}{dt} = \frac{m}{2}\frac{dv^2}{dt}$$

Substituting $v = dx/dt$ on the rhs:

$$(-kx + mg\mu_k)\frac{dx}{dt} = -\frac{1}{2}\frac{d}{dt}k(x - \frac{mg\mu_k}{k})^2$$

Hence

$$\frac{d}{dt}[\frac{m}{2}v^2 + \frac{k}{2}(x - \frac{mg\mu_k}{k})^2] = 0$$

and the trajectories are the **ellipses**

$$\frac{m}{2}v^2 + \frac{k}{2}(x - \frac{mg\mu_k}{k})^2 = const$$

We rescale the ellipses into circles by

$$s = \frac{v}{\mu_k g}\sqrt{\frac{k}{m}}; \quad y = \frac{k}{mg\mu_k}x$$

to obtain

$$s^2 + (y-1)^2 = c^2$$

LIMIT CYCLE:

The phase plane coordinates of the **slip point** $P$ are

$$v_0 = u; \quad x_0 = \frac{mg\mu_s}{k}$$

and become in the new units

$$s_0 = \frac{u}{\mu_k g}\sqrt{\frac{k}{m}}; \quad y_0 = \frac{\mu_s}{\mu_k}$$

If the block starts out at rest with respect to the conveyor belt it will follow the line $s = s_0$ to the right until it reaches $y = y_0$. It will then keep repeating a path in which it follows the circle

$$(y-1)^2 + s^2 = s_0^2 + (y_0 - 1)^2$$

until it **sticks** to the belt at

$$s = s_0, \; y = 1 - (y_0 - 1) = 2 - y_0$$

it will then follow the path along $s = s_0$ until it again reaches $y = y_0$.

The periodic trajectory is an **attractor** to trajectories which starts close to it. Such an orbit is called a **limit cycle**.

EQUILIBRIUM POINT

We could also have started the system with

$$v = 0; \quad x = \frac{mg\mu_k}{k}$$

The forces are then balanced and the block will remain at rest.

We have assumed that

$$\mu_k = const$$

in or model. If $\mu_k$ has a **minimum** as a function of velocity for some

$$v_m > u$$

the equilibrium will become unstable.

The **rosin** which a violin player rubs on the bow has the property that the kinetic friction decreases with increasing velocity up to high velocities. The purpose of rubbing with rosin is to make the undesirable equilibrium point unstable.

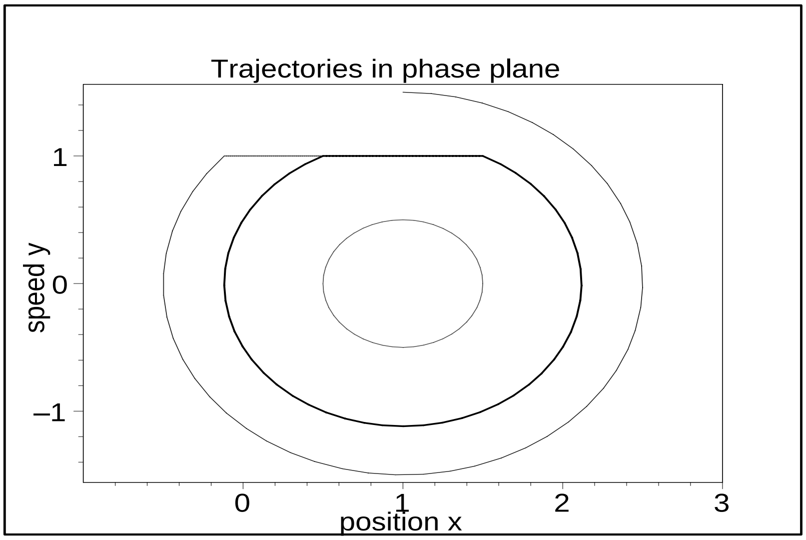
SMALL OSCILLATIONS

Next, start the block at rest with respect to the spring support with extension $x$ close to, but not at, the equilibrium value. The block will then exercise simple harmonic motion around the equilibrium. If the spring has a small damping the trajectories will slowly spiral in towards equilibrium (e.g. if we don't rub rosin on the block and the belt!).

Some trajectories are shown below. A Maple worksheet producing the trajectories is given at
http://www.physics.ubc.ca/ birger/n206l2.mws .

SUMMARY
We have

- discussed the laws of dry friction
- analyzed a slider block system
- defined the phase plane
- showed that it exhibited
  - a limit cycle




– an equilibrium point which could be stable or unstable

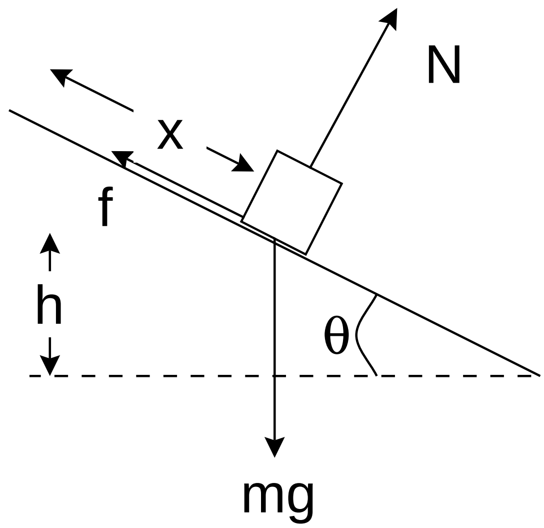
– small oscillations about the equilibrium

FURTHER READING

- **laws of friction** see Rabinowicz[15]
- **snow avalanches** see McClung and Shaerer[13]
- **slider block model for earth quakes**
  - "classic" see Burridge and Knopoff[3]
  - text see Sholtz[18]

**Example problems**

**Problem 2.2.1**
(Problem 3 of problem set 2 2000)
A block is put on an inclined plane as shown in the figure
The angle $\theta$ of inclination is increased slowly from zero until the static angle of repose is reached and the block starts to move (see figure). The angle is




then held constant. Find the subsequent motion assuming the coefficients of dynamic and static friction are $\mu_k$ and $\mu_s$ respectively

**Problem 2.2.2**
(Problem 3 of 2002 problem set 1)
A particle slides down an inclined plane which forms an angle $\theta = 30^o$ with the horizontal (see figure). It starts out with a speed of 1 m $s^-1$ and stops after it has traveled 1 m along the plane. What is the kinetic angle of repose?

**Problem 2.2.3**
(Problem 2 of 2002 problem set 3)
An object is projected upwards along an inclined plane which makes $45^o$ with the vertical. The initial speed is 10 $ms^{-1}$. The coefficient of kinetic friction is 0.1. How far up the plane will it reach? What is the speed when it comes back where it started? Assume the coefficient of static friction is too small to hold the object.

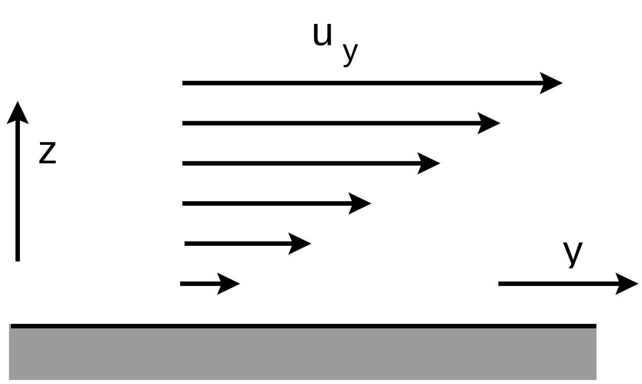
## 2.3 Drag. Dimensional analysis

LAST TIME

- discussed the laws of dry friction
- analyzed a slider block system
- defined the phase plane for this system
- showed that it exhibited
  - a limit cycle
  - an equilibrium point which could be stable or unstable
  - small oscillations about the equilibrium

TODAY discuss **drag**, i.e. the resistance to motion that is felt by a body moving through a fluid.

VISCOSITY
The concept of viscosity dates back to Newton who considered **laminar** fluid flow. In this type of flow a fluid moves in parallel layers in such a way that we have a **velocity profile**.




A body moving in the $y-$direction relative to the fluid sets up a velocity gradient $du_y/dz$. Newton assumed that

$$\frac{F_y}{A} = -\eta \frac{du_y}{dz}$$

A fluid obeying the above law is called **Newtonian**. The constant of proportionality is the **viscosity**. The tangential force/area is defined as the **shear stress**.

UNITS OF VISCOSITY

- In SI-units the unit of viscosity is

$$N\ s\ m^{-2} = Pa\ s = kg\ m^{-1}s^{-1}$$

($N =$ Newton, $Pa =$ Pascal)

- Another unit of viscosity is

$$1\ centipoise = 10^{-3}kg\ m^{-1}s^{-1} = 10^{-2}g\ cm^{-1}s^{-1}$$

- Some people prefer to use the **kinematic viscosity** $\nu$ defined as

$$\nu = \frac{\eta}{\rho}$$

where $\rho$ is the mass density of the fluid ($kg\ m^{-3}$)

- The SI unit of kinematic viscosity is $m^2 s^{-1}$.

- A non-SI unit of kinematic viscosity is the centistoke:

$$1\ centistoke = 10^{-6}m^2\ s^{-1} = 10^{-2}cm^2 s^{-1}$$

PROPERTIES OF VISCOSITY

- The important fluids air,water and oil are all Newtonian to a very good approximation.

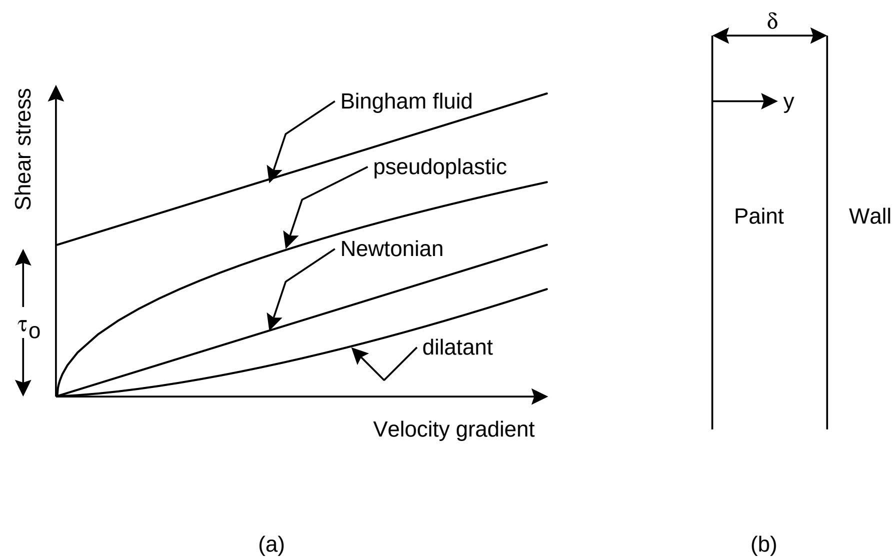
- Viscosity of water $\approx 1$ *centipoise*. The viscosity of air at room temperature is
$\approx 2 \times 10^{-5} Pa\ s = 2 \times 10^{-2} centipoise.$

- There are many Non-Newtonian fluids.

If shear stress increase with $du_y/dz$

- slower than linear $\Rightarrow$ fluid is dilatant

- linear $\Rightarrow$ fluid is Newtonian




- faster than linear ⇒ fluid is pseudoplastic.
- If there is a stress-threshold for flow ⇒ we have what is known as a Bingham fluid.

Comments:

- Examples of Bingham fluids: **Paints, pastes, jellies**. Stress threshold prevents paint from running if spread thinly. It can still be spread by applying brush.
- Example of pseudoplastic: **blood**. Decreased resistance to flow at higher speeds allows it to flow in narrow arteries.
- example of dilatant: **corn starch dissolved in water**. Solution flows easily, but fluid behaves as solid under strong impact.

STOKES FLOW

We return to the problem of estimating the drag force on falling object in Newtonian fluid.

- Expect drag force to increase with viscosity $\eta$ of fluid.
- Expect drag to increase with size of object. Parameterize size by diameter $d$.
- Expect drag force to depend on speed $u$ of object
- At **very** low speeds flow surrounding object **laminar.**
- Laminar the flow is steady flow. We therefore don't expect the inertia of fluid to matter.
- If acceleration plays no role drag should not depend on the mass density $\rho$ of the fluid. The only fluid property that matters is the vscosity.

DIMENSIONAL ANALYSIS
When the flow is laminar expect the drag-force to obey a law of the form

$$F = constant \; d^{\alpha} u^{\beta} \eta^{\gamma} \tag{1}$$

Dimensional analysis has nothing to tell us about the multiplying constant. However, as we shall see, it allows us to determine the constants $\alpha$, $\beta$ and $\gamma$, while the constant will depend on the **shape** of the object!

Let [ ] denote **"dimension of"**. We have

$$[F] = kg \; m \; s^{-2}; \; [d] = m;$$

$$[u] = m \; s^{-1}; [\eta] = kg \; m^{-1} s^{-1}$$

Dimensional consistency of (1) requires

$$\begin{array}{llcll} mass & \rightarrow & 1 & = & \gamma \\ length & \rightarrow & 1 & = & \alpha + \beta - \gamma \\ time & \rightarrow & -2 & = & -\beta - \gamma \end{array}$$

Hence $\alpha = \beta = \gamma = 1$. We get **Stokes law** (For spherical objects $const. = 3\pi$)

$$F = const. \eta d u$$

TYPES OF FLOW

If the speed of the object passing through the fluid is increased the nature of the flow undergoes a number of qualitative changes.

An excellent elementary discussion of what happens next is given in the Feynman lectures[5] (Volume II sect 41).

- Even for very low speeds vortices start to form around moving objects.
- As the speed increases the vortices shed into a "von Kármán vortex street".
- At still higher speed we get intense vorticity in thin boundary layers.

On what do the critical speeds $u_c$ for transitions to different regimes depend?

REYNOLDS NUMBER

Turbulent flow is unsteady and inertia now plays a role. Viscosity and size of object will still be important. Try

$$u_c = const.\eta^{\alpha}\rho^{\beta}d^{\gamma}$$

$[u_c] = m\ s^{-1};\ [\eta] = kg\ m^{-1}s^{-1},$
$[\ \rho = density] = kg\ m^{-3};\ [d] = m$
Dimensional consistency now requires

$$\begin{array}{lcccl} mass & \rightarrow & 0 & = & \alpha + \beta \\ length & \rightarrow & 1 & = & -\alpha - 3\beta + \gamma \\ time & \rightarrow & -1 & = & -\alpha \end{array}$$

Giving $\alpha = 1;\ \beta = -1;\ \gamma = -1$ or

$$u_c = const\frac{\eta}{\rho d}$$

We define the **Reynolds number as**

$$\boxed{Re = \frac{u\rho d}{\eta}}$$

DRAG COEFFICIENT

We now assume a drag force law

$$F = d\ u\eta f(Re)$$

where $f$ is a function of the geometry of the object. In the case of a sphere at very low Reynolds number , the drag force is given by $f(Re) = 3\pi$. We could equally well have used drag law

$$F = \rho u^2 d^2 C_D(Re)$$

By comparing we find that

$$f(Re) = Re\ C_D(Re)$$

$C_D$ is the customary definition of the **drag coefficient**

PHYSICAL SIMILARITY

An important consequence of the above argument is the idea that flows with the same Reynolds numbers are **similar**. This allows us to estimate important properties in complicated situation by using **scale models**.

EXAMPLE

Suppose we need to be able to estimate the terminal velocity of an object which is to be dropped from high altitude. When the terminal velocity $u_m$ is reached the drag force $F_o$ on the object is equal and opposite to the force of gravity:

$$F_o = mg$$

We have at our disposal an 1:10 scale model ($d_o = 10 d_m$) of the object and a wind tunnel in which we can measure the force $F_m$ on the model for an adjustable flow speed $u_m$.

For physical similarity the Reynolds numbers in the two situations must be the same

$$\frac{u_o \rho_o d_o}{\eta_o} = \frac{u_m \rho_m d_m}{\eta_m}$$

If the density $\rho$ and the viscosity $\eta$ is the same in the two cases

$$u_m d_m = u_o d_o$$

The drag force

$$F = \rho u^2 d^2 C_D(Re)$$

will then be the same in the two cases.

**The terminal velocity is one tenth of the air speed in the wind tunnel, when the drag force is $mg$.**

Suppose next one wishes to make the experiment in a water tank instead of a wind tunnel. We have

$$\eta_m = 10^{-3} Pa\ s; \ \ \eta_o = 2\ 10^{-5} Pa\ s$$

$$\rho_m = 10^3\ kg\ m^{-3}; \ \ \rho_o = 1.29\ kg\ m^{-3}$$

Physical similarity then requires that

$$u_m d_m \frac{1000}{10^{-3}} = u_o d_o \frac{1.29}{2\ 10^{-5}}$$

or $u_m = 0.64 u_o$ for a 10:1 scale model. The ratio between the drag forces is then:

$$\frac{F_m}{F_o} = \frac{1000}{1.29} 0.64^2 \frac{1}{10^2} = 3.2$$

**If the flow speed in the tank is adjusted so that the force is $3.2\,mg$ the speed will be 0.64 times the terminal speed of the object in air!**

Alternatively, we could increase **pressure** of wind tunnel. The viscosity of a gas is approximately independent of pressure. The density is proportional to pressure. Increase the pressure by a factor of 10, the velocity should be kept unchanged for Reynolds number to stay the same. The drag force on the model would be $1/10^{th}$ of the real force!

BEHAVIOR OF DRAG COEFFICIENT

- For most object shapes the drag force is approximately $\propto u$ for $Re \leq 1$. The drag coefficient is then inversely proportional to the Reynolds number.

- In the approximate range $10 < Re < 10^3$ the drag coefficient $C_D(Re)$ is approximately proportional to the inverse square root of $Re$.

- For Reynolds numbers up to about $10^5$ $C_D(Re)$ stays about constant.

- If the Reynolds number is increased further there is a sudden drop in the drag coefficient and the behavior then depends, not only on the Reynolds number, but on the roughness of the sphere's surface (hence the dimples on the golf-ball).

- If the velocity of flow approaches that of sound, the above analysis breaks down.

SUMMARY

- We defined the **viscosity** of a fluid and **Newtonian** and **non-Newtonian fluids**.

- **Dimensional analysis** was used to analyze **drag**.

- We defined the **Reynolds number** and the **drag coefficient**.

- We introduced concepts of **physical similarity** and **scaling**.

- We reviewed the empirical behavior of the **drag coefficient**.

We have seen that the equations of motion in mechanics problems involve one or more differential equations. If you don't know how to solve these equations, or if the solution is complicated, you can often save yourself a lot of work by using a software package such as Maple. You will find some hints on this in the Maple worksheet located at
http://www.physics.ubc.ca/ birger/n206l4.mws (or .html)

**Example problems**

### Problem 2.3.1

(Problem 3 of 1999 problem set 3)
An object with mass $m$ characteristic size $d$ is falling in air $\rho = 1.29\ kg\ m^{-1}$, $g = 10m\ s^{-2}$. The drag coefficient is $C_D = \sqrt{1000/Re}$ for $Re < 1000$,

$C_D = 1$ for $Re > 1000$. ($Re$ is the Reynolds number). The viscosity of air is $\eta = 2\ 10^{-5} Pa\ s$. What is the terminal velocity if

**a:** $m = 1\ kg,\ d = 0.1m$.

**b:** $m = 10^{-3} kg,\ d = 10^{-2} m$.

**c:** $m = 10^{-6} kg,\ d = 10^{-3} m$.

Compare with the speed the object would have reached in free fall from a height of 100 m and zero drag.

**Problem 2.3.2**
(Problem 2 of 2000 problem set 2)
A 1kg object is dropped from height 1000m with zero initial velocity. Assume the drag coefficient $C_D$ defined in Lecture2.3 is *constant* = 1. What is the speed when it hits the ground, and how long does it take to get there? Assume the density of air is 1.29 $kg\ m^-3$, the diameter of the falling object is 0.07 $m$.

**Problem 2.3.3**
(Problem 1 of 2001 problem set 2)
Assume that the drag coefficient of a spherical object moving through a fluid such as air or water can be fitted to

$$C_D = \frac{\alpha}{Re} + \beta$$

$\alpha = 7.75$ and $\beta = 0.17$. The drag force is related to the drag coefficient by

$$F = \rho u^2 d^2 C_D$$

$\rho (\approx 1.29\ kg\ m^{-3}$ for air at STP) mass density of fluid

$d =$ diameter of object

$u =$ speed of object relative to the fluid

$Re$ is the Reynolds number

$$Re = \frac{u \rho d}{\eta}$$

$\eta(\approx 2 \times 10^{-5}\ Pa\ s$ for air at STP) viscosity of fluid

Find the terminal velocity of a falling

**a:** Soap bubble of diameter $d = 0.01\ m$ and mass $10^{-7} kg$

**b:** Rain drop of diameter $10^{-4} m$ and mass $0.5 \times 10^{-9} kg$

**c:** Basket ball of diameter $0.25\ m$ and mass $0.6\ kg$.

**Problem 2.3.4**
(Problem 2 of 2001 problem set 2)

**a:** In the three cases of the previous problem, how long will it take for the objects listed above to reach 50% of the terminal velocity starting from rest.

**b:** How far have the objects fallen when 50% of the terminal velocity is reached? (This may be a good place to practice the numerical methods discussed in class using Maple, if you don't have the time to do this, you can get a rough estimate by assuming that, when the objects start falling the drag coefficient is small, and the distance traveled is of the order $gt^2/2$.)

**Problem 2.3.5**
(Problem 3 of 2001 problem set 2)
A particle of mass $M$ and initial speed $v$ is moving horizontally. How far will the body travel before it comes to rest if the only net force on the particle is

**a:** a force of friction

$$F = Mg\mu_k$$

**b:** a drag force proportional to the speed of the particle.

**c:** a drag force proportional to the square of the speed?

**Problem 2.3.6**
(Problem 4 of 2002 problem set 2)

A particle of mass $m$ is moving horizontally with initial velocity $v_0$. It is subject to a drag force

$$F = \alpha v + \beta v^2$$

How far will it travel before it comes to rest?

**Problem 2.3.7**
(Problem 1 of 2002 problem set 2)
A badminton bird falls vertically from rest. Let $y(t)$ be the distance it has traveled. Experimentally it was found that $y(t)$ can be fitted to the formula

$$y(t) = \frac{v_T^2}{g} \ln(\cosh(\frac{gt}{v_T}))$$

where

$$g = 9.81 \text{m s}^{-2}, v_T = 6.8 \text{m s}^{-1}$$

Plot the drag force on the bird for velocities $< v_T$

**Problem 2.3.8**
(Problem 1 of 2001 midterm)
**a:**
Two particles move along the same path under the influence of a conservative force. The potential energy along the path is the same for both particles, but one of them has twice the mass of the other. What is the ratio of the times taken to traverse the path?
**b:**
Two particles follow the same path under the influence of a conservative force. They have the same mass but the potential energy of the first is twice that of the other. What is the ratio of the times taken to traverse the path?

## 2.4 Divergence and curl. Stokes theorem.

LAST Lectures

- Discussed **friction**.
- Defined **viscosity** and **Newtonian** and **non-Newtonian fluids**.

- **Dimensional analysis** was used to analyze **drag**.
- Defined the **Reynolds number**. .
- Reviewed the behavior of the **drag coefficient**.

TODAY I start to discuss problems in 2 and 3 dimensions. In one dimension forces that only depend on **position** (not on velocity) are always **conservative**. In Lecture 2.1 I argued that in higher dimension force is **conservative** if its line integral between two points

$$-\int_{\vec{r}_i}^{\vec{r}_f} F \cdot d\vec{r}$$

is **independent of the path**.

POSITION DEPENDENT FORCE FIELDS

Today I will restrict my attention to forces that depend on position alone (i.e. not on the speed of the particle). In this case we assign a force vector $\vec{F}$ to every point $\vec{r}$ in space. We call $\vec{F}(\vec{r})$ a **vector field**.

In lecture2.1 we introduced the vector operator $\nabla$ , or **nabla** (or del) by

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

When $\nabla$ operates on a **scalar** we call it the **gradient**.

We also showed in lecture 2.1 that if the line integral over the force was independent of the path we could construct a potential $V(\vec{r})$ so that

$$\nabla V = -\vec{F}$$

The three figures below illustrate three different force fields. The first is the field associated with the force

$$F_x = y; \quad F_y = x$$

and is clearly not conservative. The two others are the fields associated with the potentials

$$V = -x^2 - y^2; \;\; V = -x^2 + y^2$$

respectively and are conservative. The figures were produced from the Maple worksheet
http://www.physics.ubc.ca/~birger/n206l5a.mws

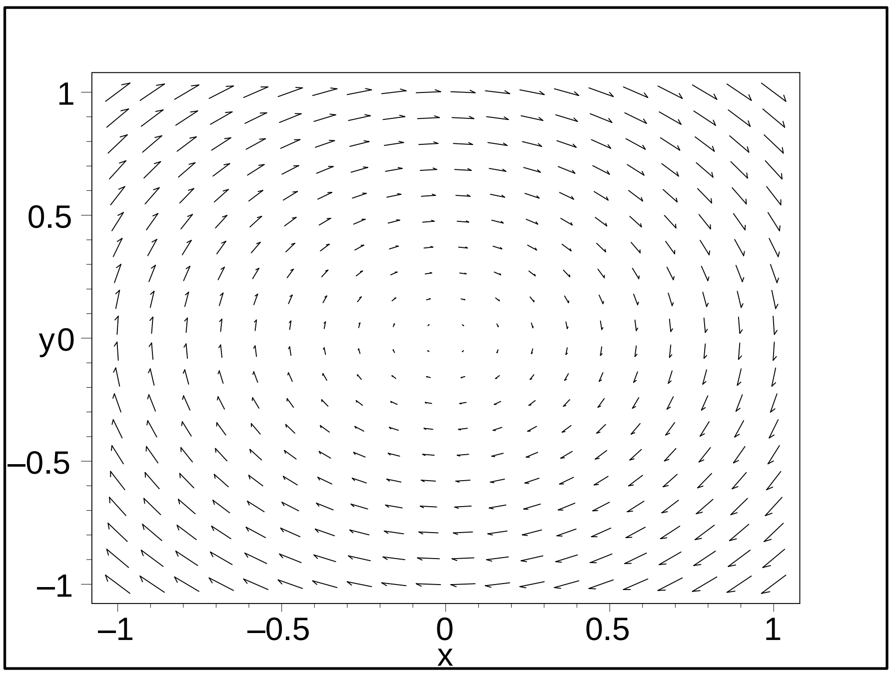



Figure 1: A nonconservative force field.

THE DIVERGENCE AND THE CURL
Just as in ordinary vector multiplication there are two ways of applying the operator $\nabla$ to a vector field:

The **divergence** is defined as the **scalar**

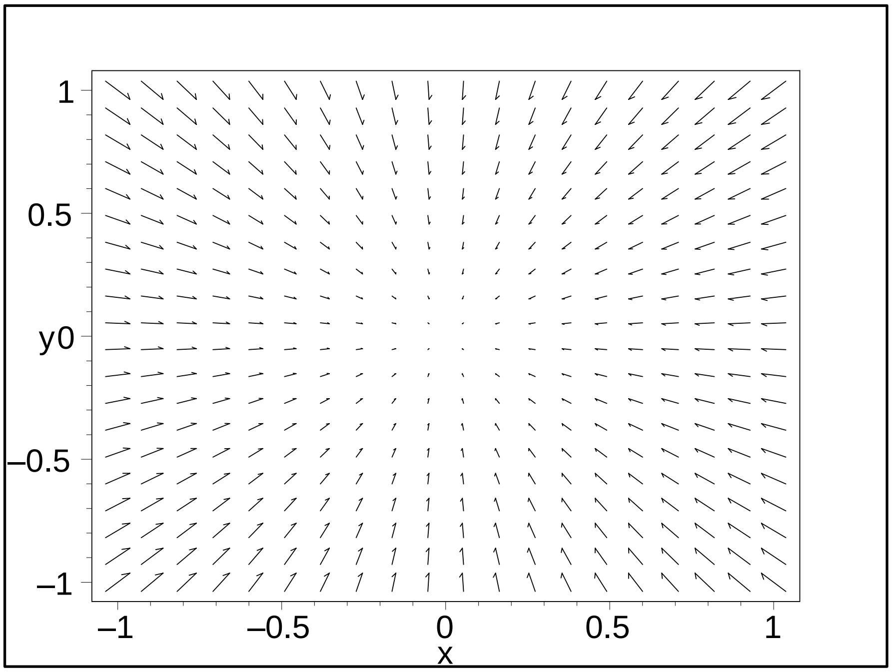



Figure 2: A conservative force field.

$$\nabla \cdot \vec{F} \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

We will not have much to say about the divergence for the time being.

The **curl** (sometimes called rot) is defined as the **vector product**

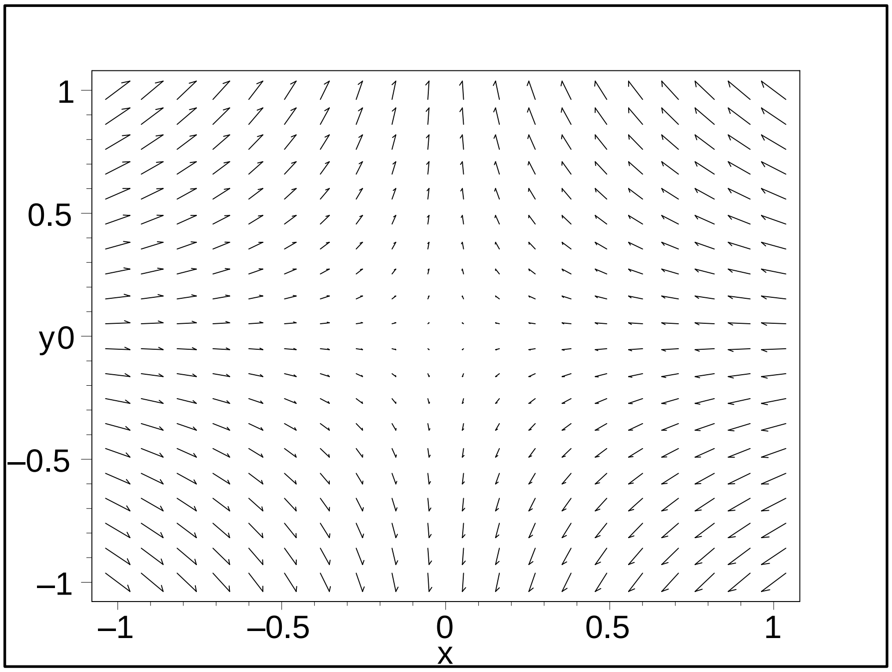



Figure 3: Another conservative force field.

$$\nabla \times \vec{F} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{\mathbf{i}}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \hat{\mathbf{j}}(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \hat{\mathbf{k}}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})$$

We now argue that a necessary and sufficient condition for a force field to be conservative is that

$$\nabla \times \vec{F}(\vec{r}) = 0$$

**everywhere** in the region of interest. You will get a more detailed discussion in MATH 217 (or 317).

First note that the curl of a gradient is identically zero!.

$$\nabla \times \nabla V = 0$$

To see this substitute

$$F_x = -\frac{\partial V}{\partial x};\ F_y = -\frac{\partial V}{\partial y}; F_z = -\frac{\partial V}{\partial z};$$

into

$$\nabla \times \nabla V = \hat{\mathbf{i}}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \hat{\mathbf{j}}(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \hat{\mathbf{k}}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})$$

Since

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}$$

we see that **each second derivative occurs twice with opposite sign**. Hence the terms add up to zero!

I will next attempt to give a physical interpretation of the curl of a vector field in terms of the line integral along a closed path around a surface element.

Consider a closed line integral around a little square with sides $dx$ and $dy$ in the $x - y - plane$.

$$\oint_{infinitesimal} \vec{F} \cdot d\vec{r}$$

$$= \int_x^{x+dx} F_x(x', y)dx' + \int_y^{y+dy} F_y(x + dx, y')dy'$$

$$- \int_x^{x+dx} F_x(x', y + dy)dx' - \int_y^{y+dy} F_y(x, y')dy'$$

$$= (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})dxdy$$

In the last step I have used

$$F_y(x + dx, y') = F_y(x, y') + dx\frac{\partial F_y}{\partial x}$$

$$F_x(x', y + dy) = F_x(x', y) + dy\frac{\partial F_x}{\partial y}$$

Next we note that

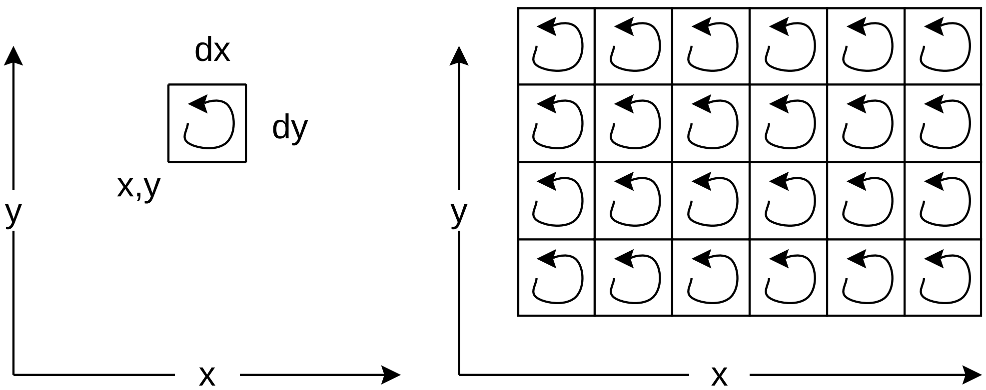
$$= (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) = (\nabla \times \vec{F})_z$$

and introduce the **vector surface element**

$$d\vec{a} = \hat{n} dx dy$$

where $\hat{n}$ is the **unit normal** to the surface area element ($\hat{\mathbf{k}}$ here). We find

$$\oint_{infinitesimal} \vec{F} \cdot d\vec{r} = \nabla \times F \cdot d\vec{a}$$




If we now consider the line integrals around a large loop as the sum over line integrals over many little loops we get **Stokes theorem**

$$\oint \vec{F} \cdot d\vec{r} = \int_A \nabla \times \vec{F} \cdot d\vec{a}$$

**The closed loop integral over any vector function $\vec{F}$ is the area integral over the projection of $\nabla \times \vec{F}$ on the normal to the enclosed surface A! If the curl=0 everywhere the force field is conservative!**

We illustrate Stokes' law and how to evaluate the curl in the Maple work sheet available at
http://www.physics.ubc.ca/~birger/n206l5b.mws

If you check out this work sheet you will learn that Stokes law must be applied with some caution. You may easily verify that the force field

$$F_x = y/(x^2 + y^2);\ F_y = -x/(x^2 + y^2);\ F_z = 0$$

has $\nabla \times \vec{F} = 0$, but the line integral of the force around the origin is clearly nonzero (see figure below). The problem arises from the singularity at the origin where the curl is undefined. Vector fields of this type are important in understanding vortices in fluid flow, and they play an important rôle in the theories of superfluidity and superconductivity.

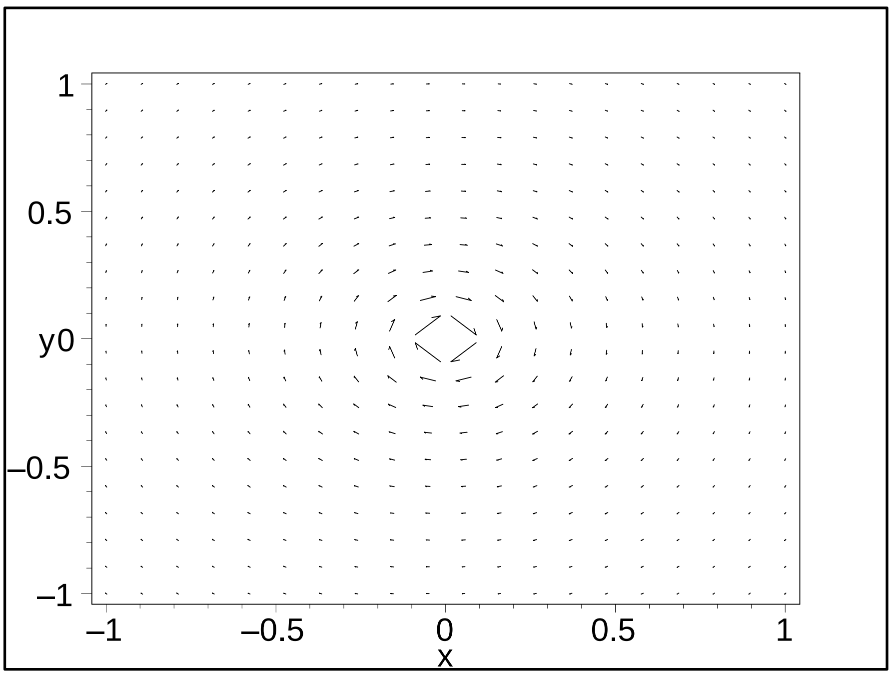



Figure 4: The curl of this field is zero everywhere, (except at the origin), but it is still non-conservative!

SUMMARY
We have

- discussed force fields in two and three dimensions.
- defined the divergence and curl

- demonstrated Stokes theorem
- showed that a conservative potential has no curl
- showed that if the curl of a field vanishes identically in a region the force field is conservative.
- showed by an example that singularities in the force field can be treacherous.
- showed how force fields and vector calculus are handled by Maple.

**Example problems**

**Problem 2.4.1**
(Problem 4 of 1999 problem set 4)

**a:** For what value of the constant c is the force field

$$\vec{F} = xy\hat{\mathbf{i}} + cx^2\hat{\mathbf{j}} + z^3\hat{\mathbf{k}}$$

conservative?

**b:** Is the force field

$$\vec{F} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z^3\hat{\mathbf{k}}$$

conservative?

**Problem 2.4.2**
(Problem 4 of 2000 problem set 2)

**a:** Is the force field

$$\vec{F} = axyz\hat{\mathbf{i}} + cyx^2\hat{\mathbf{j}} + z^3\hat{\mathbf{k}}$$

conservative for any values of the constants $a$ and $c$?

**b:** Is the force field

$$\vec{F} = zy\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

conservative?

# 3 Lagrangian mechanics.

## 3.1 Generalized coordinates and forces

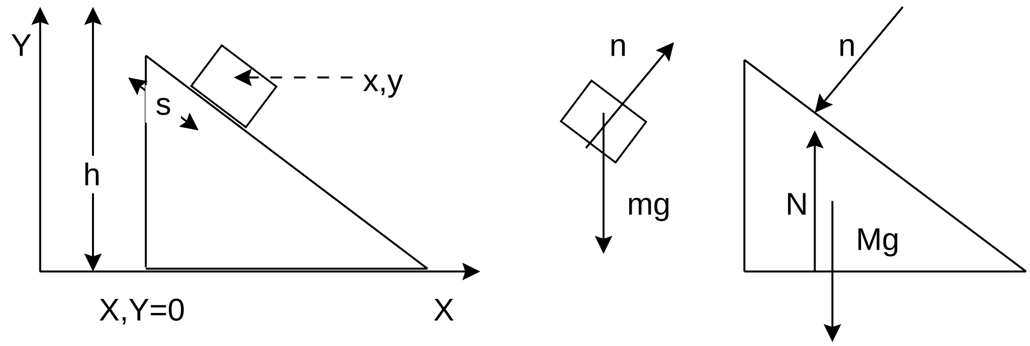
LAST TIME

- Discussed force fields in two and three dimensions and defined the divergence and curl.
- Demonstrated Stokes theorem.
- Showed that $\nabla \times \vec{F} = 0$ for conservative force field.
- Showed that if the curl of a field vanishes identically in a region where the force field is conservative.
- Found that singularities in the force field can be treacherous.
- Showed how force fields and vector calculus is handled by Maple.

TODAY
Start moving away from Newtonian mechanics and introduce an alternative **Lagrangian** approach. The motivation for this is:

- When solving complicated problems it is often easier to work with **scalar** quantities such as the kinetic and potential energies than it is to work with **vector** quantities such as forces and torques.
- Mechanics problems commonly involve **constraints**, most commonly due to rigidity of bodies in contact. These constraints are maintained by normal forces and other reaction forces. Often these forces are not of any particular interest to us and life would be easier if we could avoid making explicit references to them altogether.

EXAMPLE: Consider the following problem:
A block slides without friction down a wedge which in turn is free to move without friction horizontally as shown in figure. Let $X, Y$ and$x, y$ be the coordinates of representative points on the wedge and block respectively.




Let $s$ represent distance down the inclined plane of the wedge relative some position at height $h$. The constraint conditions are then

$$\begin{aligned} x &= X + s\cos\alpha \\ y &= h - s\sin\alpha \\ Y &= 0 \end{aligned} \tag{2}$$

where $\alpha$ is the angle of the wedge. The constraints are maintained by **normal forces** $\vec{N}, \vec{n}$ (see free body diagrams above). The equations of motion are then

$$\begin{aligned} M\ddot{X} &= -n\sin\alpha \\ M\ddot{Y} &= -Mg + N - n\cos\alpha \\ m\ddot{x} &= n\sin\alpha \\ m\ddot{y} &= -mg + n\cos\alpha \end{aligned} \tag{3}$$

The sets of equations (2) and (3) constitute seven equations altogether. We can use five of them to eliminate the constraint forces $N$ and $n$ and the coordinates $Y, x, y$. We are left with two **equations of motion** for $X$ and $d$ respectively. We find after some algebra

$$(M+m)\ddot{X} + m\ddot{s}\cos\alpha = 0$$

$$m\ddot{s} + M\ddot{X}\cos\alpha = mg\sin\alpha$$

The variables $s$ and $X$, which remain after we have eliminated the constraints are examples of **generalized coordinates**. We refer to the number of generalized coordinates as the **number of degrees of freedom** (two in the present problem[1]. We can choose generalized coordinates in many different ways, but their number is an intrinsic property of the problem and always the same.

EXAMPLE

Consider the dumbbell depicted to the left below. Assume that the distance $r$ is fixed. We have one equation of constraint

$$r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

There are thus 5 degrees of freedom. A possible choice of generalized coordinates would be the coordinates $x, y, z$ of the center of mass and the polar coordinate $\theta$ and $\phi$ of the vector connecting the two masses.

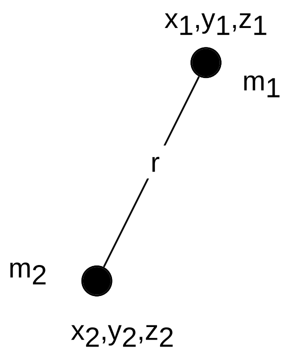
The distances between the vertices of a triangle is fixed. There are thus 3 equations of constraints and 9 coordinates. This leaves us with 6 degrees of freedom. A possible choice of generalized coordinates would be $x, y, z$ coordinates of the center of mass, polar coordinates $\theta, \phi$ of the normal $\hat{n}$ to the plane of the triangle, and the angle $\psi$ of rotation of the triangle with respect to $\hat{n}$.

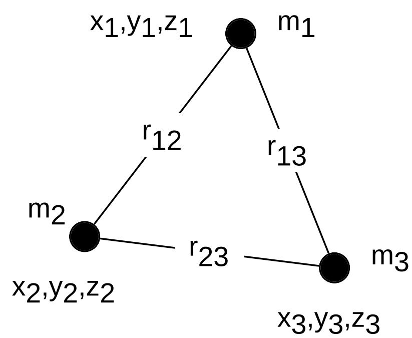
SOME DEFINITIONS

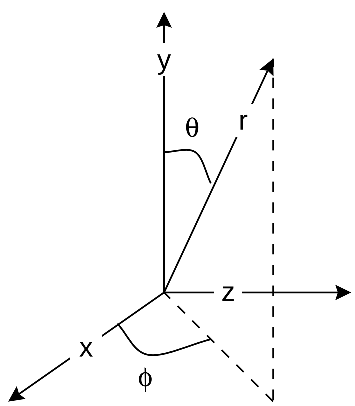
Suppose we have a mechanics problems described by $M$ coordinates $x_1, x_2 ... x_M$ in an inertial frame of reference. Let there be $M - N$ equations of constraints so that the systems can be described by $N$ generalized coordinates $q_1, q_2 \cdots q_N$. If the relationship between the generalized and the old coordinates can be written

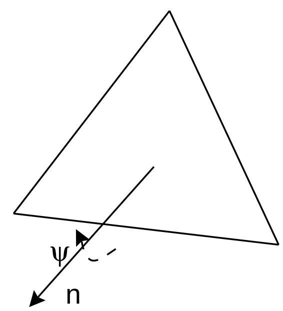
$$\begin{aligned} x_1 &= x_1(q_1, q_2 \cdots q_N, t) \\ x_2 &= x_2(q_1, q_2 \cdots q_N, t) \\ &\cdots \\ &\cdots \\ x_M &= x_M(q_1, q_2 \cdots q_N, t) \end{aligned} \tag{4}$$

[1]Beware! In courses of thermodynamics such as PHYS 203, 313 degrees of freedom are counted differently. Sometimes two thermodynamic degrees of freedom make up one mechanical degree)




we say that the constraints are **holonomic**. For the constraints to be holonomic the functions $x_1(\cdots q_i \cdots), x_m(\cdots q_i \cdots)$ must not depend explicitly on the **generalized velocities**

$$\dot{q}_i = \frac{dq_i}{dt}$$

The constraint of **rolling without slipping** is that the velocity is zero at the point of contact, between a rolling object and the plane on which it rolls. Such constraints are in general **non-holonomic**.

For the constraints to be holonomic they must be given by equations not inequalities. The condition that a system of particles are confined within a box of a certain volume is a **non-holonomic** constraint.

Note that the conditions relating the coordinates in the inertial frame to the generalized coordinates may depend explicitly on time (e.g the generalized coordinates may be coordinates in a frame of reference which is moving non uniformly). The constraints are still holonomic. If there is no explicit time dependence the constraints are **scleronomic** otherwise they are **rheonomic**.

We have if **all constraints are holonomic**

$$\dot{x}_i(q_1 \cdots q_N, \dot{q}_1 \cdots \dot{q}_N, t) = \sum_k \frac{\partial x_i(q_1 \cdots q_N, t)}{\partial q_k}\dot{q}_k + \frac{\partial x_i(q_1 \cdots q_n, t)}{\partial t}$$

or

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i(q_1 \cdots q_N)}{\partial q_k}$$

This relationship will prove useful in what follows (it is referred to as dot cancelation in Hand and Finch[9]. Note that dot cancelation will in general be violated if constraints are velocity dependent!

The projections of the forces on the generalized coordinates are called **generalized forces**. Let $F_i$ be the component of the system force acting on the $i$th coordinate. The $k$th generalized force is then

$$\mathcal{F}_k = \sum_i F_i \frac{\partial x_i}{\partial q_k}$$

If the forces are **conservative** they can be derived from a potential

$$V(x_1, x_2 \cdots x_M) = \mathcal{V}(q_1, q_2 \cdots q_N)$$

$$F_i = -\frac{\partial V}{\partial x_i}$$

we find for the generalized force

$$\mathcal{F}_k = -\sum_i \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} = -\frac{\partial \mathcal{V}}{\partial q_k}$$

we will modify the definition of generalized force slightly in lecture 3.5

SUMMARY We have introduced some new concepts which will be very important in what follows. In particular we have defined:

- generalized coordinates, velocities and forces.

- holonomic, scleronomic and rheonomic constraints.
- degrees of freedom.

Next time we will use these concepts and define the **Lagrangian** and derive the **Lagrangian equations of motion**.

**Example problems**

**Problem 3.1.1**
(Problem 1 of 2000 problem set 2)
Consider a chain of particles connected in such a way the distance between the successive particles in the chain is kept constant but the angle of between bonds is random (freely jointed chain, a very idealized model of a polymer). How many degrees of freedom are there if the chain has N particles and the ends are free. Find a set of generalized coordinates to describe the situation

**(a)** For a chain in three dimensions.

**(b)** For a chain constrained to lie in a plane.

## 3.2 Lagrangian equations of motion

LAST TIME

- Introduced generalized coordinates, velocities and forces.
- Defined holonomic, scleronomic and rheonomic constraints.
- Showed how to determine number of degrees of freedom.

TODAY
we will use these concepts and define the **Lagrangian** and derive the **Lagrangian equations of motion**.

Let us assume that we are dealing with a system which can be described by the Cartesian coordinates

$$x_1, x_2 \cdots x_M$$

in an inertial reference frame. The kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{M} m_i \dot{x}_i^2$$

The equations of motion are given by Newton's second law

$$\dot{p}_i = m_i \ddot{x}_i = F_i$$

where $p_i$ is the $i$th component of the momentum, and $F_i$ is the component of the force acting in the direction of $x_i$. These forces may include **holonomic forces of constraint**. Let us assume that when these have been eliminated we have $N$ degrees of freedom described by generalized coordinates

$$\begin{aligned} x_1 &= x_1(q_1, q_2 \cdots q_N, t) \\ x_2 &= x_2(q_1, q_2 \cdots q_N, t) \\ &\cdots \\ &\cdots \\ x_M &= x_M(q_1, q_2 \cdots q_N, t) \end{aligned}$$

We now imagine that these equations have been substituted into the expression for the kinetic energy

$$T(\cdots \dot{x}_i \cdots) = \mathcal{T}(\cdots q_k \cdots, \cdots \dot{q}_k, t)$$

We have

$$\frac{\partial \mathcal{T}}{\partial q_k} = \sum_{i=1}^{M} m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k} = \sum_{i=1}^{M} p_i \frac{\partial \dot{x}_i}{\partial q_k}$$

Next

$$\dot{x}_i(\cdots q_k \cdots, \cdots \dot{q}_k \cdots, t) \equiv \frac{dx(\cdots q_k \cdots, t)}{dt} = \sum_{k=1}^{N} \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}$$

and we find

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i(q_1 \cdots q_N, t)}{\partial q_k}$$

and

$$\frac{d}{dt}\frac{\partial x_i}{\partial q_k} = \sum_{j=1}^{N} \frac{\partial^2 x_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 x_i}{\partial q_k \partial t} = \frac{\partial \dot{x}_i}{\partial q_k}$$

We use these result to write

$$\frac{\partial \mathcal{T}}{\partial q_k} = \sum_{i=1}^{M} p_i \frac{d}{dt} \frac{\partial x_i}{\partial q_k} \tag{5}$$

and

$$\frac{\partial \mathcal{T}}{\partial \dot{q}_k} = \sum_{i=1}^{M} m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \sum_{i=1}^{M} p_i \frac{\partial x_i}{\partial q_k} \tag{6}$$

Next take the time derivative of (6)

$$\frac{d}{dt} \left( \frac{\partial \mathcal{T}}{\partial \dot{q}_k} \right) = \sum_{i=1}^{M} \left( \dot{p}_i \frac{\partial x_i}{\partial q_k} + p_i \frac{d}{dt} \frac{\partial x_i}{\partial q_k} \right) \tag{7}$$

For the first term inside the sum on the right hand side of(7) we use the definition of generalized force from lecture 3.1

$$\mathcal{F}_k = \sum_{i=1}^{M} F_i \frac{\partial x_i}{\partial q_k}$$

while for the second term we use (9) to find

$$\frac{d}{dt} \left( \frac{\partial \mathcal{T}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{T}}{\partial q_k} = \mathcal{F}_k$$

We know assume that the generalized force can be split up into two contributions

$$\mathcal{F}_k = -\frac{\partial \mathcal{V}(\cdots q_k, \cdots)}{\partial q_k} + Q_k$$

The first contribution is a **conservative force** derived from a **velocity independent potential**. The second term represents "left-overs" such as e.g. friction or drag forces.

We also define the **Lagrangian**

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

Since we have assumed that the potential and the constraints are velocity independent

$$\frac{\partial \mathcal{V}}{\partial \dot{q}_k} = 0$$

we find **Lagrangian equation of motion**

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k$$
$$k = 1, 2 \cdots N$$

Note that there is one equation for each degree of freedom. In the special case that all forces are conservative and derived from a velocity independent potential

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

You might think that we have only taken something relatively simple (Newton's equations) and made it into something complicated (Lagrange's equations). Actually, this is not the case: we will find that in most cases the Lagrangian approach is the easiest to work with, when solving problems of even moderate difficulty. Furthermore, we will find that the more complicated the problem the greater the advantage of using the Lagrangian approach.

EXAMPLE
As our first example consider the slider block problem of the lecture 3.1. The kinetic energy is in terms of the generalized coordinates $s$ and $X$ is

$$\mathcal{T} = \frac{M\dot{X}^2}{2} + \frac{m}{2}[(\dot{X} + \dot{s}\cos\alpha)^2 + \dot{s}^2\sin^2\alpha]$$

The potential energy is

$$\mathcal{V} = mg(h - s\sin\alpha)$$

We have

$$\frac{\partial \mathcal{L}}{\partial \dot{s}} = m\cos\alpha\dot{X} + m\dot{s}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{X}} = (m + M)\dot{X} + m\cos\alpha\dot{s}$$

$$\frac{\partial \mathcal{L}}{\partial X} = 0$$

$$\frac{\partial \mathcal{L}}{\partial s} = mg \sin \alpha$$

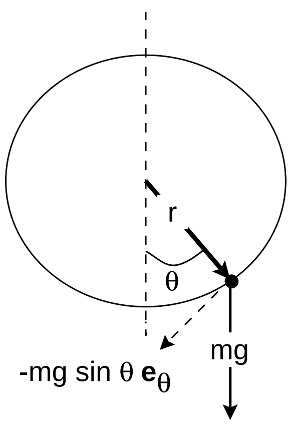
The equations of motion are thus

$$(m + M)\ddot{X} + m \cos \alpha \, \ddot{s} = 0$$

$$m \cos \alpha \ddot{X} + m\ddot{s} - mg \sin \alpha = 0$$

These equations of motion are the same as we found last time using Newtonian mechanics. The main difference is that there is no need to bother about the free body diagram or the normal forces $N$ and $n$.

EXAMPLE
**The pendulum**




Instead of using the Cartesian coordinates of the mass it is convenient describe the motion by the angle $\theta$. The length of the pendulum, $r$, is assumed to be constant (i.e. is a holonomic constraint).

The kinetic and potential energies are

$$\mathcal{T} = \frac{mr^2\dot{\theta}^2}{2}; \mathcal{V}(\theta) = -mgr \cos \theta$$

With $\mathcal{L} = \mathcal{T} - \mathcal{V}$ the equation of motion becomes

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}) = -mgr \sin\theta - mr^2 \frac{d\dot{\theta}}{dt} = 0$$

giving the equation of motion:

$$\ddot{\theta} + \frac{g}{r} \sin\theta = 0$$

We will discuss the properties of the solutions to the equation of motion in lectures3.7 and 3.8

SUMMARY
We have defined the Lagrangian

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

and derived the Lagrangian equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k, \quad k = 1 \cdots N$$

where $Q_k$ are generalized nonconservative forces. If all forces are conservative $Q_k = 0$.

**Example problems**

**Problem 3.2.1**
(Problem 1 of 2000 problem set 3)
A bead of mass $m$ slides without friction along the hyperbola $xy = c = const$. Gravity acts in the negative $y$ direction. Use the constraint to eliminate the coordinate $x$ from the kinetic energy, and write down the Lagrangian, and the equation of motion for the coordinate $y$.

**Problem 3.2.2**
(Problem 2 of 2000 problem set 3)
Find the equations of motion for an "elastic pendulum": a particle of mass $m$ is attached to an elastic string of stiffness $K$ and un-stretched length $l_0$. Assume the mass moves in a fixed vertical plane.

**Problem 3.2.3**
(Problem 3 of 2000 problem set 3)
The point of support of a simple pendulum is being elevated at constant acceleration a so that the height of the support is $at^2/2$ the vertical velocity $at$. The acceleration of gravity is $g$. Find the differential equation of motion for the motion of the pendulum in the accelerating frame.

**Problem 3.2.4**
(Problem 4 of 2000 problem set 3)
In class we found the equations of motion for a block of mass $m$ that is free to slide down a frictionless wedge of mass $M$ the angle of the wedge is $\theta$. The wedge in turn is free to slide on a smooth horizontal surface. Solve the equations of motion assuming the block and wedge starts rest and the time is too short for the block to reach the bottom of the wedge.

**Problem 3.2.5**
(Problem 1 of 2001 problem set 4)
**a:** A weightless spring of stiffness $k$ is connected to a mass $m$. The system is moving horizontally. The kinetic energy of the system is

$$T = \frac{m}{2}\dot{x}^2$$

while the potential energy is

$$V = \frac{k}{2}x^2$$

Show that the Lagrangian equation of motion is

$$m\ddot{x} + kx = 0$$

The period of oscillation of this system is

$$\tau = 2\pi\sqrt{\frac{m}{k}}$$

**b:** Assume that the mass $M$ of the spring is significant and is distributed uniformly along the spring. Also assume that the velocity of any part of the spring is proportional to its distance from the point of suspension. Find the period of oscillation of the mass spring system.

**Problem 3.2.6**
(Problem 2 of 2001 problem set 4)
The mass spring system of problem 3.2.5 is suspended vertically so that gravity plays a role. Also the un stretched length of the spring is not zero but $L$ giving for the potential energy of the mass $m$

$$V = -mgx + \frac{k(x-L)^2}{2}$$

**a:**
What is now the period of oscillation, assuming the spring is mass-less?
**b:**
The spring is not mass-less but distributed uniformly along its length as in problem 1b. Find the period of oscillation.

**Problem 3.2.7**
(Problem 3 of 2001 problem set 4)
Two blocks of equal mass m are connected by a flexible rope of length $L$. One block is placed on a smooth horizontal table, the other hangs over the edge. Find the Lagrangian equation of motion with the height $h$ of the hanging mass as a generalized coordinate, if

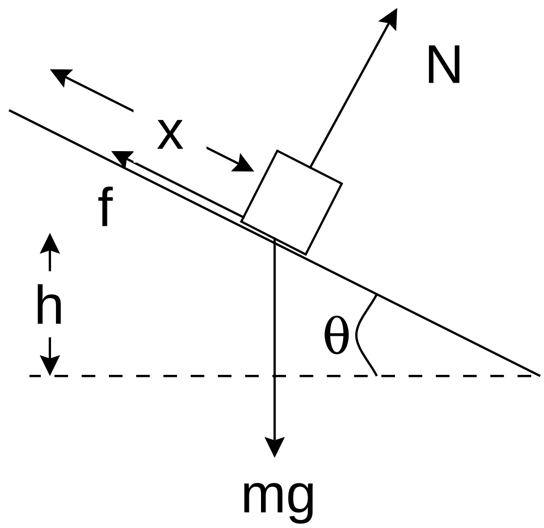
- **a:** The mass of the rope can be neglected.
- **b:** The mass $M$ of the rope is uniformly distributed along its length.

**Problem 3.2.8**
(Problem 2 of 2002 problem set 3)

A block is put on a frictionless inclined plane as shown in the figure. The plane is moving up and down vertically with amplitude

$$z = a \sin \omega t$$

**a:** Find the equation of motion using $x$ as a generalized coordinate.
**b:** Under what conditions on $a$ and $\omega$ will the block start to rattle, because the normal force is inadequate to keep the block down on the plane.




**Problem 3.2.9**
(Question 3 of 2002 problem set 3)
Write down the Lagrangian for a double pendulum restricted to move under gravity in a vertical plane. A mass $m_1$ is connected by a light rod of length $l_1$ to a fixed support and a mass $m_2$ is connected to $m_1$ with a rod of length $l_2$. Use as generalized coordinates the angle $\theta_1$ and $\theta_2$ of the rods with the vertical. Find the generalized momenta associated with these coordinates.

**Problem 3.2.10**
(Question 2 of 2002 midterm)
Two particles, each of mass m are connected by a massless spring with spring constant $k$ and un-stretched length $l_0$. The masses are constrained to move along the $x-$axis of the system.
**a:** Select a set of generalized coordinates and write down the equations of motion for the system.
**b:** Solve the equations of motion assuming that initially the particle are separated by the distance $l_0$. The particle to the left is initially at rest, while the other particle has initial velocity $v_0$ to the right.

**Problem 3.2.11**
(Question 3 of 2002 midterm)

A pendulum of mass $m$ is constrained to move in a vertical plane and has a fixed support. The length $r$ of the pendulum is varied sinusoidally

$$r = r_0 + a\sin(\omega t)$$

**a:** Find the equation of motion using the angle $\theta$ with the vertical as generalized coordinate.
**b:** Simplify the equation of motion in the small angle approximation $\cos\theta \approx 1$, $\sin\theta \approx \theta$.

**Problem 3.2.12**
(Question 2 of 2001 midterm)
The potential energy of a particle moving in the $x - y$ plane is

$$\mathcal{U}(x, y) = \frac{k}{2}(x^2 - 2y)$$

with $k > 0$, i.e. the potential energy is quadratic in $x$ and linear in $y$. There is no friction. The kinetic energy is

$$\mathcal{T} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

**a:** Write down the equations of motion.
**b:** The particle starts with initial velocity $\dot{x}(0) = v_0, \dot{y}(0) = 0$ from the point $x_0 = 0, y_0 = 0$. Find the subsequent motion and describe the motion qualitatively.

**Problem 3.2.13**
(Question 3 of 2000 midterm)
A bead of mass $m$ can slide without friction along a horizontal circular hoop of radius $r$

$$(x - x_0)^2 + y^2 = r^2$$

The $x-$ component of the center of the hoop undergoes forced harmonic motion

$$x_0 = a\sin\omega t$$

Write down the equation of motion for the bead using the angle $\theta$ as generalized coordinate.

$$x - x_0 = r\cos\theta;\ y = r\sin\theta$$

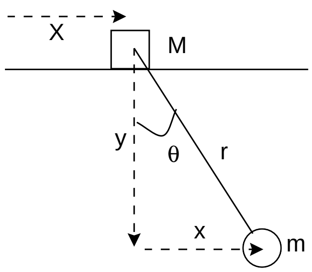
**Problem 3.2.14**

(Question 1 of 1999 midterm)

A bead of mass $m$ slides without friction along a helix, which in cylindrical coordinates $(r, \phi, z)$ can be described as

$$\phi = \omega z, \;\; r = a$$

where $\omega$ and $a$ are constants ($2\pi/\omega$ is the pitch of the helix). Gravity acts in the negative $z$ direction. The bead starts with zero velocity at height z=0. Find $z(t)$ for the subsequent motion.




**Problem 3.2.15**

(Question 2 of 1999 midterm)

The point of support of a simple (rigid) pendulum (length $r$ mass $m$) is forced to move in the horizontal direction according to

$$x = a\sin(\omega t)$$

where $\omega$ is constant and $t$ is time. Find how the movement of the support will modify the differential equation of motion for the angle $\theta$.

## 3.3 Calculus of variations.

LAST TIME

We defined the Lagrangian

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

and derived the Lagrangian equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k, \quad k = 1 \cdots N$$

where $Q_k$ are generalized nonconservative forces. If all forces are conservative $Q_k = 0$.

TODAY
We will derive the **Euler-Lagrange** equations of **calculus of variations**.

The generic problem of variational calculus is the following:
Consider a function

$$L(z(t), \frac{dz}{dt}, t)$$

which depends on another function $z(t)$ in region

$$t_1 \leq t \leq t_2$$

Typically $L$ is a function of dimension **energy**, or represents an incremental **cost, gain or profit.**

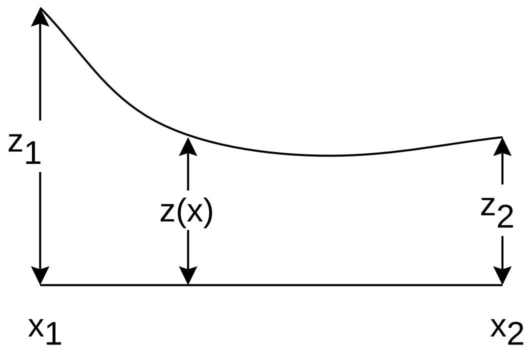
**Problem:**
Find the **function** $z(t)$ for which the **action**

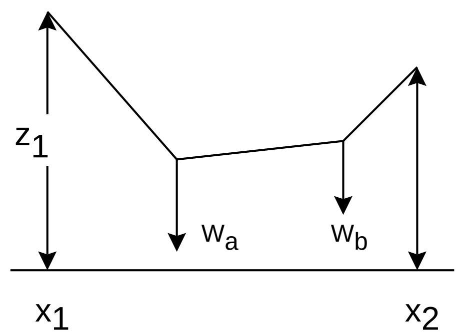
$$S = \int_{t_1}^{t_2} L(z(t), \frac{dz}{dt}, t)dt$$

is an **extremum** (e.g. maximum, minimum). To make this problem well posed we must add to it boundary conditions at the ends.

Calculus of variation plays a role in this course because we can reformulate **Lagrangian dynamics** as a problem in calculus of variations. This reformulation turns out to be particlar important when considering extensions of classical mechanics such as **general relativity, quantum mechanics** and **statistical mechanics**. The calculus of variation comes into its own right in a number of economically significant **optimization problems**. The connection between mechanics and variational calculus can thus be helpful in finding solutions to problems which at first sight appear quite "non-mechanical".

**Example 1:**
A particle slides under the influence of gravity along a frictionless **slide** $z(x)$ starting from rest. The particle starts at height $z_1$, horizontal position $x_1$.




It ends up at $z_2$, $x_2$. How do we design the slide $z(x)$ so that the particle traverses it in the shortest possible time?

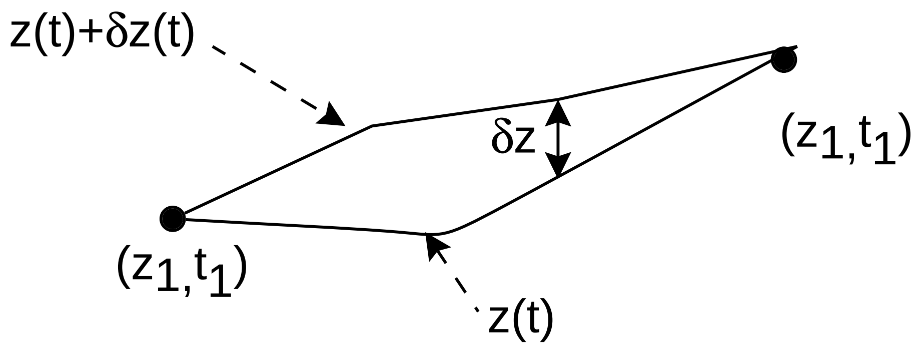
**Example 2:**
A rubber band is strung between to poles a horizontal distance $d$ apart. The height of the pole at $x = 0$ is $z_1$. At the other end the height is $z_2$. The elastic energy of the band depends on its length S. The band carries a load $W(x)$. What is the shape $z(x)$ of the band which minimizes the gravitational + elastic energy?

**Example 3:**
A clothesline is hung between to fixed supports a horizontal distance $d$ apart. The height of the support at $x = 0$ is $z_1$. At the other end the height is $z_2$. The length of the clothesline is $S$. The clothesline carries a load $W(x)$. What is the shape $z(x)$ of the clothesline which minimizes the gravitational potential energy?

Problem 3 is different from the other two problems because there is a **constraint** (length of the line), in addition to boundary conditions at ends of the clothesline.

We will first consider the situation where there are no additional constraints.

Return to the generic problem:




Let us use the notation

$$z' \equiv \frac{dz}{dt}$$

rather than $\dot{z}$ since $t$ need not represent time. Consider two different functions

$$z(t),\ and\ z(t) + \delta z(t)$$

where $\delta z$ is infinitesimal. The derivatives too will be infinitesimally different in the two cases

$$z'(t)\ and\ z' + \delta z'$$

where

$$\delta z' = \frac{d}{dt}\delta z$$

We require that at the ends

$$0 = \delta z(t_1) = \delta z(t_2)$$

The **variation** in S is

$$\delta S = \int_{t_1}^{t_2} dx[L(z + \delta z, z' + \delta z', t) - L(z, z', t)]$$

For extremum

$$0 = \delta S = \int_{t_1}^{t_2} dt[L(z + \delta z, z' + \delta z', t) - L(z, z', t)]$$

**for an arbitrary variation** $\delta z$.
We have

$$\delta S = \int_{t_1}^{t_2} dt[\delta z(t)\frac{\partial L}{\partial z} + (\frac{d}{dt}\delta z)\frac{\partial L}{\partial z'}]$$

We integrate the last term by parts

$$\int_{t_1}^{t_2} dt(\frac{d}{dt}\delta z)\frac{\partial L(z,z',t)}{\partial z'}$$

$$= \delta z(t)\frac{\partial L(z,z',t)}{\partial z'}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\delta z(t)\frac{d}{dt}\left(\frac{\partial L(z,z',t)}{\partial z'}\right)$$

Since

$$\delta z(t_1) = \delta z(t_2) = 0$$

we must have

$$\delta z(t)\frac{\partial L(z,z',t)}{\partial z'}\bigg|_{t_1}^{t_2} = 0$$

We are left with

$$\delta S = \int_{t_1}^{t_2} dt\delta z(t)[\frac{\partial L(z,z',t)}{\partial z} - \frac{d}{dt}(\frac{\partial L(z,z',t)}{\partial z'})]$$

For $\delta S$ to be zero for arbitrary variations $\delta z(t)$ the expression in the square bracket [ ] must be identically zero. This gives us the

**EULER-LAGRANGE EQUATION**

$$\boxed{\frac{\partial L(z,z',t)}{\partial z} - \frac{d}{dt}(\frac{\partial L(z,z',t)}{\partial z'}) = 0}$$

The **total** derivative with respect to $t$ involves derivatives with respect to **both** explicit and implicit dependence:

$$\frac{d}{dt}(\frac{\partial L(z,z',t)}{\partial z'})$$

$$= \left( \frac{\partial}{\partial t} + \frac{dz}{dt}\frac{\partial}{\partial z} + \frac{d^2 z}{dt^2}\frac{\partial}{\partial z'} \right) \frac{\partial L(z, z', t)}{\partial z'}$$

The above results can easily be generalized to the case where there is more than one dependent variable. Suppose we want to find extremal values of the integral

$$S = \int_{t_1}^{t_2} L(z_1 \cdots z_N, \frac{dz_1}{dt} \cdots \frac{dz_N}{dt}, t) dt$$

assuming that $z_1 \cdots z_N$ have fixed values at the endpoints $t_1, t_2$ of the integration.

We now carry out **independent** variations

$$z_i \Rightarrow z_i + \delta z_i$$

for **each dependent variable** $z_i$. Following the same procedure as before we find

$$\frac{d}{dt}\frac{\partial L}{\partial \frac{dz_i}{dt}} - \frac{\partial L}{\partial z_i} = 0$$

for all $i = 1 \cdots N$.

SUMMARY

- We isolated a generic class of **variational problems**.
- The goal was to find a **function** $z(x)$ for which the integral over some property $L(z, dz/dx, x)$ has an extremal value.
- We derived the **Euler-Lagrange** equation for the solution to the problem.
- If we have several dependent variables $z_1(t), z_2(t), z_N(t)$ we get $N$ Euler-Lagrange equations, one for each dependent variable.

**Example problems**

**Problem 3.3.1**
(Problem 3 of 2000 problem set 4)
The principle of least action can be extended to "Lagrangians" that contains higher time derivatives than first of the generalized coordinate $q_i$. Show that, if

$$S = \int_a^b dt \mathcal{L}(q_i, \dot{q}_i, \ddot{q}_i) = extremum$$

subject to fixed values of $q_i$ and $\dot{q}_i$ at the ends, the corresponding Euler-Lagrange equation becomes

$$\frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) + \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Apply this result to obtain the equation of motion for

$$\mathcal{L} = -\frac{mq\ddot{q}}{2} - q^2$$

## 3.4 Examples of calculus of variations problems.

LAST TIME

- We isolated a generic class of **variational problems**.
- The goal was to find a **function** $z(t)$ for which the integral over some property $L(z, dz/dt, t)$ had an extremal value.
- We derived the **Euler-Lagrange** equation for the solution to the problem.
- If we have several dependent variables $z_1(t), z_2(t), z_N(t)$ we get $N$ Euler-Lagrange equations, one for each dependent variable.

Today we will begin with working out as an example the slide problem introduced last time. The problem is called the **brachistochrone problem** and is historically important (Bernoulli late 1600's):

Assume that the particle starts at rest at horizontal position $x = 0$, height $z = 0$. Suppose the lower end of the slide is at $x = d,\ z = -h$ (we let $z$ be negative in the downwards direction).
¿From conservation of energy

$$mgz = -\frac{mv^2}{2}$$

We have a choice to calculate either $x(z)$ or $z(x)$. Selecting $x$ as the dependent variable we write

$$v^2 = v_x^2 + v_z^2 = v_z^2(1 + (\frac{dx}{dz})^2)$$

This gives ($x' \equiv \frac{dx}{dz}$)

$$t = \int_0^h \frac{dz}{v_z} = \int_0^h dz\sqrt{\frac{1 + (x')^2}{-2gz}}$$

or

$$S \equiv t;\ L(x, x', z) \equiv \sqrt{\frac{1 + (x')^2}{-2gz}}$$

The Euler Lagrange equation now reads

$$\frac{\partial L(x, x', z)}{\partial x} - \frac{d}{dz}(\frac{\partial L(x, x', z)}{\partial x'}) = 0$$

$L$ does not depend **explicitly** on $x$ so

$$\frac{d}{dz}\frac{\partial}{\partial x'}\sqrt{\frac{1 + (x')^2}{-2gz}} = 0$$

or

$$\frac{\partial}{\partial x'}\sqrt{\frac{1 + (x')^2}{-2gz}} = c_1$$

$$\frac{x'}{\sqrt{(1+(x')^2)(-2gz)}} = c_1$$

where $c_1$ is a constant. Remembering that

$$x' = \frac{dx}{dz} = \frac{1}{dz/dx}$$

we find

$$-z[1+(\frac{dz}{dx})^2] = c = const$$

or

$$\frac{dz}{dx} = \sqrt{\frac{c+z}{-z}} \tag{8}$$

We note that

- This equation can be integrated to yield a solution with one additional constant $c_2$.
- The constants $c$ and $c_2$ (or combinations of them) can be determined from the heights at $x = 0$ and $x = d$.
- For small $x$ and $z$, $\frac{dz}{dx} \propto z^{-1/2}$. i.e. the optimum slide will start out with vertical slope.

Using the method of "the inspired guess" we see that equation (1) can be solved by making the substitution

$$z = -c\sin^2\frac{\theta}{2}$$

into the differential equation

$$\frac{dz}{dx} = \frac{dz}{d\theta}\frac{d\theta}{dx} = \pm\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

There is a sign ambiguity here. We let positive $\theta$ correspond to positive $x$, negative $z$, and pick the minus sign above. Next with

$$\frac{dz}{d\theta} = -c\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

we find

$$c \sin^2 \frac{\theta}{2} \frac{d\theta}{dx} = 1$$

Employing the trigonometric relation

$$\sin^2(\frac{\theta}{2}) = \frac{1 - \cos\theta}{2}$$

we obtain with

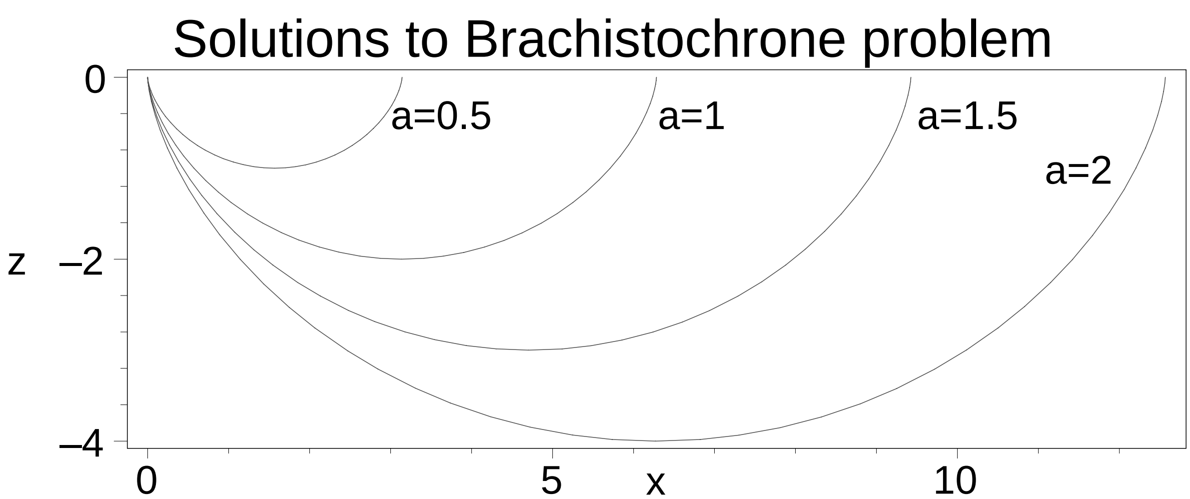
$$a = \frac{c}{2}$$

$$dx = a(1 - \cos\theta) d\theta$$

and we obtain the parametric relations (letting $x = z = 0$ correspond to $\theta = 0$)

$$x = a(\theta - \sin\theta); \quad z = -a(1 - \cos\theta)$$

which are equations for a **cycloid**, the trajectory in the $x - z$-plane of a point on the rim of a wheel of radius a rolling along the $x-$axis. We plot the curves for some values of $a$ below




We will encounter these curves again in lecture 4.1

We discuss the solution in more detail in the Maple worksheet
http://www.physics.ubc.ca/~birger/n206l9a.mws

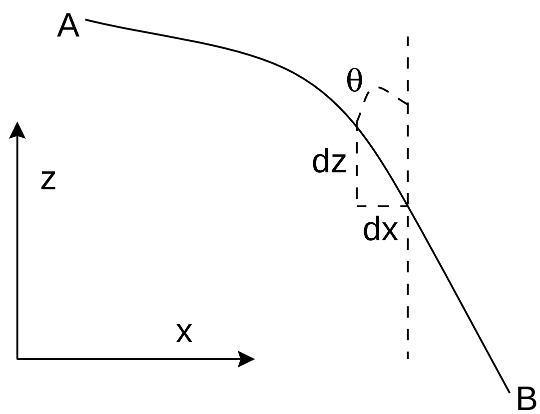
EXAMPLE

**Fermat's principle in optics.**

The (group) velocity of a light ray in a medium with **index of refraction** $n$ is

$$v = \frac{c}{n}$$

where $c$ is the velocity of light in vacuum. Fermat's principle states that the path taken by a ray of light in an inhomogeneous medium is the one which can be traversed in shortest possible time.




We assume that a ray starts at the point $A = (x_1, z_1)$ and ends at the point $B = (x_2, z_2)$. Hence

$$t = \frac{1}{c}\int_{z_1}^{z_2} dz\; n(x,z)\sqrt{1 + (\frac{dx}{dz})^2} = minimum$$

To be specific we assume that the index of refraction depends on $z$ only. Writing

$$L(x, \frac{dx}{dz}, z) = n(z)\sqrt{1 + (\frac{dx}{dz})^2}$$

and obtain

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \frac{dx}{dz}} = \frac{n(z)\frac{dx}{dz}}{\sqrt{1 + (\frac{dx}{dz})^2}}$$

Since $L$ doesn't depend on $x$ the Euler-Lagrange equation is

$$\frac{d}{dz}\frac{\partial L}{\partial \frac{dx}{dz}} = 0$$

or

$$\frac{n(z)\frac{dx}{dz}}{\sqrt{1+(\frac{dx}{dz})^2}} = constant. \tag{9}$$

We note (see figure above) that

$$\tan\theta = \frac{dx}{dz}; \;\; \sin\theta = \frac{\frac{dx}{dz}}{\sqrt{1+(\frac{dx}{dz})^2}}$$

This allows us to rewrite (2) as

$$n(z)\sin\theta = const$$

which you may recognize as **Snell's law** of geometrical optics. E.g. if light passes from one medium with index of refraction $n_1$ to another with index $n_2$ we must have

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

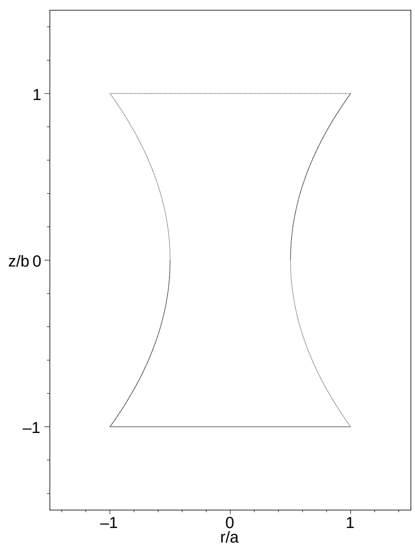
SUMMARY
We have given two historically important examples from the calculus of variation:

- the Brachistochrone problem
- Fermat's principle and Snell's law.
- We obtained numerical solutions to the first problem using Maple.

**Further reading:** The book by Fowkes and Mahony [6], offers several worked examples of the application of calculus of variations to non-mechanical problems. The two Mathematical methods books, Riley, Hobson and Bence [19] and Arfken and Weber[2] both have chapters on calculus of variation.

**Example problems**

**Problem 3.4.1**
(Problem set 5 2001)
Variational principle for a soap film.




Two parallel rings with equal radius $a$ are placed with their centers $2b$ apart on the $z-$axis. An axially symmetric soap film is stretched between the rings (see figure). Since (neglecting gravity) the free energy of the film is proportional to the surface area, the stable shape will be one that minimizes this quantity. Let $r(z)$ be the radius of the film at height $z$

**a:**
Show that the total surface area is given by

$$S = 2\pi \int_{-b}^{b} r\sqrt{1 + (\frac{dr}{dz})^2}dz$$

**b:**
The problem now is to find the function $r(z)$ that minimizes $S$. Use the "energy" first integral to find a first order differential equation for $r(z)$.

**c:**
Integrate the differential equation and show that the solution can be written

$$\alpha r = \cosh(\alpha z + \beta)$$

The curve $r(z)$ is called a catenary and is also the curve describing a heavy chain or cable of fixed length hanging from fixed supports (e.g. a power line hanging between two hydro poles). Here

$$\cosh(x) = \frac{e^x + e^{-x}}{2}; \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

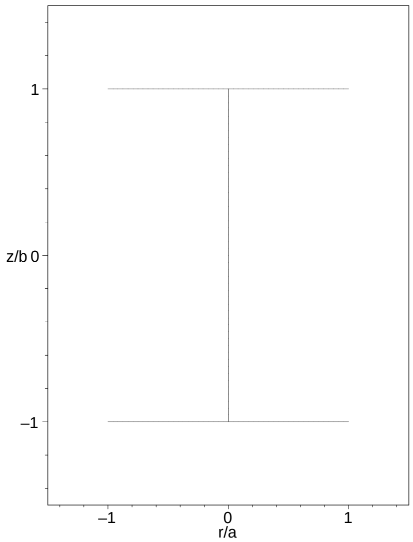
is the hyperbolic cosine and sine respectively, satisfying

$$\cosh^2(x) - \sinh^2(x) = 1; \quad \frac{d\cosh(x)}{dx} = \sinh(x); \quad \frac{d\sinh(x)}{dx} = \cosh(x)$$

**d:**
The constants $\alpha$ and $\beta$ are determined by the boundary conditions $r(\pm b) = a$. Because of the symmetry of the problem $\beta = 0$. Choosing the unit of length to be $a$, we are left with the transcendental equation

$$\alpha = \cosh(b\alpha)$$

by plotting the left and right hand side of the equation above for some values of $b$ show that the equation admits **two** solutions when $b$ is smaller than a critical value $b_c$. At $b_c$ there is one solution while for $b > b_c$ there are no real solutions. Find $b_c$ numerically to three significant figures
**e:**
For some values of $b$ the minimum surface consists of films stretched over the two rings and connected by a "wormhole" of infinitesimal radius (see figure). This is commonly referred to as Goldschmidt's solution.
Which of the three possible surfaces (corresponding to the two solutions of the transcendental equation and Goldschmidt's solution)has the smallest area if $b < b_c$, but close to that value?
**f:**
Pick a value of $b < b_c$ which is not close to $b_c$. Which surface has now the smallest area?
**Bonus question:**
You have probably learned in a differential equations course that trajectories don't cross. Here we have two solutions starting at $r = a, z = -b$ meeting at $r = a, z = +b$ (or three if we admit Goldschmidt's solution). How come?




## 3.5 Hamilton's principle

LAST TIMES

- We isolated a generic class of **variational problems**.
- The goal was to find a **function** $z(x)$ for which the integral

$$S = \int_{x_1}^{x_2} L(z(x), \frac{dz}{dx}, x) dx$$

over some property $L(z, dz/dx, x)$ had an extremum.

- We derived the **Euler-Lagrange** equation

$$\frac{\partial L(z, z', x)}{dz} - \frac{d}{dx}(\frac{\partial L(z, z', x)}{\partial z'}) = 0$$

for the solution to the problem.

- We analyzed, as an example, the brachistochrone problem and Fermat's principle.

PRINCIPLE OF LEAST ACTION
(or Hamilton's principle)

The Euler-Lagrange equation for the variational principle and the Lagrangian equation of motion (derived in lecture 3.2 for a conservative system) are of the same form. To see this let the independent variable $x$ represent time and the dependent variable $z$ be a generalized coordinate

$$L(z, z', x) \Rightarrow \mathcal{L}(q, \dot{q}, t)$$

This suggests that Newtonian mechanics can be derived from a variational principle. We now reformulate the mechanics of conservative systems to achieve this. We assume that a conservative system is characterized by a **Lagrangian** which is a functional of the generalized coordinates and velocities of the particles constituting the system. The actual trajectory is then the one for which the **action**

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L} dt$$

is extremal subject to boundary conditions at the endpoints. The integration is with respect to time. Usually the action will be a minimum, hence the name principle of least action.

We consider this to be a **postulate** taking the place of Newton's laws as the foundation of mechanics. It then remains to construct the Lagrangian -something which, of course, depends on the problem at hand. Also, we are not saying that Newton was wrong- we want the Lagrangian formulation to reproduce Newton's law when applicable.

**Caveat:** I mention in passing that the actual path is not always a minimum for the entire path, but only for sufficiently short segments. This is no problem in practice. In deriving the equation of motion we only use the extremum condition.

The Euler-Lagrange equation is linear in the Lagrangian. We can multiply $\mathcal{L}$ by a constant without changing the equation of motion. We choose the Lagrangian to have dimension of **energy**.

**Example**:
Particle in one dimension subject to velocity independent force:
We put $v = dx/dt$. If the Lagrangian is

$$\mathcal{L} = \mathcal{L}(x, v, t)$$

the equation of motion is

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial v}) = 0$$

We write $\mathcal{T} = \frac{1}{2}mv^2$ for the kinetic energy

$$\frac{\partial \mathcal{T}}{\partial x} = 0; \ \frac{d}{dt}(\frac{\partial \mathcal{T}}{\partial v}) = m\frac{dv}{dt} = ma$$

where $a$ is the acceleration. Similarly if $V(x)$ is the potential energy

$$\frac{\partial \mathcal{V}}{\partial x} = -f; \ \frac{\partial \mathcal{T}}{\partial v} = 0$$

If we put $\mathcal{L} = \mathcal{T} - \mathcal{V}$ we see that equation of motion becomes the familiar

$$f = ma$$

GENERALIZED COORDINATES
In our previous example $x$ was the Cartesian coordinate of the particle. It need not be, as discussed lecture 3.1. We may use any set of generalized coordinates that amount to imposing holonomic constraints on Cartesian coordinates.

MANY DEGREES OF FREEDOM
Most often we are dealing with systems requiring a number of generalized coordinates to describe the motion.

Suppose $N$ coordinates are required to specify the motion (after we have substituted for the holonomic constraints). We say that the system has $N$ **degrees of freedom**.
The variational principle is now

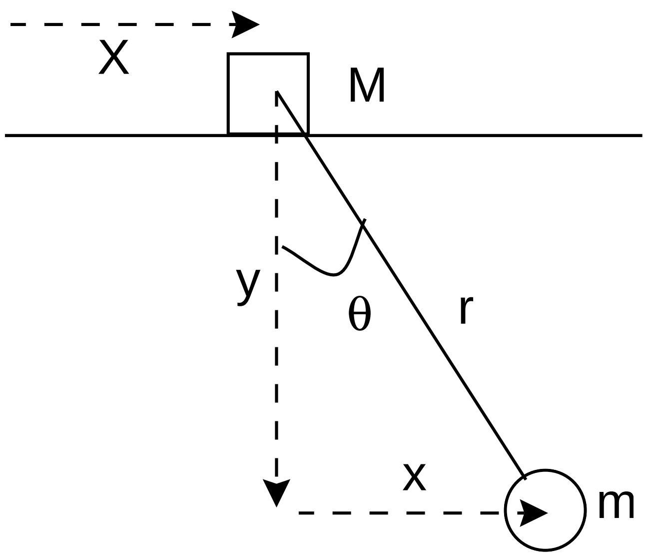
$$\delta \mathcal{S} = \delta \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2 \cdots q_N, \dot{q}_1, \dot{q}_2, \cdots \dot{q}_N) dt = 0$$

We can carry out the variation independently for each of the coordinates and obtain a set of $N$ Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$

i.e. one equation for each coordinate.

**Example**
PENDULUM WITH MOVABLE SUPPORT




A pendulum of length $r$, mass $m$. Its support has mass $M$ and it can slide without friction horizontally (coordinate $X$). The horizontal and vertical components of the pendulum mass are

$$x = X + r\sin\theta;\ y = -r\cos\theta$$

The velocity components are

$$\dot{x} = \dot{X} + r\cos\theta\ \dot{\theta};\ \dot{y} = r\sin\theta\ \dot{\theta}$$

Hence the Lagrangian is

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

$$= \frac{M}{2}\dot{X}^2 + \frac{m}{2}[\dot{X}^2 + r^2\dot{\theta}^2 + 2\dot{X}\dot{\theta}r\cos\theta] + mgr\cos\theta \qquad (10)$$

We will come back to the equations of motion for this system later.

GENERALIZED FORCES AND MOMENTA.

When the kinetic energy is on the form

$$\mathcal{T} = \frac{1}{2}m\dot{x}^2$$

and $\mathcal{V}(x)$ is velocity independent, the equation of motion can be written

$$\frac{\partial \mathcal{L}}{\partial x} = f = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\frac{dv}{dt} = \dot{p}$$

where $p$ is the **momentum** and $f$ the **force**

In the general case:

$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ = **generalized momentum**

In lecture 3.1 I defined the generalized conservative force as the partial derivative of the potential energy with respect to the generalized coordinate. We now modify the definition so that

$f_i = \frac{\partial \mathcal{L}}{\partial q}$ = **generalized force**

The Lagrangian equations of motion can thus be written:

$$f_i = \frac{dp_i}{dt}$$

**If the Lagrangian does not depend explicitly on one of the coordinates the corresponding generalized force is zero and the corresponding generalized momentum is conserved!**

**Example** THE PENDULUM

$$\mathcal{L} = \frac{mr^2\dot{\theta}^2}{2} + mgr\cos\theta$$

The generalized force is

$$f_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = -mgr \sin\theta$$

Physically the generalized force associated with the angle $\theta$ is the **torque!**
The generalized momentum is

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

which we recognize as the angular momentum. The Lagrangian equation of motion is thus just

**Rate of change of angular momentum=torque**

**Example**
PENDULUM WITH MOVABLE SUPPORT
The Lagrangian (1) doesn't depend explicitly on $X$ hence

$$p_X = \frac{\partial \mathcal{L}}{\partial \dot{X}} = (M+m)\dot{X} + m\dot{\theta}r\cos\theta$$

is conserved. A bit of reflection will convince you that this is just the equation for the conservation of linear momentum in the $x-$direction.

So there is nothing new!
We could have obtained the above results without resorting to Lagrangians. However, if the system is complicated the Lagrangian approach offers the possibility of proceeding in a systematic fashion, without having to worry about free body diagrams, normal forces, or pseudo forces due to acceleration of coordinate system.
The systematic, algebraic, approach makes it much easier to avoid errors!

SUMMARY
We have

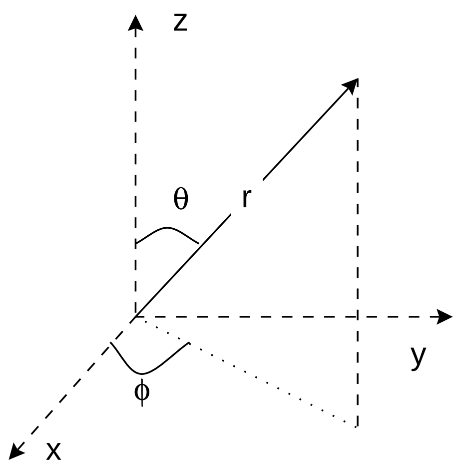
- developed Lagrangian dynamics from the principle of least action
- shown that for a conservative system with velocity-independent forces we could reproduce Newtonian dynamics if we put for the Lagrangian

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

- introduced generalized momenta
- modified the definition of generalized force
- shown that if Lagrangian did not depend on a generalized coordinate the corresponding momentum was conserved.

**Example problems**

**Problem 3.5.1**
(Problem 1 of 2002 problem set 3)




The velocity $\vec{v}$ of a particle in spherical coordinates can be written

$$\vec{v} = \dot{r}\hat{e}_r + r\sin\theta\ \dot{\phi}\hat{e}_\phi + r\dot{\theta}\hat{e}_\theta$$

The potential energy is $kr^2/2$ with $k > 0$. The particle is constrained to the plane $(\theta = \pi/2)$
**a:** Write down the Lagrangian for the system. Are the generalized momenta associated with the coordinates $r, \phi$ conserved?
**b:** Find an expression for the law of conservation of energy. For given values of the conserved generalized momenta what is the smallest value of the energy?
**c:** Describe the trajectory of the particle in case **b:**.

## 3.6 Conservation of energy. Galilean relativity

LAST TIME

- Developed Lagrangian dynamics from Hamilton's principle.
- Showed that for a conservative system with velocity-independent forces we could reproduce Newtonian dynamics if we put for the Lagrangian

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

- Introduced
  - generalized momenta
  - and modified the concept of generalized force
- Showed that if Lagrangian did not depend on a generalized coordinate the corresponding momentum was conserved.

TODAY
we discuss how the law of conservation of energy appears in Lagrangian dynamics,

Consider a Lagrangian which does not depend **explicitly** on **time**

$$\mathcal{L} = \mathcal{L}(q_1 \cdots q_N, \dot{q}_1 \cdots \dot{q}_N)$$

Let us compute the total time derivative of this Lagrangian

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{N} \left( \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i \right) \tag{11}$$

We assume that the generalized coordinates and velocities also satisfy the Lagrangian equation of motion:

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 \tag{12}$$

Substituting (2) into (1) yields

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{N} \left( \dot{q}_i \frac{d}{dt} (\frac{\partial \mathcal{L}}{\partial \dot{q}_i}) + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i \right)$$

$$= \sum_{i=1}^{N} \frac{d}{dt} (\dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i})$$

It follows that

$$\frac{d}{dt} \left( \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} \right) = 0$$

Just as we defined last time the generalized momentum as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

we now define the **energy** as

$$\boxed{E = \sum_{i=1}^{N} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = \sum_{i=1}^{N} p_i v_i - \mathcal{L}}$$

**We conclude that with the energy defined as above, and if the Lagrangian does not depend explicitly on time, then energy is conserved!**

EXAMPLE
Suppose the kinetic energy is

$$\mathcal{T} = \frac{m\dot{q}^2}{2}$$

and

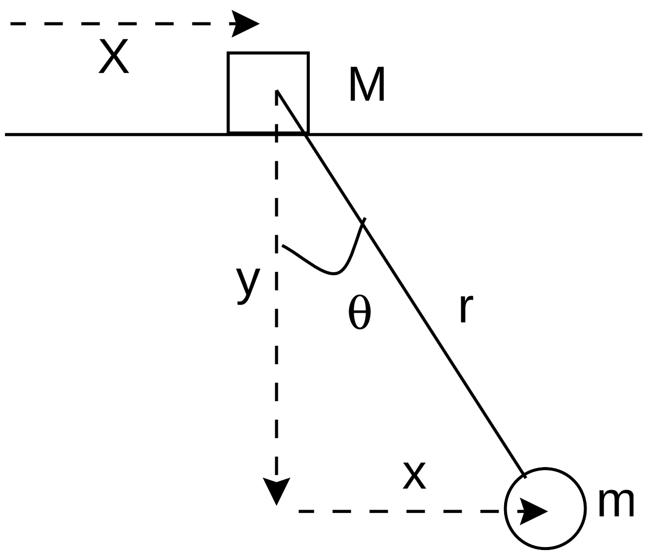
$$\mathcal{L} = \mathcal{T}(\dot{q}) - \mathcal{U}(q)$$

We have

$$E = m\dot{q}^2 - \frac{m\dot{q}^2}{2} + \mathcal{U}(q) = \mathcal{T} + \mathcal{U}$$

as expected

EXAMPLE




Let us next consider the pendulum with movable support that we encountered in the last lecture 3.5:

$$\mathcal{L} = \mathcal{T} - \mathcal{U}$$
$$= \frac{M}{2}\dot{X}^2 + \frac{m}{2}[\dot{X}^2 + r^2\dot{\theta}^2 + 2\dot{X}\dot{\theta}r\cos\theta] + mgr\cos\theta$$

where the generalized coordinates are $\theta$ and $X$. Again it is easy to see that

$$\dot{\theta}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} + \dot{X}\frac{\partial\mathcal{L}}{\partial\dot{X}} = 2\mathcal{T}$$

So again

$$\mathcal{E} = \mathcal{T} + \mathcal{U}$$

We next argue that our new definition of energy agrees with what we had in Newtonian mechanics whenever the Lagrangian is on the form

$$\mathcal{L}(\cdots q_i \cdots, \cdots \dot{q}_j \cdots) = \mathcal{T} - \mathcal{U}$$

where the kinetic energy is a **quadratic** function of the velocities

$$\mathcal{T} = \sum_{i,j=1}^{N} a_{ij}(q_1 \cdots q_N)\dot{q}_i\dot{q}_j \tag{13}$$

and the potential energy $\mathcal{U}$ is independent of the velocities. The two examples above are special cases of this situation. We have

$$\sum_{k=1}^{N} \dot{q}_k \frac{\partial \mathcal{T}}{\partial \dot{q}_k} = \sum_{k,j=1}^{N} a_{kj}(q_1 \cdots q_N)\dot{q}_k\dot{q}_j + \sum_{i,k=1}^{N} a_{ik}(q_1 \cdots q_N)\dot{q}_i\dot{q}_k = 2\mathcal{T} \tag{14}$$

Hence

$$E = 2\mathcal{T} - \mathcal{T} + \mathcal{U} = \mathcal{T} + \mathcal{U}$$

The result (3) is a special case of Euler's theorem for homogeneous functions.

ADDING A TOTAL TIME DERIVATIVE TO THE LAGRANGIAN

We have earlier seen that multiplying the Lagrangian by a constant has no effect on the equations of motion. We next show that adding a total time-derivative of a function of the coordinates has no effect on the equations of motion.

Consider two systems: one described by the Lagrangian $\mathcal{L}(q, \dot{q}, t)$ and the other by

$$\mathcal{L}'(q, \dot{q}, t) = \mathcal{L}(q, \dot{q}, t) + \frac{d}{dt}f(q, t)$$

(Here $'$ indicates "new function", not time derivative.) The **action** associated with the second system is

$$\mathcal{S}' = \int_{t_1}^{t_2} dt\mathcal{L}' = \int_{t_1}^{t_2} dt(\mathcal{L} + \frac{d}{dt}f(q, t)) = \mathcal{S} + f(q(t_2), t) - f(q(t_1), t)$$

Since the coordinates at the end-points are held fixed in our variational principle we see that the two systems have the same equation of motion!

Question: What is wrong with adding to the Lagrangian a total derivative of the form

$$\frac{d}{dt}f(q, t)$$

GALILEAN RELATIVITY

Some of you have encountered the special theory of relativity in PHYS 200. In that course you learned how to carry out a Lorentz transformation to a coordinate system which moves with uniform relative velocity $V$ with respect to another system. In classical mechanics the situation is a bit simpler. Suppose in a given coordinate system the position of a particle is $\vec{r}$ and the velocity is $\vec{v}$. In a coordinate system which moves with constant velocity $\vec{V}$ with respect to the first the position is $r'$ and the velocity $v'$.

$$\vec{r} = \vec{r'} + \vec{V}t$$

$$\vec{v} = \vec{v'} + \vec{V}$$

in classical mechanics observers in both systems can agree on the time ($t = t'$).

The kinetic energy of the particle in the original coordinate system is

$$\frac{mv^2}{2} = \frac{m(v')^2}{2} + m\vec{V} \cdot \frac{d\vec{r'}}{dt} + \frac{mV^2}{2}$$

This looks different from the contribution to the Lagrangian

$$\mathcal{T}' = \frac{m(v')^2}{2}$$

we would have had constructed the Lagrangian starting from the primed system. However, since the difference

$$m\vec{V} \cdot \frac{d\vec{r'}}{dt} + \frac{mV^2}{2} = \frac{d}{dt}(\vec{V} \cdot \vec{r'} + \frac{mV^2 t}{2})$$

is a total time derivative (when $\vec{V}$ is a constant) it has no effect on the dynamics. The equivalence of the primed an unprimed systems is called the principle of Galilean relativity.

SUMMARY

We have

- defined the energy of a Lagrangian system
- shown that if the Lagrangian does not depend explicitly on time the energy is conserved

- demonstrated that it is harmless to add a total time derivative

$$\frac{d}{dt}f(q,t)$$

to the Lagrangian

- discussed the principle of Galilean relativity.

**Example problems**

**Problem 3.6.1**
(Question 1 of 1999 problem set 4)
A particle of mass m moves towards a plane separating two regions one with potential energy $U_1$ another with potential energy $U_2$. Assume that $U_1 > U_2$.

**a:** Assume the particle starts out in region 1 towards the separating plane with speed $v_1$, and at an angle $\theta_1$, with respect to the plane normal. Find the speed and angle $\theta_2$ with respect to the normal in region 2.

**b:** The particle starts in region 2 towards the separating plane. What are the conditions on $\theta_2$ and $v_2$ for the particle to be reflected at the boundary.

**Hint** The momentum component parallel to the plane is conserved.

**Problem 3.6.2**
(Question 1 of 2000 problem set 4)
Consider the Lagrangian ($\vec{v} = d\vec{r}/dt$)

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - V(\vec{r})$$

**a:** What is the equation of motion?
**b:** What is the energy?
**c:** What is the momentum

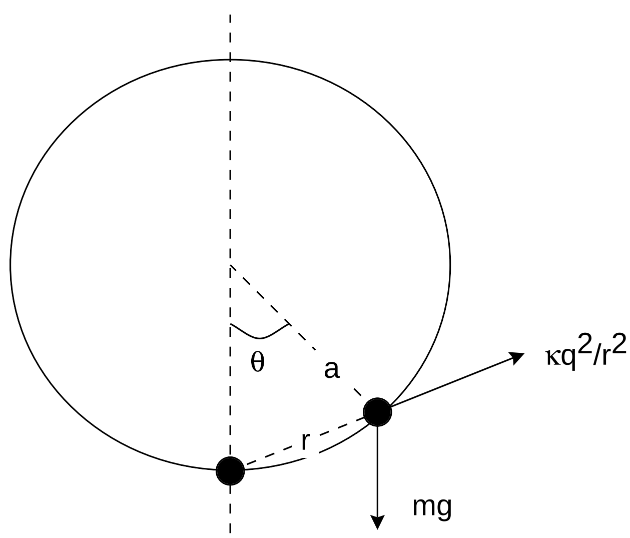
**Problem 3.6.3**
(Question 2 of 2000 problem set 4)

Consider the variational principle

$$\int_A^B \frac{(1+(\frac{dy}{dx})^2))^{1/2}}{y} dx = minimum$$

Where $A$ corresponds to $x = -1, y = 1$, $B$ to $x = 1, y = 1$ Use the "energy" first integral to find an expression for the curve. Try to simplify your expression.

**Problem 3.6.4**
(Question 1 of 2000 problem set 5)




A particle with charge $q$ and mass $m$ moves along a circle of radius $a$ in a fixed vertical plane. Another charge $q$ is fixed at the bottom position of the circle. The Coulomb potential energy if the charges are separated by a distance $r$ is

$$U(r) = \frac{\kappa q^2}{r}$$

**a:** Write down the Lagrangian with the angle $\theta$ as a generalized coordinate.

**b:** Find the equation of motion. Show that $\theta = \pi$, $\dot{\theta} = 0$ is a solution.

**c:** Under what condition is $\theta = \pi$ a stable equilibrium position (minimum in the potential energy)? It is convenient to express the condition in terms of dimensionless parameter

$$\lambda = \frac{\kappa q^2}{mga^2}$$

**d:** When $\theta = \pi$ is unstable, what is then the stable equilibrium position?

**Problem 3.6.5**

(Question 1 of 2002 midterm)

A particle of mass $m$ is constrained to move in the $x-y-$ plane. The potential energy of the particle is $U = 0$ for $y > 0$, $U = const. = U_0$ for $y < 0$. Initially $y > 0$ and the velocity of the particle is $v_x = v \sin\theta$, $v_y = -v\cos\theta$ with $0 < \theta < \pi$, i.e. the particle is heading for the lower region at an angle $\theta$ with the vertical. There is no friction or drag in the region $y > 0$, but there is a drag force in the region $y < 0$

$$\vec{F} = -\mu\vec{v}$$

where $\mu$ is a constant.

Assume that $\theta = \pi/4$, and that in appropriate units $m = v = 1$, $U_0 = -1$

**a:** What is the angle with the vertical in the lower region?

**b:** Assume that in the units used above $\mu = 1$ and that the particle enter the lower region at the origin of our coordinate system. Where does the particle stop?

**Problem 3.6.6**

(Question 3 of 2002 final)

In particle accelerators the particles typically reach relativistic speeds. Assume the Lagrangian for a particle in a constant force field to be

$$\mathcal{L} = -m_0c^2\sqrt{1 - \frac{v^2}{c^2}} + qx$$

where $v = \dot{x}$. The particle starts from rest at $x = 0$ at time $t = 0$

**a:** What is the momentum of the particle at position x?

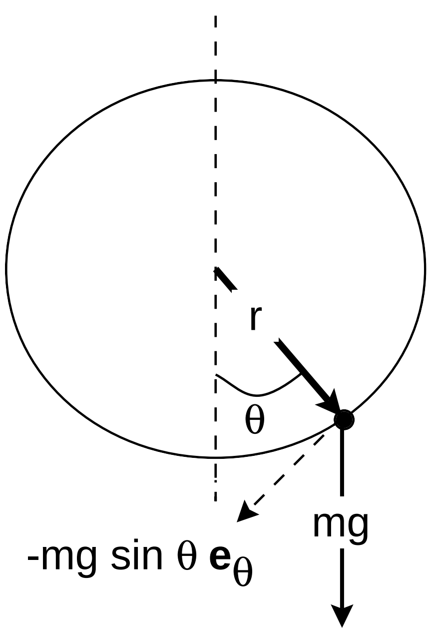
**b:** Find a formula for the velocity of the particle at time $t$. What is the limiting speed as $t \to \infty$.

## 3.7 The pendulum

LAST TIME

- Defined the energy of a Lagrangian system
- Showed that if the Lagrangian does not depend explicitly on time the energy is conserved
- Showed that adding a total time derivative to the Lagrangian is harmless
- Discussed the principle of Galilean relativity.

TODAY we will discuss a simple conservative system, the pendulum.




As shown in lecture 3.2 the Lagrangian for the pendulum can be written

$$\mathcal{L} = \mathcal{T} - \mathcal{U} = \frac{mr^2\dot{\theta}^2}{2} + mgr\cos\theta$$

The equation of motion is

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}) = -mgr\sin\theta - mr^2\frac{d\dot{\theta}}{dt} = 0$$

giving:

$$\ddot{\theta} + \frac{g}{r}\sin\theta = 0$$

SMALL AMPLITUDE APPROXIMATION
If $\theta << 1$ then $\sin\theta \approx \theta$ and

$$\frac{d^2\theta}{dt^2} + \frac{g}{r}\theta$$

This solution describes **simple harmonic motion**

$$\theta = A\cos(\omega t - \phi)$$

where

$$\omega = \sqrt{\frac{g}{r}}$$

and the constants $A$ and $\phi$ are determined from the **initial condition**:
E.g. if $\theta = 0,\ t = 0$ then $\phi = \pm\frac{\pi}{2}$. If $d\theta/dt = 0,\ t = 0$ then $\phi = 0$, or $\pi$ (depending on whether the pendulum is moving to the left or right initially)
The period is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g}}$$

FINITE AMPLITUDE

If the amplitude **is not small** we have to solve the **nonlinear** equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{r}\sin\theta = 0$$

(nonlinear because sin $\theta$ is a nonlinear function of the dependent variable $\theta$.)

We will see in lecture 3.8 that we can find solutions in terms of **elliptic functions**. Even if we know nothing about such functions all is not lost!

1. We can still solve the differential equation numerically, see e.g the Maple worksheet at
http://www.physics.ubc.ca/ birger/n206l4.mws (or .html)

2. Much information about the solution can be gained from the law of conservation of energy.

It is hard to make sense of a numerical solution unless one has first a qualitative idea of what is going on. Do second part first!

CONSERVATION OF ENERGY

We can without any loss og generality choose the potential energy to be zero for $\theta = \pi/2$

$$\mathcal{U}(\theta) = \int_{\pi/2}^{\theta} d\theta' mgr \sin\theta' = -mgr\cos\theta$$

The kinetic energy is

$$\mathcal{T} = \frac{1}{2} mr^2 (\frac{d\theta}{dt})^2$$

The total energy along a trajectory is thus

$$\mathcal{E} = \mathcal{T} + \mathcal{U} = \frac{1}{2} mL^2 (\frac{d\theta}{dt})^2 - mgr\cos\theta = const$$

Solve for the angular velocity

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2\mathcal{E}}{mr^2} + \frac{2g}{r}\cos\theta}$$

implicit equation for $\theta$ in terms of $t$

$$t = \pm \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\frac{2\mathcal{E}}{mL^2} + \frac{2g}{r}\cos\theta'}}$$

Signs $\Rightarrow$ movement to the left$(-)$ or right$(+)$ .

DIMENSIONLESS VARIABLES

Now is a good time to get rid of some of the constants cluttering up our equations. After all we don't want to redo the calculation each time we go to Mars (or change the mass or length of the pendulum).

Define **dimensionless time** $x$

$$t = \sqrt{\frac{r}{g}}x$$

The differential equation for $\theta$ becomes

$$\frac{d^2\theta}{dx^2} + \sin\theta = 0$$

Next, introduce the **dimensionless energy** $\epsilon$

$$\mathcal{E} = mgr\epsilon$$

Equation for $\theta$ becomes less forbidding

$$x = \pm\int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{2(\epsilon + \cos\theta')}}$$

THE DIFFERENT CASES

Solving the integral still poses a problem, but we can learn a lot about the qualitative properties, without performing the integration.

In a "real" world the expression inside the square root must be positive

$$\epsilon + \cos\theta > 0$$

We have

$$-1 \leq \cos\theta \leq 1$$

- The smallest possible value of $\epsilon$ is $-1$.

- If $-1 < \epsilon < 1$ there are critical angles

  $$\theta_c = \pm\cos^{-1}(-\epsilon)$$

  The pendulum will oscillate between $-\theta_c$ and $\theta_c$.

- $1 < \epsilon$. The pendulum becomes a rotor!

OSCILLATIONS NEAR EQUILIBRIUM:

If $\epsilon = -1$, the pendulum stays at rest at the bottom, **equilibrium**, position $\theta = 0$.
On the other hand if

$$\epsilon + 1 = \frac{\delta}{2}$$

with $\delta << 1$ Then $\theta_c << 1$ and

$$x = \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{2(\epsilon + \cos\theta')}} \approx \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\delta - (\theta')^2}}$$

$$= \sin^{-1}(\frac{\theta}{\sqrt{\delta}}) - \sin^{-1}(\frac{\theta_0}{\sqrt{\delta}}))$$

By suitably choosing the constant $\theta_0$ this becomes

$$\theta = \sqrt{\delta}\sin(x - x_0)$$

We recover the simple harmonic motion with period $2\pi$!

THE UNSTABLE EQUILIBRIUM

If $\epsilon = 1$, the position $\theta = \pi$ is an **unstable equilibrium**.

An infinitesimal perturbation is enough for the pendulum to fall off the top.

The pendulum will take an infinite time to reach the top position again. To see this note that the integral

$$\int^{\pi} \frac{d\theta'}{\sqrt{2(1 + \cos\theta')}} = \infty$$

It follows that the period of oscillation will approach $\infty$ as $\epsilon \to 1$ from below.

Similarly the period of rotation will approach $\infty$ as $\epsilon$ approaches unity from above!

ROTATING PENDULUM

If $\epsilon > 1$ the pendulum will be rotating. The period in reduced units will be

$$X = \int_0^{2\pi} \frac{d\theta}{\sqrt{2(\epsilon + \cos\theta)}}$$

We can calculate the period for large $\epsilon$ by Taylor expanding with $z = \frac{1}{\epsilon}\cos\theta$

$$\frac{1}{\sqrt{1+z}} = 1 - \frac{z}{2} + \frac{3z^2}{8} - \frac{5z^3}{16} + \frac{35z^4}{128}\cdots$$

We have

$$\int_0^{2\pi} d\theta \cos^n\theta = 0; \quad for\ n\ odd$$

$$\int_0^{2\pi} d\theta \cos^2\theta = \pi$$

$$\int_0^{2\pi} d\theta \cos^4\theta = \frac{3\pi}{4}$$

Giving for the period in reduced units

$$X = \frac{\pi}{\sqrt{2\epsilon}}(2 + \frac{3}{8\epsilon^2} + \frac{105}{512\epsilon^4}\cdots)$$

This series will diverge as $\epsilon \to 1$.

TURNING POINTS

Suppose we start the pendulum from $\theta = 0$ with positive angular velocity. It will then proceed according to

$$x = \int_0^{\theta} \frac{d\theta'}{\sqrt{2(\epsilon + \cos\theta')}}$$

until time $X_{1/4}$ when it reaches $\theta_c$ . It will then pick up the negative root

$$x = X_{1/4} - \int_{\theta_c}^{\theta} \frac{d\theta'}{\sqrt{2(\epsilon + \cos\theta')}}$$

Time will still run forwards since $d\theta' < 0$. It will continue this way until $\theta = -\theta_c$ and $x = 3X_{1/4}$ pick up the positive root again until $\theta_c$ and so on...

The period of oscillation in reduced units will be

$$X = 4\int_0^{\theta_c} \frac{d\theta}{\sqrt{2(\epsilon + \cos\theta)}}$$

The integrand diverges as $\theta$ approaches $\theta_c$ but the integral is still finite.

SUMMARY

- We have illustrated some properties of conservative systems using the pendulum as an example.
- The total energy played a crucial role in determining the qualitative properties of the behavior.
- At the lowest energy the pendulum was at rest in equilibrium.
- At higher energy the pendulum would do simple harmonic motion about the equilibrium.
- At higher energies still there was a change-over to a new type of behavior (rotation).

**Example problems**

**Problem 3.7.1**
(Question 1 of 1999 problem set 3)
Consider a pendulum in which a mass is connected to a fixed support by a **string** rather than a rod. In class derived the equation of motion in reduced units

$$2\frac{d^2\theta}{dx^2} + \sin\theta = 0$$

and showed that the energy parameter

$$\epsilon = (\frac{d\theta}{dx})^2 - \cos\theta$$

was a constant of the motion. If a string is used rather than a rod the pendulum will collapse if the tension in the string turns negative. For which values of $\epsilon$ will this happen?

**Problem 3.7.2**
(Question 2 of 1999 problem set 4)
Determine the period of oscillation as a function of energy when a particle of mass $m$ moves in a field where the potential energy is

$$V(x) = V_0 \tan^2(x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Check that you get the expected result in the limit of small amplitudes when $\tan(x) \approx x$

**Problem 3.7.3**
(Question 1 of 2001 midterm)
A particle is constrained to the sphere $r = 1$ and the potential energy is

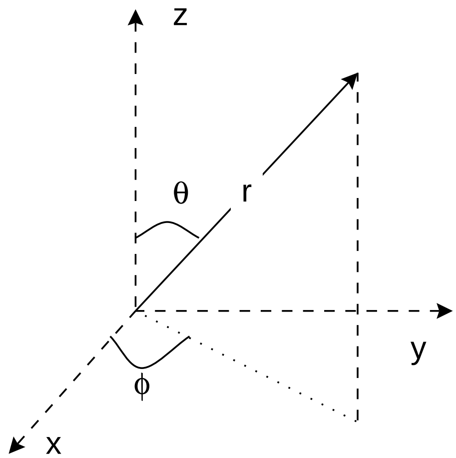
$$k \cos^2(\theta)$$

where $k > 0$, i.e. the potential energy is a minimum near the equator not at the top and bottom position. You may take the mass $m$ and $k$ to be 1. Any friction or drag on the particle can be neglected.
**a:** Write down the Lagrangian for the system. The generalized momentum $p_\phi$ associated with the coordinate $\phi$ is conserved. Can the angular velocity $\dot{\phi}$ change sign?
**b:** Find an expression for the law of conservation of energy $E$. For a given non-zero value of $p_\phi$, and allowed values of $E$, show that the particle can never reach the top ($\theta = 0$) or bottom ($\theta = \pi$) position.
**c:** Given a non-zero value of $p_\phi$ and an allowed value of the energy, locate the smallest and largest values of $\theta$ for the orbit?

**Problem 3.7.4**
(Question 2 of 2000 midterm)

The velocity $\vec{v}$ of a particle in spherical coordinates can be written

$$\vec{v} = \dot{r}\hat{e}_r + r\sin\theta\, \dot{\phi}\hat{e}_\phi + r\dot{\theta}\hat{e}_\theta$$




The potential energy is $mgr\cos\theta$. The particle is constrained to the surface ($r = const$) (spherical pendulum).
**a:** Write down the Lagrangian for the system, and conservation laws for momentum and energy.
**b:** How can the largest and smallest angles $\theta_{max}, \theta_{min}$ be found? You don't need to solve the equation satisfied by the angles.
**c:** Can the angular velocity $\dot{\phi}$ ever change sign?

**Problem 3.7.5**
(Question 5 of 2000 final)
The potential and kinetic energy of a particle with mass m are given by

$$\mathcal{U}(q) = kq^4, \quad \mathcal{T} = \frac{m}{2}\dot{q}^2, \quad k > 0$$

**a:** Describe qualitatively the motion for a given value of the energy $E = \mathcal{T} + \mathcal{U}$.
**b:** Find an expression for the period of oscillation in terms of a definite integral (you don't need to evaluate the integral).
**c:** By which factor will the period change if

1. The energy is doubled, but $m$ and $k$ kept constant.

2. The mass is doubled, but $E$ and $k$ kept constant.

3. The constant $k$ is doubled, but $E$ and $m$ kept constant.

## 3.8 Lessons from the pendulum

Today we want to follow up the discussion of the pendulum by using what we learned to make some important generalizations. More details in the form of a maple worksheet is available at
http://www.physics.ubc.ca/~birger/n206l14.mws The first topic is:

ELLIPTIC INTEGRALS.
Elliptic integrals are defined as integrals over rational functions $R(x, \sqrt{y})$ where y is a polynomial in x of order 3 or 4.

$$\text{ellipticintegral} = \int_a^b R(\sqrt{y}, x)dx$$

An exhaustive description of elliptic integrals is given in Chapter 17 of Abramowitz and Stegun [1965]. Of special interest to us are the **incomplete elliptic integrals** of the first kind which in Maple are defined as

$$\text{EllipticF}(z, k) = \int_0^z \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

and the complete elliptic function of the first kind, which is obtained simply by setting the upper limit of integration to unity

$$\text{EllipticK}(k) = \text{EllipticF}(1, k)$$

In 2.2 we found for the dimensionless time passed for the pendulum to swing from $\theta = 0$ to the maximum position $\theta = \Theta$

$$x = \int_0^\Theta \frac{d\theta}{\sqrt{2(\epsilon + \cos(\theta))}}$$

we will assume that $-1 < \epsilon < 1$ so that the pendulum swings between two angles $-\Theta$ and $\Theta$ with $\cos(\Theta) = -\epsilon$. The period in reduced units is $\tau = 4x$. By a few substitutions this expression can be converted to a complete elliptic integral of the first kind. First rewrite the integral for the period

$$\tau = 2\sqrt{2}\int_0^\Theta \frac{d\theta}{\sqrt{(\cos(\theta) - \cos(\Theta))}}$$

and introduce the new integration variable

$$t = \frac{\sin(\theta/2)}{\sin(\Theta/2)}$$

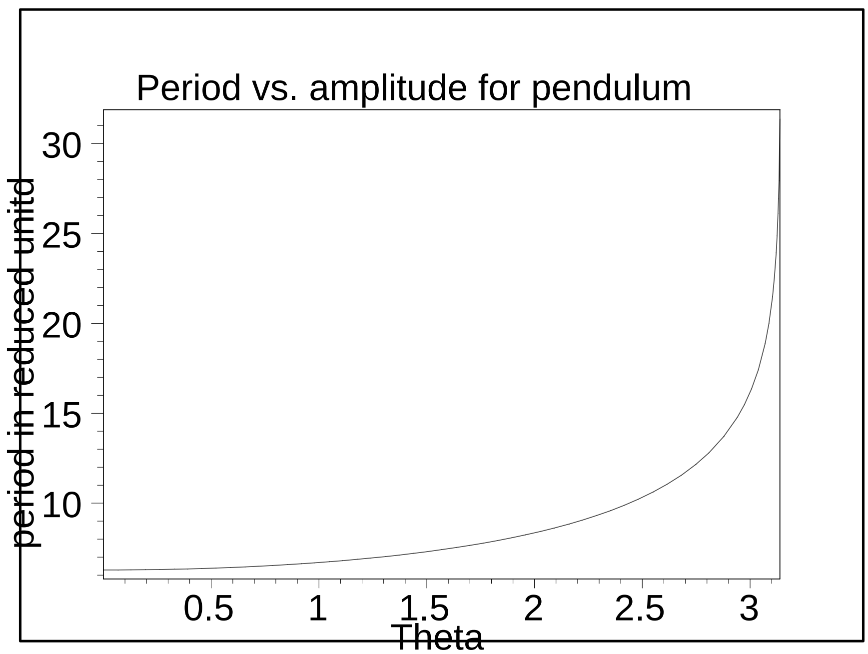
and the constant

$$k = \sin(\frac{\Theta}{2})$$

We find that the integral then becomes on the form given for the complete elliptic integral of the first kind and

$$\tau = 4\,\mathrm{ellipticK}(k)$$

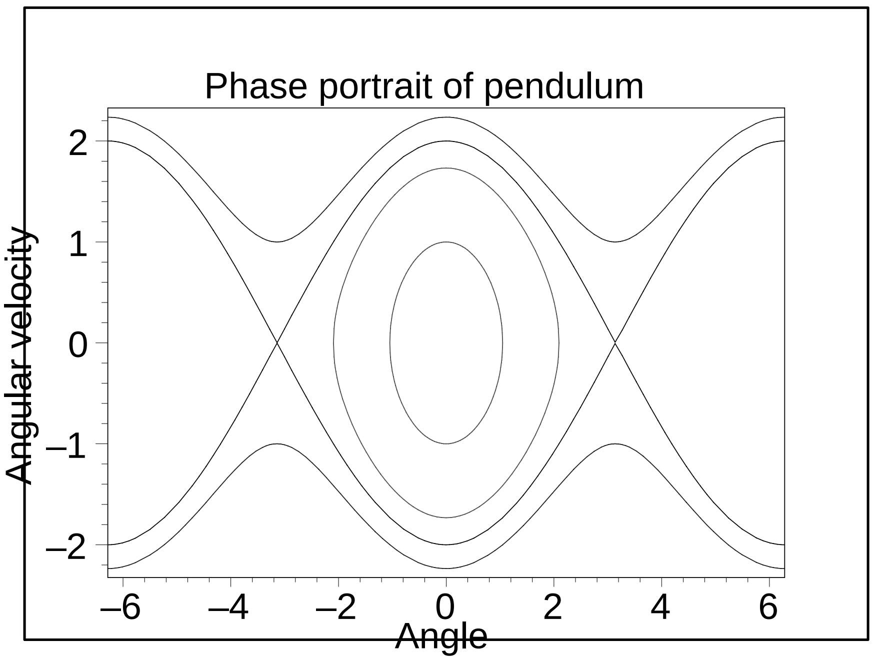
The elliptic function can be called by Maple and evaluated numerically. It is then a simple matter to plot the period of the pendulum in reduced units as a function of the amplitude and a plot the period as a function of the amplitude is given below. For small amplitudes the period Theta becomes $2\Pi$ in agreement with the small amplitude approximation!




TRAJECTORIES IN PHASE PLANE

The phase plane of the pendulum has the angle $\theta$ and the angular velocity

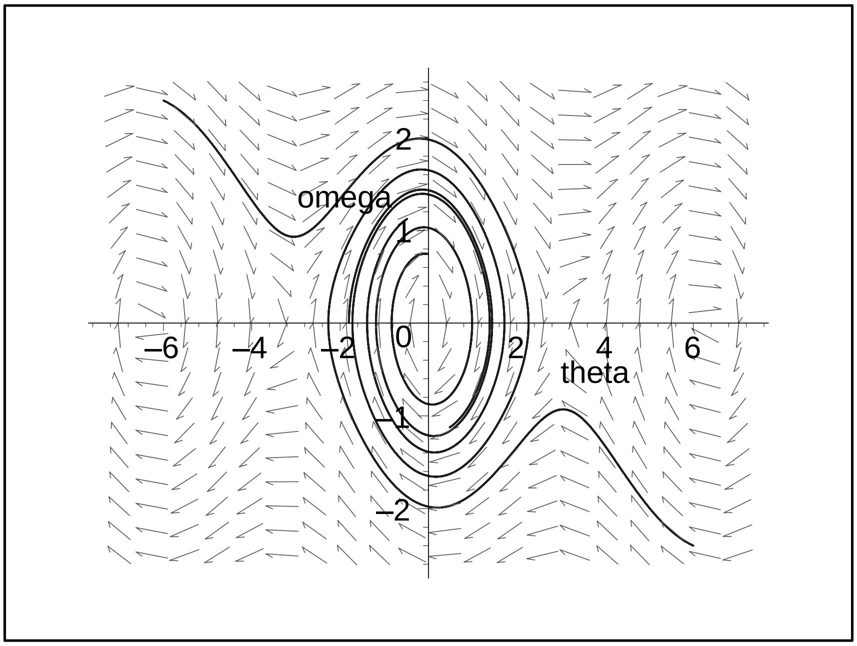
$\omega$ as coordinate axes. In the figure below I used the law of conservation of energy to construct the trajectories:




The top curve corresponds to a pendulum rotating counterclockwise, while the bottom curve corresponds to clockwise rotation. The two intersecting curves correspond to the case where the energy $\epsilon$ is zero. In this case the pendulum just barely makes it to the top position. Traversing these trajectories takes an infinitely long time and they separate regions of the phase plane associated with oscillatory and rotary motions. Such a trajectory is referred to as a separatrix. Inside the separatrix we have oscillatory back and forth motion.

Maple has a package DEtools which contains commands that allow us to analyze the behavior in the phase plane even if the exact result is not known. We illustrate this by introducing a damping term. In the figures below employ the command **phaseportrait** is employed to illustrate the use of this command. In both cases the damping proportional to the angular velocity. In the first figure the damping is weak, while in the second figure the motion is over damped. See the maple worksheet at

http://www.physics.ubc.ca/~birger/n206l14.mws
for details.

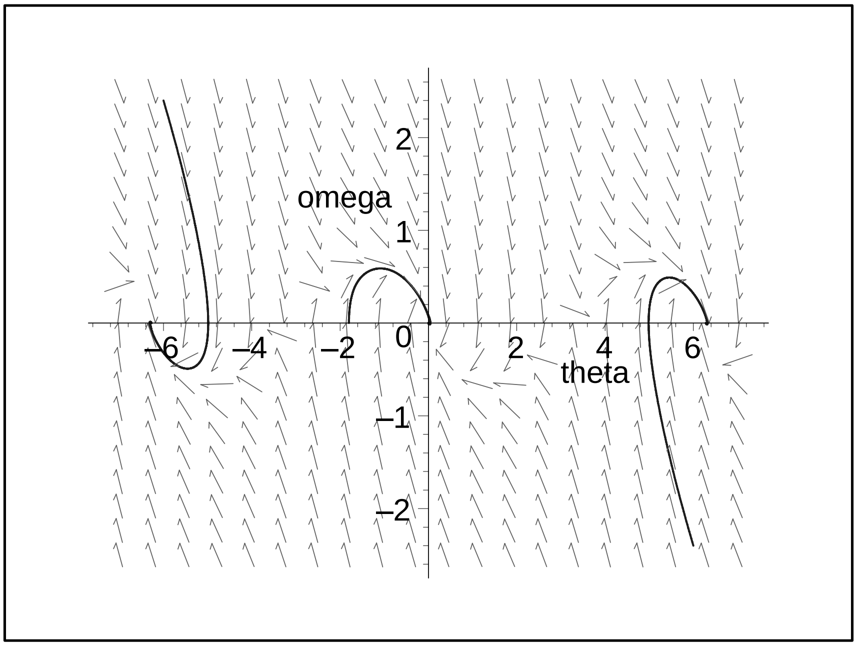



Next some terminology. Let us describe the system by two coupled differential equations

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = f(\omega, \theta)$$

The **equilibrium points** where the time derivatives of both $\omega$ and $\theta$ are zero are **singular points** of the differential equations. Singular points where there are two trajectories arriving at the equilibrium point, and two trajectories leaving it are called **saddle points**. If there are trajectories starting arbitrarily near the equilibrium point that moves away from it, the equilibrium is **unstable.** The saddle point is an example of an unstable equilibrium. In the case of zero damping the trajectories starting near the point $\omega = \theta = 0$ will undergo harmonic motion around the equilibrium point. Such a singular point is called a **center** (or a vortex). Since trajectories near




a center stays in the neighborhood without approaching or moving away from the singular point we say that the center is **neutrally stable.** When the damping coefficient is positive all trajectories starting near the point $\omega = \theta = 0$ will approach the equilibrium point making it **asymptotically stable,** or an **attractor**. The equilibrium when damping is small is made approached through **damped oscillations**. The equilibrium is then called a **focus.** When the damping coefficient becomes larger the motion is **over damped** and the equilibrium point is called a **node.** If the damping coefficient changes sign the trajectories change direction and the equilibrium becomes an **unstable focus** or **node.**

The type of singularities can be determined by linearizing the differential equation around the singular point

$$\frac{d\omega}{dt} \approx a\omega + b\theta$$

$$\frac{d\theta}{dt} \approx c\omega + e\theta$$

The solution will be on the form

$$\omega = \alpha_1 \, e^{(\lambda_1 t)} + \beta_1 \, e^{(\lambda_2 t)}$$

$$\theta = \alpha_2 \, e^{(\lambda_1 t)} + \beta_2 \, e^{(\lambda_2 t)}$$

where $\lambda_1, \ \lambda_2$ are roots of the characteristic equation

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

If the **real part of the roots are both negative** the equilibrium is asymptotically stable. If at least one root has a positive real part it is unstable. If the roots are purely imaginary further investigation may be required, but if the system is conservative the singular point is a center. If the roots are real we are dealing with a node, if they are complex we have a focus. An equilibrium point which is asymptotically stable is an attractor. Attractors are not necessarily points in phase space. In lecture 2.2 we considered a stick-and-slip slider block problem in which all trajectories, sufficiently close by, approached a limit cycle. Later on we will encounter more complicated such structures called strange attractors. In much of this course we will limit ourselves to conservative systems. The law of conservation of energy then forbids trajectories of different energy from approaching each other. Attractors, strange or otherwise, foci and nodes are forbidden for conservative systems. The only equilibrium points allowed for conservative systems are centers and saddle points!

**Problem 3.8.1**

(Question 2 of 1999 problem set 3)
Calculate the period $X$ in reduced units for a pendulum with amplitude $\pi/3$ ($60^o$). Compare with the period in the small amplitude approximation.

**Problem 3.8.2**

(Question 1 of 2001 problem set 3)
As the amplitude is increased in a mass spring oscillating system, nonlinearities in the spring becomes increasingly important. An important such system is the human eardrum, where it has been known for a long time that the ear can "hear" frequencies which are not present in the incident signal and sum and differences of frequencies present in the acoustic input. Let us model an eardrum by a nonlinear oscillating system by the equation of motion

$$m\ddot{x} + R\dot{x} + k_1 x + k_2 x^2 + k_3 x^3 = g(t)$$

where $R, k_1, k_2, k_3$ are constants

**a:** Show by dividing the equation by a constant and employing a suitable unit of time that the equation can be simplified to

$$\ddot{x} + x + \alpha\dot{x} + ax^2 + bx^3 = f(t)$$

**b:** Show that if $a \neq 0$ and positive we can choose a unit of length so that $a = 1$.

**c:** Consider first the case $\alpha = b = f = 0, a = 1$. Plot the potential energy function for this system in the range $x = -2..1$. Imagine that the system is put in motion with $\dot{x} = 0$ and some positive value of $x$. What is the maximum value of $x$ for period1c oscillations to occur? What happens if this value is exceeded?

**d:** Consider next the case $a = -1$. How does the behavior of the system change qualitatively?

**Problem 3.8.3**
(Question 3 of 2001 problem set 3)
Consider the nonlinear oscillator

$$\ddot{x} + x + x^3 = 0$$

**a:** Plot the potential energy function for this system in the range $x = -2..2$. Imagine that the system is put in motion with $\dot{x} = 0$ and some positive value of $x$. Describe qualitatively the subsequent behavior of the system.

**b:** Answer the same questions as in **a:** for

$$\ddot{x} - x + x^3 = 0$$

## 3.9 The forced pendulum.

LAST TIMES
Discussed the unforced pendulum as an example of an **integrable system**, where we could use the energy first integral to classify the different types of solution.
TODAY
We will discuss the forced oscillations of the pendulum. Since the Lagrangian now is time-dependent, energy is no longer conserved, and the situation becomes more complicated. We will not have time to go into all the details, and refer to the very readable presentation of **the Pendulum lab** by Franz-Josef Elmer of the University of Basel for more details:
http://monet.physik.unibas.ch/ elmer/pendulum/index.html

EQUATIONS OF MOTION OF FORCED PENDULUM

**Horizontal forcing**
Assume first that the point of support of a simple (rigid) pendulum (length $r$ mass $m$) is forced to move in the horizontal direction according to

$$X = a\sin(\omega t)$$

where $\omega$ is constant and $t$ is time. We first need to find how the movement of the support will modify the equation of motion for the angle $\theta$. In lecture 3.5 we calculated the kinetic and potential energy of a pendulum with a support that was free to move horizontally

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

$$= \frac{M}{2}\dot{X}^2 + \frac{m}{2}[\dot{X}^2 + r^2\dot{\theta}^2 + 2\dot{X}\dot{\theta}r\cos\theta] + mgr\cos\theta$$

In the present situation the coordinate $X$ no longer represents a degree of freedom, but rather a **constraint**, giving

$$\mathcal{L} = \frac{m}{2}[\dot{X}^2 + r^2\dot{\theta}^2 + 2\dot{X}\dot{\theta}r\cos\theta] + mgr\cos\theta$$

$$= \frac{m}{2}[a^2\omega^2\cos^2(\omega t) + r^2\dot{\theta}^2 + 2a\omega\cos(\omega t)\dot{\theta}r\cos\theta] + mgr\cos\theta$$

We have

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = mr^2\dot\theta + mar\omega\cos(\omega t)\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgr\sin\theta - mar\omega\cos(\omega t)\dot\theta\sin\theta$$

If we add to the equation of motion a phenomenological damping force, proportional to the angular velocity, we obtain for the case of horizontal forcing

$$\boxed{\ddot\theta - \frac{a}{r}\omega^2\sin(\omega t)\cos\theta + \frac{g}{r}\sin\theta + \gamma\dot\theta = 0}$$

**Vertical forcing**

Next consider the case of vertical forcing. Following the procedure of lecture 3.5 we put

$$x = r\sin\theta; \quad y = Y - r\cos\theta$$

$$\dot x = r\dot\theta\cos\theta; \quad \dot y = \dot Y + r\dot\theta\sin\theta$$

with

$$Y(t) = a\sin(\omega t); \quad \dot Y = a\omega\cos(\omega t)$$

We find

$$\mathcal{T} = \frac{m}{2}(r\dot\theta^2 + a^2\omega^2\cos^2(\omega t) + 2ar\omega\dot\theta\cos(\omega t)\sin\theta)$$

$$\mathcal{V} = mg(a\sin(\omega t) - r\cos\theta)$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = mr^2\dot\theta + mar\omega\cos(\omega t)\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = mar\omega\dot\theta\cos(\omega t)\cos\theta - mgr\sin\theta$$

which yields the equation of motion for vertical forcing

$$\boxed{\ddot\theta - \frac{a}{r}\omega^2\sin(\omega t)\sin\theta + \frac{g}{r}\sin\theta + \gamma\dot\theta = 0}$$

where we have again added a phenomenological forcing term.

SMALL AMPLITUDE APPROXIMATION
We first consider the case of small amplitudes. We may then approximate

$$\cos\theta \approx 1; \sin\theta \approx \theta$$

**Horizontal forcing**
The equation of motion is now

$$\ddot{\theta} + \frac{g}{r}\theta + \gamma\dot{\theta} = \frac{a}{r}\omega^2 \sin(\omega t)$$

which is the familiar equation for the forced harmonic oscillator. If the damping term is nonzero, the system settles, after a **transient**, into simple periodic motion, with frequency equal to that of the driving. The amplitude of the forced oscillation is

$$\boxed{\theta_{max} = \frac{\frac{a\omega^2}{r}}{\sqrt{(\omega^2 - \frac{g}{r})^2 + (\omega\gamma)^2}}}$$

The phase of the forced motion differs from that of the forcing, depending on the damping. If the damping is zero, we will have a superposition of oscillations at the natural frequency $\sqrt{g/r}$ and the forcing frequency $\omega$. When the two frequencies are equal (resonance) the motion becomes unstable, and the amplitude grows linearly with time, without bound, until nonlinear terms in the equation of motion come into play. Since you are familiar with the forced harmonic motion from other courses, I will not pursue this case further.

If we increase the amplitude of the forcing, nonlinear terms in the equation of motion take over. What then happens, is that the resonance frequency shifts towards smaller frequencies (longer period). As shown in the *pendulum lab*, this foldover effect leads to **instabilities** and **hysteresis**. Another nonlinear effect is that the motion no longer is purely sinusoidal. This opens the possibility of **super harmonic resonance**, where the driving frequency is a fraction of the fundamental frequency of the pendulum.

**Vertical forcing**
The small amplitude approximation is now

$$\ddot{\theta} + \frac{g}{r}\theta + \gamma\dot{\theta} = \frac{a\omega^2}{r}\sin(\omega t)\theta$$

This differential equation is a special case of **Hill's equation**

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0$$

where the coefficients $a(t), b(t)$ are **periodic**, with period $T$. It has many important applications. To mention a few: in solid state physics the Schrödinger equation for a particle in a periodic potential can be put in this form, if we let $t$ be a spatial coordinate, and $T$ the period of the lattice. The effect of Jupiter on the orbits of the other planets can be approximated by a periodic perturbation, as can the effect of a nearby moon on the debris in the rings of Saturn. Periodic pumping is a technique learned early by most children trying to get a swing to move higher, when no parent is available to push (although finding the proper equations is not as easy as doing it).

If the damping term is zero the equation of motion is an example of the **Mathieu equation**

$$\ddot{x} + (a - 2q\cos(2t))x = 0$$

STABILITY OF TRAJECTORIES
An important qualitative question in problems involving Hill's equation, or the Mathieu equation, is whether the trajectories are **stable**.

It is often convenient to write higher order differential equations as a set of coupled first order equations. Let us introduce $y = \dot{x}$ as a new variable so that Hill's becomes

$$\dot{y} = -a(t)y - b(t)x$$

$$\dot{x} = y$$

A set of $n$ linear first order equations can then be written on matrix form, as

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \cdots \\ \dot{y}_n \end{pmatrix} = A(t) \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix}$$

The general solution to this set of equations is a linear combination of $n$ linearly independent solutions

$$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 \cdots \; c_n\vec{y}_n$$

We define the fundamental solution matrix

$$Y(t) = (\vec{y}_1, \vec{y}_2, \cdots, \vec{y}_n)$$

The choice of the **basis**, $\vec{y}_i$. is not unique, but if $X(t)$ and $Y(t)$ are two different solution matrices, one can always find a constant matrix $C$ connecting them

$$X(t) = Y(t)C$$

In fact, since

$$X(t) = Y(t)Y^{-1}(0)X(0)$$
$$C = Y^{-1}(0)X(0) \qquad (15)$$

If the matrix $A$ of a system of linear differential equations is periodic.

$$A(t+T) = A(t)$$

$Y(t+t)$ and $Y(t)$ are solutions to the **same** differential equation, and we must have

$$Y(t+T) = Y(t)C$$

If you have access to a numerical procedure for solving the differential equation you can find the matrix $C$ by using (2) and assume that $Y(0)$ is the unit matrix. The elements of $Y(T) = C$ can then be found by systematically integrating the differential equation for each $i, i = 1 \cdots n$ assuming $y_i(0) = 1, \; y_j(0) = 0 \; for \; i \neq j$.

The stability of the system can be investigated by computing the eigenvalues $\lambda_i$ of $C$.

- If **all** eigenvalues $|\lambda_i| < 1$, all solutions will approach $\vec{y} = 0$ as $t \to \infty$. The system is then **asymptotically stable**.
- If one or more eigenvalues have $|\lambda_i| = 1$ (and they are distinct) and the remainder of the eigenvalues have $|\lambda_j| < 1$ the system is **neutrally stable**. The solution will remain bounded for all times.
- If $|\lambda_i| > 1$ for one or more value of $i$ the system is **unstable** the amplitude will grow out of bounds for "almost all" initial conditions.

It can be shown that in the undamped case (Matthieu equation)the product of eigenvalues $\lambda_1\lambda_2 = 1$. The system is then either neutrally stable, or unstable, asymptotic stability is not possible.

The above method of stability analysis is illustrated in the Maple worksheet at
http://physics.ubc.ca/~birger/n2-6l16b
with results that agrees with the discussion of parametric resonance in the *Pendulum lab*. With damping, the solution of linearized equation either decays to zero (stable case) or grows without bounds (unstable case). The existence unstable regions is commonly referred to as **parametric resonance**. The feature that the solution remains unstable, even in the presence of damping, distinguishes parametric resonance from the ordinary resonance of the harmonic oscillator. The main resonance occurs when the driving frequency is **twice** the natural frequency, this is another distinguishing feature as is the existence of super harmonic resonances when the driving frequency is 1/2,1/3 etc of the natural frequency

$$\omega_0 = \sqrt{\frac{g}{r}}$$

In the unstable region the solution will grow until the nonlinear terms in the equation of motion take over, and force the solution to remain bounded. Just as in the case of horizontal forcing one sees a **foldover effect** with **hysteresis**. In the presence of damping there are also regions in parameter space in which there are trajectories, which after a transient, settles on a **periodic orbit** with a period which is either the same as the driving frequency or a multiple of it. There is also a regime in which the orbits are **chaotic**, with no period, and never repeating itself.

We can visualize the difference regime by the following procedure:

1. Select an initial point $\theta_1, y_1$ in the $\theta - y$ plane and integrate the equation of motion over one period of the driving, generating a new point $\theta_2, y_2$.

2. Repeat the procedure many times generating the sequence

$$(\theta_1, y_1), (\theta_2, y_2)...(\theta_n, y_n)$$

3. Plot the resulting sequence of points in the $\theta - y$ plane.

The **mapping**

$$(\theta_n, y_n) \Rightarrow (\theta_{n+1}, y_{n+1})$$

is called a **stroboscopic map**, and is a special case of the **Poincaré map** in which one selects a suitable plane, **Poincaré section** in phase space which all orbits cross. The mapping of the coordinates of one crossing to the next crossing, in the same direction, is called the **Poincaré map**. We describe how one can compute stroboscopic plots in the Maple worksheet at http://physics.ubc.ca/˜birger/n206l16c.mws (or .html)

The driven pendulum behaves quite differently, depending on whether there is **damping** or not. If there is no damping the motion can be "quasiperiodic", i.e. appear to involve a small number of incommensurate frequencies. In that case the points on the stroboscopic plot will appear to lie on a smooth curve. The trajectory can also be fully **chaotic** with the mapping appearing to fill a region of the phase plane. For the same parameter values there will often be regions which are excluded from the chaotic trajectory. If the system is started with initial conditions in the excluded region, it will typically exhibit quasiperiodic behavior.

If there is damping the system will, after an initial transient, approach an attractor. This attractor is a single point in the stroboscopic plot, if its period is the same as the forcing frequency. If the period is a multiple of the forcing frequency, it consists of a finite number of points. We also can have a stroboscopic plot with a "strange attractor", that doesn't fill a two dimensional region of the phase plane, but requires more points than a curve.

Such objects are called **fractals** and typically exhibit self-similar features. The motion on the strange attractor is also called chaotic.

Both trajectories on the strange attractor, and fully chaotic maps that fill a region of the phase plane, exhibit **extreme sensitivity to initial conditions**. This implies that trajectories that start close to each other in the phase plane will diverge. Arbitrarily small rounding off errors, or other in accuracies in the numerical differential equations solver, then makes it practically impossible to integrate the motion on a chaotic orbit for very long. This doesn't mean that one cannot compute the properties of the attractor numerically, since it is an attractor, small numerical errors will not make the computed trajectory move away from it. An analogous situation occurs in weather forecasting. Even though we cannot predict the **weather** for much more than a week ahead, we may still be able to simulate the **climate**!

The forced pendulum is a good lead-in to the very rich field of study **nonlinear dynamics**. Because of time limitations we have only been able to scratch the surface of this large and important field of research, but I hope I have been able to provide something of the flavor of the subject!

Further reading: The **pendulum lab** of Franz-Josef Elmer http://monet.physik.unibas.ch/ elmer/pendulum/links.html contains several links and references to textbooks. See also Richard H. Enns and George C. McGuire[4] and Lynch[12].

**Example problem**
**Problem 3.9.1**
(Question 2 of 2001 problem set 3)
Use the Maple commands odeplot and the numeric option of dsolve (see lecture 5) to explore the behavior of the forced "drum"

$$\ddot{x} + x + x^2 + \alpha\dot{x} = f(t)$$

**a:** Assume that

$$f(t) = c\sin(\omega t)$$

with $\omega = 0.2$ and starting with $x = \dot{x} = 0$. Estimate the maximum amplitude of the forcing term $c$ before the drum ruptures, assuming zero damping $\alpha$.

**b:** Introduce a damping $\alpha = 0.2$ and answer the questions in **a:**.

# 4 Non -inertial frames of reference. Kinematics of rotation.

## 4.1 Rotating the coordinate system, Euler angles.

LAST TIMES

- Discussed the pendulum in the context of Lagrangian mechanics.
- The constraints were such that the system had one degree of freedom.
- For the unforced pendulum closed expressions for the trajectories could be found using the law of conservation of energy.
- In the case of the forced pendulum the situation became much more complicated. A rich variety of qualitatively different solutions appeared as parameter values were changed.
- We introduced the method of Poincaré section to analyze the different forms of behavior

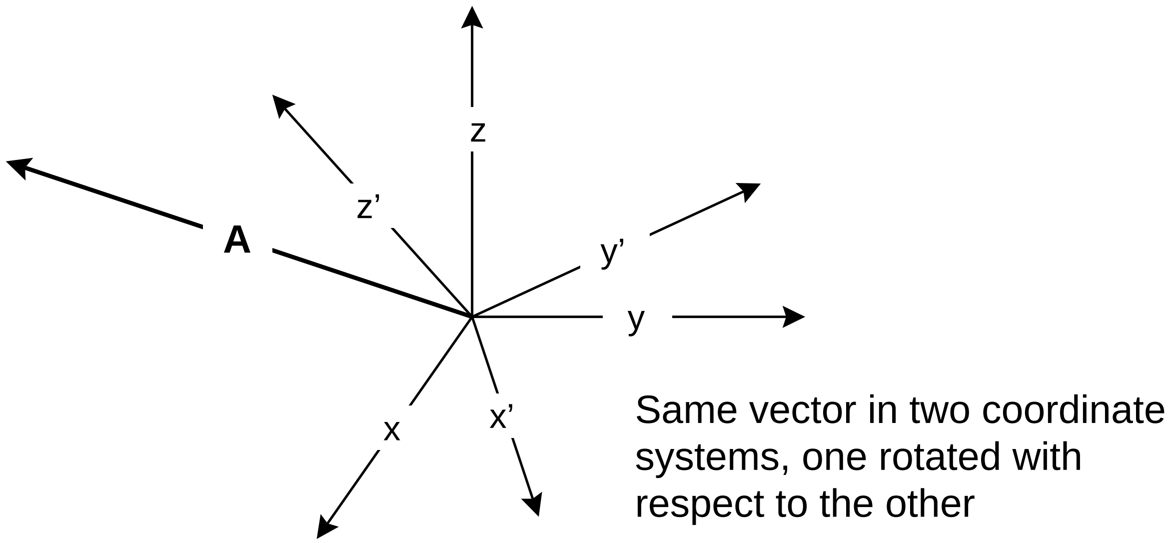
TODAY
Want to start discussing problems in three spatial dimensions. Let us begin simply by considering coordinate systems in which the axes are oriented differently:

ROTATING A CARTESIAN COORDINATE SYSTEM
Let $\vec{A}$ be the vector

$$\vec{A} = \hat{\mathbf{i}}A_x + \hat{\mathbf{j}}A_y + \hat{\mathbf{k}}A_z$$

The **same vector** in a coordinate system $(\hat{\mathbf{i}}', jh', \hat{\mathbf{k}}')$ which is rotated with respect to the first can be written

$$\vec{A} = \hat{\mathbf{i}}'A_{x'} + \hat{\mathbf{j}}'A_{y'} + \hat{\mathbf{k}}'A_{z'}$$




We have

$$A_{x'} = \hat{\mathbf{i}}' \cdot \vec{A} = (\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}})A_x + (\hat{\mathbf{i}}' \cdot \hat{\mathbf{j}})A_y + (\hat{\mathbf{i}}' \cdot \hat{\mathbf{k}})A_z$$

$$A_{y'} = \hat{\mathbf{j}}' \cdot \vec{A} = (\hat{\mathbf{j}}' \cdot \hat{\mathbf{i}})A_x + (\hat{\mathbf{j}}' \cdot \hat{\mathbf{j}})A_y + (\hat{\mathbf{j}}' \cdot \hat{\mathbf{k}})A_z$$

$$A_{z'} = \hat{\mathbf{k}}' \cdot \vec{A} = (\hat{\mathbf{k}}' \cdot \hat{\mathbf{i}})A_x + (\hat{\mathbf{k}}' \cdot \hat{\mathbf{j}})A_y + (\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}})A_z$$

In matrix form

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{j}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{k}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{k}} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$$

I will call the matrix

$$R = \begin{pmatrix} \hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{j}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{k}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{k}} \end{pmatrix}$$

the **transformation matrix**

We can reverse the role of the primed and unprimed coordinates

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{i}}\cdot\hat{\mathbf{i}}' & \hat{\mathbf{i}}\cdot\hat{\mathbf{j}}' & \hat{\mathbf{i}}\cdot\hat{\mathbf{k}}' \\ \hat{\mathbf{j}}\cdot\hat{\mathbf{i}}' & \hat{\mathbf{j}}\cdot\hat{\mathbf{j}}' & \hat{\mathbf{j}}\cdot\hat{\mathbf{k}}' \\ \hat{\mathbf{k}}\cdot\hat{\mathbf{i}}' & \hat{\mathbf{k}}\cdot\hat{\mathbf{j}}' & \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}' \end{pmatrix} \begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix}$$

The new transformation matrix is

$$R' = \begin{pmatrix} \hat{\mathbf{i}}\cdot\hat{\mathbf{i}}' & \hat{\mathbf{i}}\cdot\hat{\mathbf{j}}' & \hat{\mathbf{i}}\cdot\hat{\mathbf{k}}' \\ \hat{\mathbf{j}}\cdot\hat{\mathbf{i}}' & \hat{\mathbf{j}}\cdot\hat{\mathbf{j}}' & \hat{\mathbf{j}}\cdot\hat{\mathbf{k}}' \\ \hat{\mathbf{k}}\cdot\hat{\mathbf{i}}' & \hat{\mathbf{k}}\cdot\hat{\mathbf{j}}' & \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}' \end{pmatrix}$$

In more compact notation if

$$\vec{A}' = R\vec{A}$$

and

$$\vec{A} = R'\vec{A}'$$

then R' is the **inverse** of R

$$R' = R^{-1}$$

UNIT MATRIX

When I multiply a matrix with its inverse the result is the unit matrix

$$R^{-1}R = RR^{-1} = \mathbf{1}$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is customary to write $\delta_{ik}$ for the components of the unit vector and call it the **Kronecker** $\delta$

$$\delta_{ik} = \begin{cases} 1 & for \quad i = k \\ 0 & \quad\quad i \neq k \end{cases}$$

TRANSPOSE OF A MATRIX

If you turn the rows of a matrix into columns and columns into rows the effects is to transpose the matrix. For example the transpose of

$$R = \begin{pmatrix} \hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{j}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{k}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{k}} \end{pmatrix}$$

is

$$R^T = \begin{pmatrix} \hat{\mathbf{i}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{i}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{i}} \cdot \hat{\mathbf{k}}' \\ \hat{\mathbf{j}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{j}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{j}} \cdot \hat{\mathbf{k}}' \\ \hat{\mathbf{k}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{k}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' \end{pmatrix}$$

We see that for rotations of the coordinate system

$$R^T = R' = R^{-1}$$

i.e. the transpose is the same as the inverse

ROW AND COLUMN VECTORS

When using matrix notation there will be two kind of vectors:

Column vectors such as

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

and row vectors such as

$$\vec{B}^T = (B_x, B_y, B_z)$$

The transpose of a row vector is a column vector and *vice versa*

The scalar product of two vectors is then the matrix product

$$\vec{B}^T \vec{A} = (B_x, B_y, B_z) \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$= B_x A_x + B_y A_y + B_z A_z = \vec{A}^T \vec{B}$$

Suppose we make a coordinate transformation

$$\vec{A}' = R\vec{A}; \ \vec{B}' = R\vec{B}$$

To transpose a matrix product we reverse the order

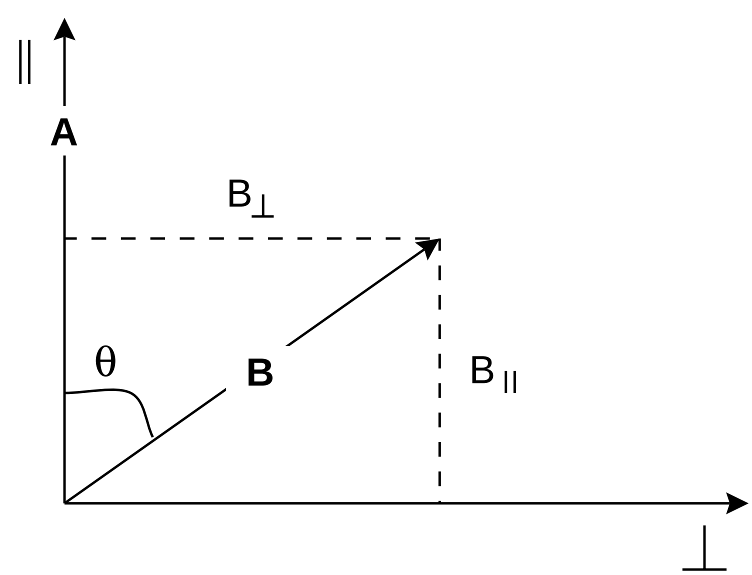
$$(R\vec{B})^T = \vec{B}^T R^T$$

Recall that the inverse of the transformation matrix is its transpose. We find

$$(\vec{A}')^T \vec{B}' = \vec{A}^T R^T R \vec{B} = \vec{A}^T \vec{B}$$

The scalar product does not depend on the orientation of the coordinate axes.
Another way of saying the same thing:
**The scalar product is invariant under rotation**
We may use this fact by choosing to evaluate products in the most convenient coordinate system.

DECOMPOSITION INTO PARALLEL AND PERPENDICULAR COMPONENTS




$$\vec{B} = (B_{||}, B_{\perp}, 0)$$

$$B_{\parallel} = |B|\cos\theta;\ B_{\perp} = |B|\sin\theta$$

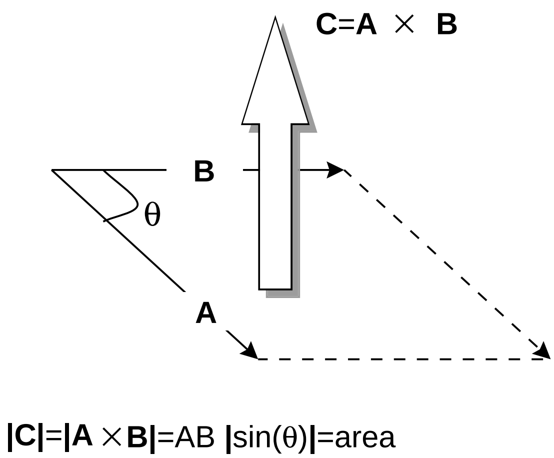
Find

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$|\vec{A}\times\vec{B}| = AB|\sin\theta|$$

DOT AND CROSS PRODUCT

Cross-product = **area** spanned by the two vectors

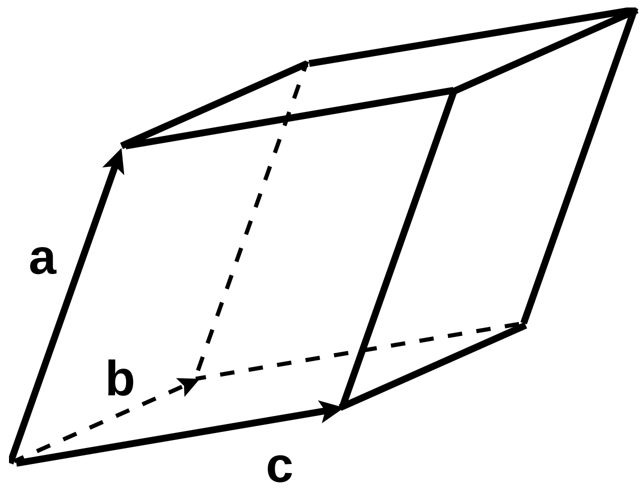



Scalar triple product = **volume** spanned by three vectors:

CONSTRUCTION OF TRANSFORMATION MATRIX
EXAMPLE 1: Rotation about $z-$axis by $\phi$.

$$\hat{\mathbf{i}}'\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}'\cdot\hat{\mathbf{j}} = \cos\phi;\ \hat{\mathbf{i}}'\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}'\cdot\hat{\mathbf{i}} = \sin\phi$$

$$\hat{\mathbf{k}}'\cdot\hat{\mathbf{k}} = 1$$




$$R_1 = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

EXAMPLE 2:
Rotation about $x-$axis by $\theta$:

$$\hat{\mathbf{j}}' \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}}' \cdot \hat{\mathbf{k}} = \cos\theta;\ \hat{\mathbf{j}}' \cdot \hat{\mathbf{k}} = -\hat{\mathbf{k}}' \cdot \hat{\mathbf{j}} = \sin\theta$$

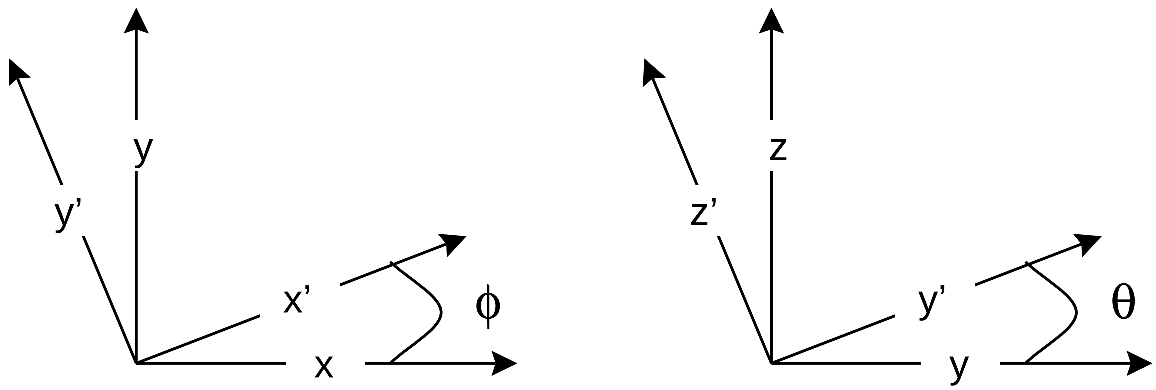
$$\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} = 1$$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

If we carry out a series of rotations the net result can be worked out using the rules of matrix multiplication. For instance a rotation about the $x-$axis followed by a rotation about the $z-$axis is represented by the rotation matrix

$$R = R_1 R_2$$

(note the order of successive rotations right to left.) The order of rotations matters. Consider the case where $\theta = \phi = 90^o$




x-y plane after rotation by angle ϕ about z-axis

y-z plane after rotation by angle θ about x-axis

$$R = R_1 R_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

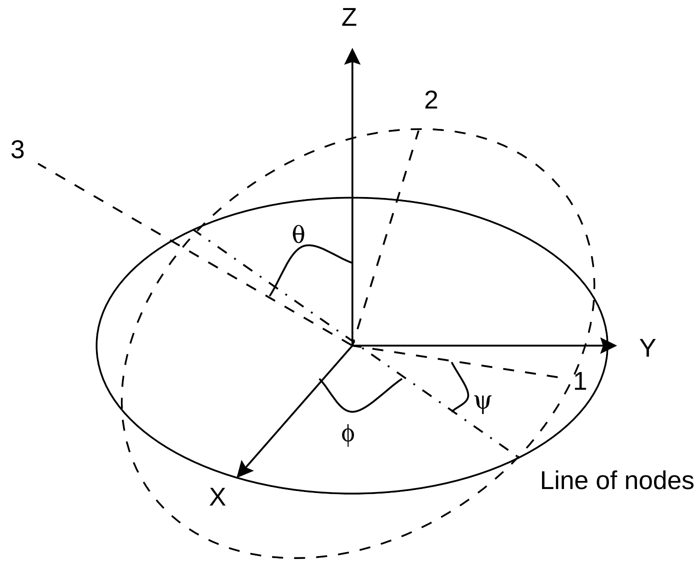
while

$$R_2 R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

We describe this situation by saying that in general rotation matrices do not **commute**. However, as we shall see in lecture 4.2 matrices corresponding to infinitesimal rotations do commute.

We wish to start on an alternative approach to describing the orientation of a body in terms of three orientational angles. A standard way to do this is through the Euler angles $\phi, \theta, \psi$.
In the figure above the three Cartesian coordinate axes labeled 1,2,3 represent the orientation of a set of axes,

**fixed on the body**, typically in principal axes directions. The three axes labeled $X, Y, Z$ are **fixed in space**. The orientation of the body-centered




coordinate system can be thought of as coming from three successive rotations:

1. by an angle $\phi$ about the $Z$-axis
2. by an angle $\theta$ about the new $x$-axis, which we will call the **line of nodes**
3. by an angle $\psi$ about new $z-$axis.

We see from the figure that a unit vector in the direction of the line of nodes can be written

$$\hat{e}_N = \hat{e}_1 \cos\psi - \hat{e}_2 \sin\psi$$

The unit vector in the direction of the space $Z-$axis can be written

$$\hat{\mathbf{k}} = \hat{e}_3 \cos\theta + \hat{e}_2 \sin\theta \cos\psi + \hat{e}_1 \sin\theta \sin\psi$$

The rotation matrix $R$ after all three rotations have been carried out is

$$R = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

SUMMARY
We have

- Discussed transformations corresponding to rotation of coordinate system
- Explored matrix notation for vectors
- Showed that rotations leave dot product invariant
- Given physical interpretation of dot and vector product
- Shown how rotation transformation can be constructed from direction cosines
- Defined the **Euler angles** describing the orientation of a rigid body.

**Note:** The Rotation matrices here correspond to the case where **the same vector** is represented **in two different coordinate systems**. We could equally well have **rotated a vector** in a **single coordinate system**. The resulting rotation matrices would be the **inverse** (or transpose) of the matrices discussed here. This may explain a discrepancy between my notation and the one used in some math texts in current use. In the present course we use these matrices almost exclusively to go from a body centered coordinate system to a space centered system when describing the orientation of a given solid body, hence our choice.

**Example problem Problem 4.1.1**
(Question 2 of 2001 problem set 1)

**a:** Find the components of the vector

$$\vec{a} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

in a coordinate system which is rotated about the $\hat{\mathbf{i}}-$axis by $60^0$ counterclockwise.

**b:** The vector $\vec{b}$ is obtained by rotating the vector $\vec{a}$ above by $60^0$ counterclockwise about the $\hat{\mathbf{i}}-$axis. What are the components of $\vec{b}$ in the $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ coordinate system.

## 4.2 Velocity and acceleration in different coordinate systems

LAST TIME

- Discussed how to rotate the coordinate system and defined the Euler angles.

TODAY

- Describe particle **trajectories** in
  - Cartesian
  - plane polar
  - cylindrical
  - spherical coordinates.
- Need to discuss velocity and acceleration in these systems.

CARTESIAN COORDINATES

The **trajectory** of a particle is just the time dependent position vector

$$\vec{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

The three unit vectors do not change in time and the velocity vector is obtained simply by differentiating the components:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

Similarly the acceleration is

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} + \frac{d^2z}{dt^2}\hat{\mathbf{k}}$$

EXAMPLE: A ROLLING CYLINDER.

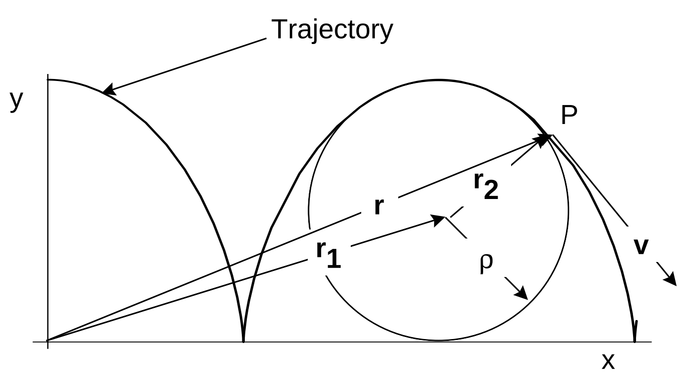
A cylinder (radius $\rho$) is rolling in the $xy-$plane
The axis moves with speed $s$ along $\hat{\mathbf{i}}$ .

Find the $x-$ and $y-$ coordinates of the position and speed of a point $P$ on the surface.

We have for the position vector to $P$,

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$

$\vec{r}_1$ is the cylinder axis
$\vec{r}_2$ extends from the axis to $P$




$$\vec{r}_1 = st\hat{\mathbf{i}} + \rho\hat{\mathbf{j}}$$

$$\vec{r}_2 = \rho\sin(\omega t)\hat{\mathbf{i}} + \rho\cos(\omega t)\hat{\mathbf{j}}$$

The rolling constraint is that P has zero speed when touching the $x-$axis
This gives $\omega = s/\rho$ We find

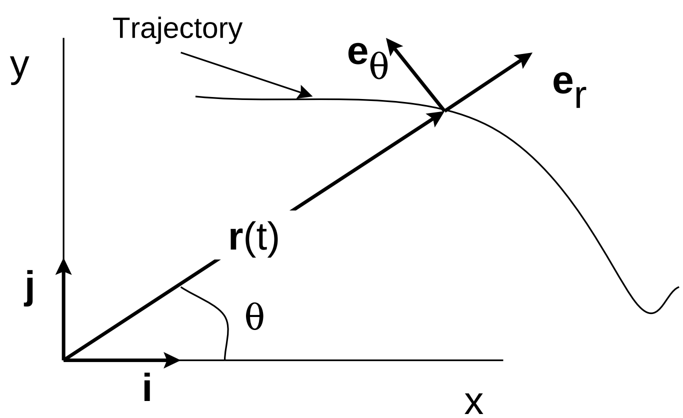
$$\vec{r} = [st + \rho \sin(\frac{st}{\rho})]\hat{\mathbf{i}} + \rho[1 + \cos(\frac{st}{\rho})]\hat{\mathbf{j}}$$

$$\vec{v} = s[1 + \cos(\frac{st}{\rho})]\hat{\mathbf{i}} - s\sin(\frac{st}{\rho})\hat{\mathbf{j}}$$

The corresponding curves in the $x - y$ plane are called cycloids. We have come across them before in the brachistochrone problem lecture 3.4.

PLANE POLAR COORDINATES
The coordinates in this system are $r$, $\theta$

$$\vec{r} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{i}} = r\vec{e}_r$$




The Cartesian components of the polar unit vectors are

$$\hat{e}_r = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$$

$$\hat{e}_\theta = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}$$

The calculation of velocity and acceleration along a trajectory is now complicated by the fact that $\hat{e}_r$ and $\hat{e}_\theta$ are changing along the path.

The unit vector time derivatives are

$$\frac{d\hat{e}_r}{dt} = \frac{d\theta}{dt}(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) = \frac{d\theta}{dt}\hat{e}_\theta$$

$$\frac{d\hat{e}_\theta}{dt} = -\frac{d\theta}{dt}(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) = -\frac{d\theta}{dt}\hat{e}_r$$

We have for the velocity

$$\vec{v} = \frac{d}{dt}(r\hat{e}_r) = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

which yields

$$\boxed{\vec{v} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta}$$

The acceleration is

$$\vec{a} = \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\hat{e}_r}{dt} + (\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2})\hat{e}_\theta + r\frac{d\theta}{dt}\frac{d\hat{e}_\theta}{dt}$$

or

$$\boxed{\vec{a} = [\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2]\hat{e}_r + [r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}]\hat{e}_\theta}$$

EXAMPLE
A racing car moves in a circular path of radius $a$ the speed of the car is $s(t)$. Find the velocity and acceleration in plane polar coordinates.

We have

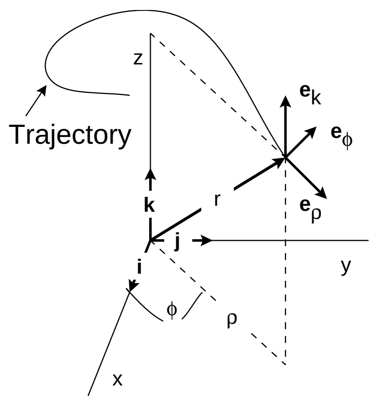
$$\frac{dr}{dt} = 0$$

$$\frac{d\theta}{dt} = \frac{s(t)}{a}$$

¿From which we find

$$\boxed{\vec{v} = s(t)\hat{e}_\theta}$$

$$\vec{a} = -\frac{s^2}{a}\hat{e}_r + \frac{ds}{dt}\hat{e}_\theta$$




Cylindrical coordinates

$\mathbf{r}=\rho\,\mathbf{e}_\rho+z\,\mathbf{e}_z$

$\mathbf{e}_\rho=\mathbf{i}\cos\phi+\mathbf{j}\sin\phi$

$\mathbf{e}_\phi=-\mathbf{i}\sin\phi+\mathbf{j}\cos\phi$

The vector $\hat{\mathbf{k}}$ does not change along the trajectory and we find

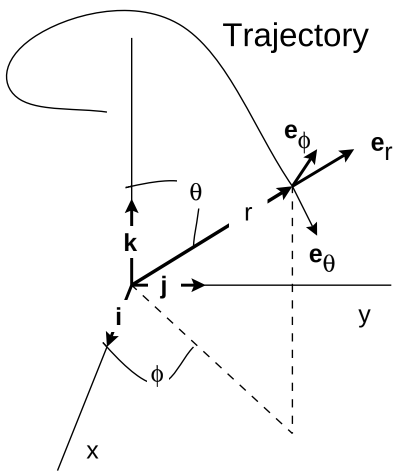
$$\vec{v} = \frac{d\rho}{dt}\hat{e}_\rho + \rho\frac{d\phi}{dt}\hat{e}_\phi + \frac{dz}{dt}\hat{e}_z$$

$$\vec{a} = [\frac{d^2\rho}{dt^2} - \rho(\frac{d\phi}{dt})^2]\hat{e}_\rho + [2\frac{d\rho}{dt}\frac{d\phi}{dt} + \rho\frac{d^2\phi}{dt^2}]\hat{e}_\phi + \frac{d^2 z}{dt^2}\hat{e}_z$$

We have

$$\frac{d\hat{e}_r}{dt} = (\frac{d\theta}{dt}\cos\theta\cos\phi - \frac{d\phi}{dt}\sin\theta\sin\phi)\hat{\mathbf{i}}$$
$$+(\frac{d\theta}{dt}\cos\theta\sin\phi - \frac{d\phi}{dt}\sin\theta\cos\phi)\hat{\mathbf{j}} - \frac{d\theta}{dt}\sin\theta\hat{\mathbf{k}}$$

# SPHERICAL COORDINATES




$x = r \sin\theta \cos\phi$
$y = r \sin\theta \sin\phi$
$z = r \cos\theta$

$\mathbf{e_r} = \mathbf{i} \sin\theta \cos\phi + \mathbf{j} \sin\theta \sin\phi + \mathbf{k} \cos\theta$
$\mathbf{e_\theta} = \mathbf{i} \cos\theta \cos\phi + \mathbf{j} \cos\theta \sin\phi - \mathbf{k} \sin\theta$
$\mathbf{e_\phi} = -\mathbf{i} \sin\phi + \mathbf{j} \cos\phi$

which simplifies to

$$\frac{d\hat{e}_r}{dt} = \frac{d\phi}{dt}\sin\theta\hat{e}_\phi + \frac{d\theta}{dt}\hat{e}_\theta$$

The velocity is given by

$$\vec{v} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

from which we find

$$\vec{v} = \frac{dr}{dt}\hat{e}_r + r\frac{d\phi}{dt}\sin\theta\hat{e}_\phi + r\frac{d\theta}{dt}\hat{e}_\theta$$

By differentiation once more one can obtain an expressions for the acceleration.

We have for the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\hat{e}_r}{dt}$$
$$+(\frac{dr}{dt}\frac{d\phi}{dt} + r\frac{d^2\theta}{dt^2})\hat{e}_\theta + r\frac{d\theta}{dt}\frac{d\hat{e}_\theta}{dt}$$
$$+(\frac{dr}{dt}\frac{d\phi}{dt}\sin\theta + r\frac{d^2\phi}{dt^2}\sin\theta + r\frac{d\phi}{dt}\cos\theta)\hat{e}_\phi$$
$$+r\frac{d\phi}{dt}\sin\theta\frac{d\hat{e}_\phi}{dt}$$

We need

$$\frac{d\hat{e}_\theta}{dt} = -\frac{d\theta}{dt}\hat{e}_r + \frac{d\phi}{dt}\cos\theta\hat{e}_\phi$$
$$\frac{d\hat{e}_\phi}{dt} = -\frac{d\phi}{dt}\hat{e}_r + \frac{d\phi}{dt}\cos\theta\hat{e}_\theta$$

which yields the unfortunately rather messy expression

$$\vec{a} = [\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2 - r(\frac{d\phi}{dt})^2\sin^2\theta]\hat{e}_r$$
$$+[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} - r(\frac{d\phi}{dt})^2\sin\theta\cos\theta]\hat{e}_\theta$$
$$+[2\frac{dr}{dt}\frac{d\phi}{dt}\sin\theta + 2r\frac{d\phi}{dt}\frac{d\theta}{dt}\cos\theta + r\frac{d^2\phi}{dt^2}\sin\theta]\hat{e}_\phi$$

SUMMARY

We have derived formulas for the velocity and acceleration of a particle in

- Cartesian
- plane polar
- cylindrical
- spherical

coordinate systems.
Clearly, particularly in the spherical coordinate system, the expressions are only only useful when, for some reason, several of the terms vanish.

The selection of which system to use depends on the symmetry of the problem!

## Example problem

**Problem 4.2.1**
(Question 1 of 1999 problem set 1)
Prove that

$$\frac{d}{dt}[\vec{r}\cdot(\vec{v}\times\vec{a})] = \vec{r}\cdot(\vec{v}\times\frac{d\vec{a}}{dt})$$

**Problem 4.2.2**
(Question 3 of 1999 problem set 1)
A sphere of radius $R$ is spinning about the $z - axis$ with angular velocity $\omega$ in an inertial frame. An object is moving with speed $v$ at $45^0\ N$ on the surface of the sphere in the southerly direction in a coordinate system fixed on the sphere. Find the components of the **velocity** and **acceleration** in the inertial frame, in spherical coordinates.

## 4.3 Non-inertial reference frames

LAST TIME
Derived formulas for velocity and acceleration in

- Cartesian
- plane polar
- cylindrical
- spherical

coordinates TODAY
Wish to discuss **non-inertial frames of reference**

ACCELERATED FRAMES OF REFERENCE
Suppose a system is described by the Lagrangian

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - \mathcal{U}(\vec{x})$$

in an inertial frame of reference. In Newtonian mechanics this the equation of motion would be

$$m\frac{d^2\vec{x}(t)}{dt^2} = f(\vec{x})$$

where

$$f(\vec{x}) = -\nabla\mathcal{U}(\vec{x})$$

We wish to be able to describe this situation in a non-inertial coordinate system where the position of a particle is $\vec{r'}$ and the second coordinate system is moving with velocity $\vec{R}(t)$ with respect to the first.

NEWTONIAN DESCRIPTION
Let $\vec{x}$ be the coordinate in the inertial frame

$$\vec{x} = \vec{r'} + \vec{R}(t) \tag{16}$$

We write

$$\vec{A}(t) = \frac{d\vec{V}}{dt} = \frac{d^2\vec{R}}{dt^2}$$

$$\tilde{f}(\vec{r'}) = f(\vec{x});\ \tilde{\mathcal{U}}(\vec{r'}) = \mathcal{U}(\vec{x})$$

The functions with ˜ (tilde) are obtained by substituting the coordinate transformation (1) in the original function.

$$m\frac{d^2\vec{r'}}{dt^2} = f(\vec{x}) - m\frac{d^2\vec{R}}{dt^2} = \tilde{f}(\vec{r'}) - m\vec{A}(t)$$

The interpretation of this result is in principle straight-forward:
**the acceleration of the primed coordinate system introduces a pseudo -force** $-m\vec{A}$

LAGRANGIAN VERSION

The particle velocity is $\vec{v}'$ in the non-inertial frame. In the inertial frame

$$\vec{w} = \vec{v}' + \vec{V}(t)$$

$$\mathcal{L} = \frac{m}{2}((v')^2 + 2\vec{v}' \cdot \vec{V} + V^2) - \tilde{\mathcal{U}}(\vec{r}')$$

The last term in the kinetic energy doesn't depend on any of the coordinates and will not affect the equations of motion. The second last term can be written

$$m\vec{v}' \cdot \vec{V} = \frac{d}{dt}(m\vec{r}' \cdot \vec{V}) - m\vec{r}' \cdot \frac{d\vec{V}}{dt}$$

Since total time derivatives don't contribute to the equations of motion we find for the transformed Lagrangian

$$\mathcal{L}' = \frac{mv'^2}{2} - \tilde{\mathcal{U}}(\vec{r}') - m\vec{r}' \cdot \vec{A}$$

The equation of motion for this Lagrangian is easily seen to be the same as what we found in the Newtonian case.

ROTATING FRAMES OF REFERENCE

Let us now bring in a further frame of reference which is rotating with respect to the primed reference. We saw in lecture 3.9 that transformation matrices associated with a **finite** angle in general don't commute, i.e. the order in which successive rotations are carried out does matter. This is not the case for **infinitesimal rotations**. Consider e.g. an infinitesimal rotation $d\phi_z$ about the $z-$axis

$$R_1 = \begin{pmatrix} \cos\phi_z & \sin\phi_z & 0 \\ -\sin\phi_z & \cos\phi_z & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d\phi_z & 0 \\ -d\phi_z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(d\phi_z^2)$$

Similarly an infinitesimal rotation $d\phi_x$ about the $x-$axis can be written

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d\phi_x \\ 0 & -d\phi_x & 1 \end{pmatrix} + \mathcal{O}((d\phi_x^2)$$

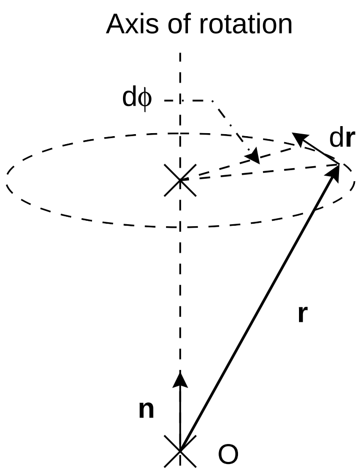
We now note that **to lowest order in the infinitesimals** (i.e. neglecting terms such as $d\phi_x^2, d\phi_z^2, d\phi_x d\phi_z$) we have

$$R = R_2 R_1 = R_1 R_2 = \begin{pmatrix} 1 & d\phi_z & 0 \\ -d\phi_z & 1 & d\phi_x \\ 0 & -d\phi_x & 1 \end{pmatrix}$$

Because of the additivity property of the rotations we can represent an arbitrary infinitesimal rotation as a **vector**

$$d\vec{\phi} = d\phi_x \hat{\mathbf{i}} + d\phi_y \hat{\mathbf{j}} + d\phi_z \hat{\mathbf{k}} = d\phi \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector in the direction of the **instantaneous axis of rotation** and $d\phi = |d\vec{\phi}|$.




From the figure above we see that the displacement $d\vec{r}$ of a point a distance $\vec{r}$ from a point $O$ on the axis of rotation can be written

$$d\vec{r} = d\phi \hat{\mathbf{n}} \times \vec{r} = d\vec{\phi} \times \vec{r}$$

The corresponding rotation matrix is

$$R = \begin{pmatrix} 1 & d\phi_z & d\phi_y \\ -d\phi_z & 1 & d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{pmatrix}$$

The velocity $\vec{v}$ of a point $P$ a distance $\vec{r}$ from a point 0 on the instantaneous axis of rotation is then

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Consider a frame of reference in which the origin $O$ is moving with respect to an inertial frame. In addition the frame rotates with angular velocity

$$\vec{\omega} = \omega \hat{\mathbf{n}}$$

about an axis through $O$. Let $\vec{r}$ be the position and $\vec{v}$ the velocity of a particle in the rotating frame and let $\vec{v}\,'$ be the velocity of the same particle in the non-rotating (primed) coordinate system moving with velocity $\vec{V}$, acceleration $\vec{A}$ with respect to the inertial frame

$$\vec{v}\,' = \vec{v} + \vec{\omega} \times \vec{r}$$

while the vector $\vec{r}$ and $\vec{r}\,'$ are the **same vector**, (but with different coordinates) in the rotating and non-rotating frames). $U(\vec{r})$ is the potential energy expressed in terms of the coordinates of the rotating frame.
We have for the Lagrangian

$$\mathcal{L} = \frac{m}{2}(v^2 + 2\vec{v} \cdot (\vec{\omega} \times \vec{r}) + (\vec{\omega} \times \vec{r})^2) - m\vec{A} \cdot \vec{r} - \mathcal{U}(\vec{r})$$

ENERGY AND MOMENTUM IN ROTATING FRAME
we use the notation

$$\frac{\partial \mathcal{L}}{\partial \vec{v}} = \hat{\mathbf{i}}\frac{\partial \mathcal{L}}{\partial v_x} + \hat{\mathbf{j}}\frac{\partial \mathcal{L}}{\partial v_y} + \hat{\mathbf{k}}\frac{\partial \mathcal{L}}{\partial v_z}$$

$$\frac{\partial \mathcal{L}}{\partial \vec{r}} = \hat{\mathbf{i}}\frac{\partial \mathcal{L}}{\partial x} + \hat{\mathbf{j}}\frac{\partial \mathcal{L}}{\partial y} + \hat{\mathbf{k}}\frac{\partial \mathcal{L}}{\partial z}$$

The generalized momentum in the rotating frame of reference is

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \vec{v}} = m\vec{v} + m\vec{\omega} \times \vec{r} = m\vec{v}'$$

i.e. the momentum has the same value as in the non-rotating frame.
The energy is

$$\mathcal{E} = \vec{p} \cdot \vec{v} - \mathcal{L} = mv^2 + m\vec{v} \cdot (\vec{\omega} \times \vec{r})$$

$$-\frac{m}{2}(v^2 + 2\vec{v}\cdot(\vec{\omega}\times\vec{r}) + (\omega\times\vec{r})^2) + m\vec{A}\cdot\vec{r} + \mathcal{U}(\vec{r})$$
$$= \frac{1}{2}mv^2 - \frac{m}{2}(\omega\times\vec{r})^2 + m\vec{A}\cdot\vec{r} + U(\vec{r})$$

i.e the energy in the rotating frame contains an extra centrifugal term.

EQUATIONS OF MOTION

We have

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + m\frac{d\vec{\omega}}{dt}\times\vec{r} + m\vec{\omega}\times\vec{v}$$

Remembering that

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = c\cdot(\vec{a}\times\vec{b})$$
$$\frac{\partial}{\partial\vec{r}}(\vec{v}\cdot(\vec{\omega}\times\vec{r})) = \frac{\partial}{\partial\vec{r}}(\vec{r}\cdot(\vec{v}\times\vec{\omega})) = \vec{v}\times\vec{\omega}$$
$$\frac{\partial}{\partial\vec{r}}(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r}) = \frac{\partial}{\partial\vec{r}}((\vec{\omega}\times\vec{r})\times\vec{\omega})\cdot\vec{r}$$
$$= 2(\vec{\omega}\times\vec{r})\times\vec{\omega}$$

Collecting terms we find

$$\boxed{\frac{md\vec{v}}{dt} = -\frac{\partial\mathcal{U}}{\partial\vec{r}} - m\vec{A} + 2m\vec{v}\times\vec{\omega} + m\vec{r}\times\frac{d\vec{\omega}}{dt} + m\vec{\omega}\times(\vec{r}\times\vec{\omega})}$$

The terms on the right hand side are

$$-\frac{\partial\mathcal{U}}{\partial\vec{r}} \Rightarrow original\ force$$
$$-m\vec{A} \Rightarrow due\ to\ accel.\ of\ non-rotating\ frame$$
$$2m\vec{v}\times\vec{\omega} \Rightarrow Coriolis\ force$$
$$m\vec{r}\times\frac{d\vec{\omega}}{dt} \Rightarrow due\ to\ non-uniform\ rotation$$
$$m\vec{\omega}\times(\vec{r}\times\vec{\omega}) \Rightarrow centrifugal\ force$$

DEFLECTION OF FALLING BODY

The problem is to find the deflection of a freely falling body from the vertical due to the rotation of the earth. Since the deflection will be small we only calculate it to lowest order in the $\omega$. We have

$$\mathcal{U} = -m\vec{g} \cdot \vec{r}$$

where $\vec{g}$ is in the $-z-$direction (downwards).
Neglecting the centrifugal force as being of order $\omega^2$

$$\frac{d\vec{v}}{dt} = 2\vec{v} \times \vec{\omega} + \vec{g}$$

Next put

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$

where

$$\vec{v}_1 = \vec{g}t$$

is the velocity in the absence of rotation, assumed to be zero initially. To lowest order in $\omega$

$$\frac{d\vec{v}_2}{dt} = 2\vec{v}_1 \times \vec{\omega} = 2t\vec{g} \times \vec{\omega}$$

If the initial height is $h$

$$\vec{r} = \vec{h} + \frac{g}{2}t^2 + \frac{t^3}{3}\vec{g} \times \vec{\omega}$$

At latitude $\lambda$

$$|\vec{g} \times \vec{\omega}| = g\omega\cos(\lambda)$$

and directed eastwards. If the time of flight is

$$t = \sqrt{\frac{2h}{g}}$$

the easterly deflection is

$$y = \frac{1}{3}(\frac{2h}{g})^{3/2} g\omega\cos\lambda$$

SUMMARY

We have obtained equations of motion or a body moving

- in a frame of reference accelerated with respect to an inertial frame of reference
- in a frame of reference rotating with respect to an inertial frame.

In the latter case we found centrifugal, Coriolis and terms due to non-uniform rotation.

We illustrated the effect of the Coriolis force by using a freely falling body as an example.

**Example problem Problem 4.3.1**
(Question 1 of 1999 problem set 5)

A projectile is fired straight up with initial speed $v_0$. Assuming $g$ is constant and ignoring air resistance, show
that the bullet will hit ground **west** of the initial point
of outward motion by an amount

$$\frac{4\omega v_0^3 \cos\lambda}{3g^2}$$

**Problem 4.3.2**
(Question 2 of 1999 problem set 5)

**a:** Find the magnitude and direction of the Coriolis force on a $1kg$ object moving north parallel to the earth surface at a latitude of $45^o$ N with speed $1km\ s^{-1}$

**b:** What is the magnitude and direction of the centrifugal

force under the same conditions as in **a**. The radius of the earth is approximately $6.4\ 10^6 m$.

**Problem 4.3.3**
(Question 1 of 2002 problem set 4)
In 1953 there was a severe flood along the Dutch coast (approximately $50^o N$). At one place water flowed from west to east with a speed of 1.25 m $s^{-1}$ through

a sea arm whose width is 4.8 km. Calculate the difference in sea levels at the two coasts. Which coast would have the highest sea level? (Apparently the actual height difference was three times larger due to wind shear).

**Problem 4.3.4**
(Question 2 of 2002 problem set 4)
A vector $\vec{A}$ has coordinates

$$A = (1, 1, 1)$$

in a Cartesian coordinate system. What are the coordinates in a system rotated with respect to the first by the Euler angles

$$\phi = \pi,\ \theta = \frac{\pi}{2},\ \psi = \frac{\pi}{4}$$

**Problem 4.3.5**
(Question 4 of 2002 final)
**a:** A particle is dropped from rest (in the reference frame of the earth rotating about its axis with angular velocity $\omega$) at height $h$ directly above you. Find the Coriolis deflection from you when it reaches the ground. You may assume that the acceleration of $g$ is constant and that $\omega$ is small enough that the Coriolis force is small compared to $mg$.
**b:** Estimate the minimum height from which it can be dropped and miss you. The latitude of Vancouver is approximately $49^o$. If you wish to duck the falling object, in which direction should you move?

**Problem 4.3.6**
(Question 3 of 2001 final)
An object with mass $m$ moves in a horizontal plane with velocity $v$ on the surface of the earth at latitude $\lambda$.
**a:**
Show that the magnitude of the horizontal component of the Coriolis force is independent of the direction of motion of the particle. Find a formula for the magnitude of the force.
**b:** Is the direction of the deviation to the right or to the left in the

1. northern hemisphere?
2. southern hemisphere?

**Problem 4.3.7**
(Question 2 of 2000 final)
A bead of mass $m$ slides without friction on a smooth circular wire in a vertical plane. The wire is rotating about its vertical diameter with angular velocity $\omega$. The acceleration of gravity is $g$
**a:**
Under what conditions is the bottom position stable?
**b:**
The bead oscillates back and forth with small amplitude about the bottom position under conditions which are stable under **a:**. What is the period of oscillation?
**c:**
When the bottom position is unstable find the stable equilibrium angle $\theta$ of the bead with respect to the vertical.

We have for the kinetic energy of a particle, expressed in terms of the coordinates in a frame of reference which is rotating with constant angular velocity $\omega$ with respect to an inertial frame

$$\mathcal{T} = \frac{m}{2}(v^2 + 2\vec{v} \cdot (\vec{\omega} \times \vec{r}) + (\vec{\omega} \times \vec{r})^2)$$

# 5 Central forces

## 5.1 Angular momentum. Central forces. Kepler's laws

. LAST TIMES

- Discussed coordinate transformations involving rotation
- Expressed velocity and acceleration in different coordinate systems
- Introduced non inertial frames of reference

TODAY
We will allow some time for all this material to settle in and instead discuss 3-dimensional motion of a pair of masses which interact by forces which only depend on the distance between them.

- Historically the most important such problem is the motion of a planet around its sun under the influence of gravity, the "Kepler problem".
- If there are no external forces the center of mass motion will be uniform and the center of mass momentum is conserved.
- Will show that trajectories associated with the relative motion are determined from laws of conservation of energy and angular momentum. Again there is no need to solve the equations of motion explicitly.

RELATIVE AND CENTER OF MASS MOTION

Let us consider two masses $M$ (e.g. the Sun) and $m$ (a planet e.g. the Earth). The position of the sun is $\vec{R}$ while the planet is at $\vec{r}$, with the velocities being $\vec{V}$ and $\vec{v}$, respectively. The position of the **center of mass** is

$$\vec{r}_{cm} = \frac{m\vec{r} + M\vec{R}}{M+m}$$

we also introduce the relative coordinate

$$\vec{r}_{rel} = \vec{r} - \vec{R}$$

We find

$$\begin{aligned}\vec{r} &= \vec{r}_{cm} + \frac{M}{M+m}\vec{r}_{rel}\\ \vec{R} &= \vec{r}_{cm} - \frac{m}{M+m}\vec{r}_{rel}\\ \vec{v} &= \vec{v}_{cm} + \frac{M}{M+m}\vec{v}_{rel}\\ \vec{V} &= \vec{v}_{cm} - \frac{m}{M+m}\vec{v}_{rel}\end{aligned}$$

KINETIC AND POTENTIAL ENERGY AND THE LAGRANGIAN

The kinetic energy of the system is

$$\mathcal{T} = \frac{m\vec{v}^2}{2} + \frac{M\vec{V}^2}{2}$$

which we can write in terms of the relative and center of mass velocities as

$$\mathcal{T} = \frac{(M+m)v_{cm}^2}{2} + \frac{\mu v_{rel}^2}{2}$$

where $\mu$ is the **reduced mass**

$$\mu = \frac{Mm}{M+m}$$

We express $\vec{r}_{rel}$ in spherical polar coordinates (see lecture 4.2. Dropping the subscript $rel$ for the relative coordinates $r, \theta, \phi$)

$$\vec{v}_{rel} = \frac{dr}{dt}\hat{e}_r + r\frac{d\phi}{dt}\sin\theta\hat{e}_\phi + r\frac{d\theta}{dt}\hat{e}_\theta$$

we find

$$\mathcal{T} = \frac{(M+m)v_{cm}^2}{2} + \frac{\mu}{2}(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2)$$

We assume that the potential energy of interaction only depends on the **magnitude** of the relative coordinate. The force associated with such a potential is called a **central force**. In the special case of the gravitational interaction the potential energy is

$$\mathcal{U}(r) = -\frac{GmM}{r}$$

where $G = 6.672\ 10^{-11}\ Nm^2kg^{-2}$. Let us describe the center of mass motion in a Cartesian coordinate system

$$v_{cm} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

The Lagrangian is thus

$$\mathcal{L} = \mathcal{T} - \mathcal{U} = \frac{(M+m)[v_x^2 + v_y^2 + v_z^2]}{2} + \frac{\mu}{2}(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2) - \mathcal{U}(r)$$

CONSERVATION CENTER OF MASS MOMENTUM

The Lagrangian doesn't depend on the position of the center of mass. The center of mass momentum is thus conserved and the components $v_x, v_y, v_z$

of the corresponding velocity components are also constant. Without loss of generality we choose to work in the reference frame where they are all zero.

ANGULAR MOMENTUM CONSERVATION

The Lagrangian doesn't depend on the angle $\phi$. The corresponding generalized momentum

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} \sin^2\theta = c = const$$

Suppose we choose coordinate axes so that the initial relative position and velocity lies in the plane $\phi = const$. We then have $c = 0$ and $\dot{\phi}$ will remain zero. The Lagrangian equation for $\theta$ is

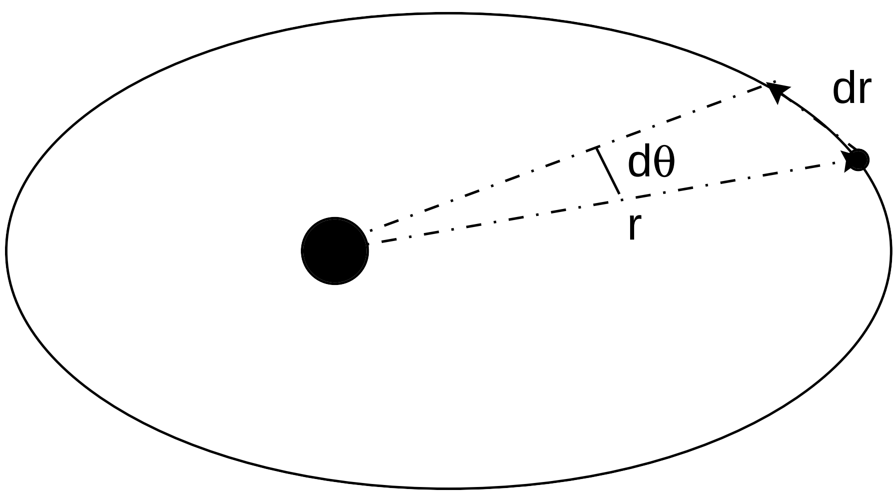
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt}\mu r^2 \dot{\theta} = \frac{\partial \mathcal{L}}{\partial \theta} = \mu r^2 \sin\theta \cos\theta \dot{\phi}^2$$

With this choice we see that since $\dot{\phi} = 0$ the **angular momentum** takes the form

$$\mu r^2 \dot{\theta} = l = const$$

and is a constant of the motion

This result can be interpreted physically as follows:




- The radius vector $\vec{r}$ from the sun to the planet remains in the plane defined by $\vec{v}_{rel}$.

- In this plane the radius vector will span an **area**

$$dA = r^2 d\theta$$

in a time interval $dt$.

- The area swept per unit time is

$$\frac{dA}{dt} = r^2\dot{\theta} = \frac{l}{\mu} = const$$

This result is known as **Kepler's second law**.

Some texts, e.g. the one by Hand and Finch employ a different but equivalent choice of coordinates. Instead of assuming that the orbit lies in the plane $\phi = const$, they place it in the plane $\theta = \pi/2$ ($x - y$ plane of the spherical polar coordinate system).

RADIAL EQUATION OF MOTION
The Lagrangian equation for $r$ is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \mu\ddot{r} = \frac{\partial\mathcal{L}}{\partial r} = \mu r(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) - \frac{d\mathcal{U}}{dr}$$

If we choose the orbit to lie in the plane $\phi = const$ and substitute the expression angular momentum $l$ we find

$$\mu\ddot{r} = \frac{l^2}{\mu r^3} - \frac{d\mathcal{U}}{dr}$$

In the special case of the Gravitational interaction we find

$$\mu\ddot{r} = \frac{l^2}{\mu r^3} - \frac{k}{r^2}$$

where $k = GMm$.

It is important to note that we substitute the angular momentum conservation law into the **equation of motion** not into the Lagrangian. If we do the substitution into the Lagrangian we would get the **wrong** equation of motion. The reason for this is that if we interprete the conservation law $l = \mu r^2 \dot{\theta}$ as a **constraint** this constraint would be **non-holonomic**!

ENERGY CONSIDERATIONS

Since the Lagrangian doesn't depend on time the energy

$$E = \mathcal{T} + \mathcal{U} = const. = \frac{\mu \dot{r}^2}{2} + \frac{l^2}{2\mu r^2} + \frac{k}{r}$$

¿From

$$v_{rel}^2 = \dot{r}^2 + r\dot{\theta}^2$$

we find

$$\dot{r}^2 = \frac{2E}{\mu} - \frac{l^2}{\mu^2 r^2} + \frac{2k}{\mu r}$$

Our analysis of this equation proceeds in an analogous way to what we did for the pendulum (lecture 3.7). For $dr/dt$ to be real the right hand side of this equation must be positive.

The values of $r$ for which $\dot{r}$ is zero are **turning points** for $r(t)$.

By inspecting the above expression we find

- If $l = 0$ we have $d\theta/dt = 0$. The orbits will head straight towards, or away from, the center of mass.
  - If $E > 0$ there is no turning point for large $r$ The orbits will extend to or from $r = \infty$
  - If $E < 0$ the orbits will turn back towards the sun when

$$\frac{k}{r} = -E$$

- If $l \neq 0$ we note that

$$\lim_{r \to 0} -\frac{l^2}{m^2 r^2} = -\infty$$

  Hence, there will always be a closest approach $r = r_{min}$

- If $E > 0$ The turning point at $r = r_{min}$ is the only one. We shall show that the orbits are hyperbolas.

- If $E = 0$ the orbits still extends to and from $r = \infty$. We shall show that these orbits are parabolas.

- If $E < 0$ there is a second turning point at some $r = r_{max}$. The orbits will be bounded and we shall show that they are ellipses.

- The families of curves **hyperbolas, parabolas and ellipses** are called **conic sections**. Before we proceed we need to review conic sections!

SUMMARY
We have discussed the problem of two bodies in space interacting via a central potential and

- separated the center of mass and relative motion
- introduced the concept of reduced mass
- showed that for particles in central force field angular momentum is conserved.
- looked at Newton's law of gravitation as example of a central force field.
- started to classify the different types of orbits for the special case of a gravitational potential.

**Example problem Problem 5.1.1**
(Question 3 of 2000 problem set 6)
Two particles with mass $m_1$ and $m_2$ interact with gravitational forces. They start out from rest a distance $r$ apart and are allowed to fall into each other. How long does it take for them to collide?

## 5.2 Kepler problem continued. Properties of the orbits

LAST TIME
Discussed the two body problem with central forces and

- separated the center of mass and relative motion and introduced the concept of reduced mass
- showed that for particles in central force field angular momentum is conserved.
- showed that Kepler's second law (equal area equal time) was a consequence of angular momentum conservation.
- looked at Newton's law of gravitation as example of a central force field, and started to classify the different types of orbits for the special case of a gravitational potential.

CONSERVATION OF ANGULAR MOMENTUM

Let us recall our notation:
$M$ = mass of sun
$m$ = mass of planet
$\mu = \frac{Mm}{M+m}$ = reduced mass
$\vec{r}$ = radius vector from sun to planet
$k = GMm$ parameter describing gravitational interaction strength.

$$\vec{v}_{rel} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta = \text{ relative velocity of planet}$$

In vector notation
$\vec{l} = m\vec{r} \times \vec{v}_{rel}$ = angular momentum.
We argued last time that conservation of angular momentum implied that the orbits were restricted to the plane containing $\vec{r}$ and $\vec{v}_{rel}$ i.e. the plane perpendicular to $\vec{l}$.
The angular momentum is

$$l = \mu r^2 \frac{d\theta}{dt}$$

CONSERVATION OF ENERGY

The total energy in the c.m. frame is conserved:

$$E = \frac{\mu v^2}{2} - \frac{k}{r} = const.$$

The kinetic energy is

$$\frac{\mu v^2}{2} = \frac{\mu}{2}(\frac{dr}{dt})^2 + \frac{\mu r^2}{2}(\frac{d\theta}{dt})^2 = \frac{\mu}{2}(\frac{dr}{dt})^2 + \frac{l^2}{2\mu r^2}$$

Substitution into the expression for the energy gives

$$\frac{dr}{dt} = \pm\sqrt{\frac{2E}{\mu} + \frac{2k}{\mu r} - \frac{l^2}{\mu^2 r^2}}$$

EQUATION FOR ORBIT
Combining the previous equation with

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2}$$

gives

$$\frac{d\theta}{dr} = \pm\frac{\frac{l}{\mu r^2}}{\sqrt{\frac{2E}{\mu} + \frac{2k}{\mu r} - \frac{l^2}{\mu^2 r^2}}}$$

$$\theta = \pm\int\frac{\frac{ldr}{r^2}}{\sqrt{2\mu[E + \frac{k}{r}] - \frac{l^2}{r^2}}}$$

This integral can be evaluated to yield

$$\theta = \cos^{-1}\frac{\frac{l}{r} - \frac{k\mu}{l}}{\sqrt{2\mu E + \frac{\mu^2 k^2}{l^2}}} + constant \tag{17}$$

CHECK FOR DIMENSIONAL CONSISTENCY
The above is a complicated expression with lots of constants. In order to see if it may be correct, let us check for dimensional consistency

$$[\frac{l}{r}] = kg\ m\ s^{-1}$$

$$[k] = m^3 kg\ s^{-2}$$

$$[\frac{k\mu}{l}] = \frac{m^3 kg^2 s^{-2}}{kg\ m^2 s^{-1}} = kg\ m\ s^{-1}$$

$$[\sqrt{2\mu E}] = \sqrt{kg\ kg\ m^2 s^{-2}} = kg\ m\ s^{-1}$$

$$[\sqrt{\frac{\mu^2 k^2}{l^2}}] = kg\ m\ s^{-1}$$

OK!

EQUATION FOR THE ORBIT

We choose the orientation $\theta = 0$ so that the integration constant in (1) is zero.

$$\cos\theta = \frac{\frac{l}{r} - \frac{k\mu}{l}}{\sqrt{2\mu E + \frac{\mu^2 k^2}{l^2}}} \tag{18}$$

Let us define

$$p = \frac{l^2}{\mu k}$$

$$\epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} \tag{19}$$

$p$ has dimension *length*, while $\epsilon$ is dimensionless. With these new constants the equation for the orbit (2) simplifies to

$$r = p - r\epsilon\cos\theta \tag{20}$$

CIRCULAR ORBITS

The expression inside the square root in (3) defining $\epsilon$ must be positive. Physically this means that there is a **minimum energy** $E_{min}$ compatible with a given angular momentum

$$E_{min} = -\frac{\mu k^2}{2l^2}$$

When $E = E_{min}$, $\epsilon = 0$. The equation for the orbit is then just

$$r = p$$

i.e. the orbit is a circle of radius $p$.

ELLIPTIC ORBITS

If $E_{min} < E < 0$, we have $0 < \epsilon < 1$. The equation (4) for the orbit will then

describe an ellipse.
Let us first note that if the plane of the ellipse is taken to be the $x-y$ plane the equation for an ellipse in Cartesian coordinates can be written

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1 \tag{21}$$

where
$x_0, y_0 =$ coordinates of **center**.
$a =$ **semi major axis.**
$b =$ **minor axis.**
The **eccentricity** is defined as the distance from the center to the **focus**

$$\epsilon = \frac{a^2-b^2}{a^2}$$

To see that (4) can be written on the form (5) we introduce Cartesian coordinates

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta; \quad y = r\sin\theta$$

Squaring (4) we obtain

$$x^2 + y^2 = (p - \epsilon x)^2$$

by rearranging terms we find

$$a = \frac{p}{1-\epsilon^2}$$

$$b = \frac{p}{\sqrt{1-\epsilon^2}}$$

$$x_0 = \frac{-p\epsilon}{1-\epsilon^2} = -\epsilon a$$

This demonstrates that the orbit is an ellipse with the sun in the focus- which is known as **Kepler's first law**.

PARABOLIC ORBIT
If $E = 0$, $\epsilon = 1$. The equation for the orbit is then

$$r = p - \epsilon x$$

Squaring both sides gives

$$x^2 + y^2 = p^2 + x^2 - 2p\epsilon x$$

or

$$y^2 = p^2 - 2p\epsilon x$$

which is the equation for a **parabola**. The closest approach to the sun is $p/2$. The $x - axis$ is the axis of the parabola. The intercepts along the $y - axis$, when $x = 0$, are $\pm p$.

HYPERBOLIC ORBITS

If $E > 0$, $\epsilon > 1$ we have

$$y^2 - (\epsilon^2 - 1)x^2 = p^2 - 2p\epsilon x$$

This expression can be written as the equation of the **hyperbola**

$$\frac{y^2}{b^2} - \frac{(x - x_0)^2}{a^2} = 1$$

where

$$x_0 = \frac{-p\epsilon}{\sqrt{\epsilon^2 - 1}}$$

$$a = \frac{p}{\epsilon^2 - 1}$$

$$b = \frac{p}{\sqrt{\epsilon^2 - 1}}$$

as $r \to \infty$

$$\frac{x}{r} = \cos\theta = \frac{1}{\epsilon}$$

The angles of the **asymptotes** of the hyperbolas are thus

$$\theta = \pm\cos^{-1}\frac{1}{\epsilon}$$

The attached Maple worksheet at http://www.physics.ubc.ca/˜birger/n206l18.mws (or .html) plots the orbits in the three cases $\epsilon < 1, \epsilon = 1, \epsilon > 1$.

TIME DEPENDENCE OF THE ORBITS

The law of conservation of energy gave us an expression for the radial velocity which we can integrate to yield

$$t = \pm \int \frac{dr}{\sqrt{\frac{2E}{\mu} + \frac{2k}{\mu r} - \frac{l^2}{\mu^2 r^2}}}$$

after some algebra using the definitions of the parameters $p$, $\epsilon$ and $a$ we find after some algebra that this equation can be rewritten

$$t = \pm \sqrt{\frac{\mu}{2|E|}} \int \frac{r dr}{\sqrt{a^2\epsilon^2 - (r-a)^2}}$$

Let us introduce the new variable $\xi$ by writing

$$r = a - a\epsilon \cos \xi$$

$$dr = a\epsilon \sin \xi d\xi$$

Substitution into the integral for $t$ gives after a bit of algebra we find

$$t = \sqrt{\frac{a^3 \mu}{k}} \int d\xi (1 - \epsilon \cos \xi)$$

This integral can easily be performed and choosing the constant of integration so that so that $t = 0$ at the closest approach to the sun (perihelion) we find the following parametric description of the orbits

$$r = a(1 - \epsilon \cos \xi\}$$

$$t = \sqrt{\frac{a^3 \mu}{k}} (\xi - \epsilon \sin \xi)$$

During one full orbit $\xi$ will increase by $2\pi$, hence we find for the period of the orbit

$$T = 2\pi\sqrt{\frac{a^3\mu}{k}}$$

Put in words we find that the square of the period is proportional to the cube of the semi-major axis which is **Kepler's third law**.
SUMMARY

- We continued our discussion of the Kepler problem.
- We showed that the laws of conservation of angular momentum and energy allowed us to solve for the orbits in polar coordinates.
- When the energy is minimum value compatible with the angular momentum the orbits are circles.
- If $E_{min} < E < 0$. The orbits are ellipses with the sun in the focus.
- When $E = 0$ the orbits are parabolas.
- If $E > 0$ the orbits are hyperbolas.
- we also found parametric expressions for the time dependence of the elliptic orbits and a simple formula for the period.

## Example problem

### Problem 5.2.1

(Question 1 of 2000 problem set 6) Two masses $m$ and $M$ are connected by a weightless string of length $a$. Mass $m$ rests on a friction-less table and the string is threaded through a small hole. Mass $M$ is connected to the other end of the string and is constrained to only move up and down. Assume that the string is long enough so that the mass $M$ will not hit the bottom of the table.
**a:** Write down the Lagrangian using the polar coordinates $r, \theta$ of the mass $m$ as generalized coordinates.

**b:** The angular momentum $l$ associated with the polar angle $\theta$ and the energy $E$ will be conserved. Eliminate $\dot{\theta}$ from the expression for the energy. Write the resulting expression as

$$E = \frac{m+M}{2}\dot{r}^2 + U_{eff}(r)$$

**c:** Use a sketch $U_{eff}(r)$ to describe the orbits of $m$ qualitatively for different parameter values.
**d:** Under what circumstances will the orbits of $m$ be circular.
**e:** How could you verify numerically for given parameter values if the orbits $r(\theta)$ are periodic? (You are not required to do the actual calculation, this is the subject of one of the end of term projects.)

**Problem 5.2.2**
(Problem 2 of 2001 final exam)
A particle of mass $m$ is attracted towards a fixed point (central force). The potential energy is

$$U(r) = -\frac{ma}{r^2}$$

where r is the distance to the fixed point and a is a constant (that is the force is proportional to the inverse **cube** of the distance). The particle starts out a distance $\vec{c}$ from the fixed point with a velocity $\sqrt{2a/c}$ in a direction $45^o$ from $\vec{c}$ (away from the fixed point).
**a:**
Will the particle

1. move in an orbit bounded by a maximum and a minimum distance from the fixed point?
2. hit the fixed point?
3. escape to infinity?

**b:**
Find an equation $r(\theta)$ for the orbit in a spherical coordinate system where the initial velocity and position lies in the plane $\phi = const.$

**Problem 5.2.3**
(Question 2 of 2000 problem set 6)
A particle moves in a central potential of the form

$$U(r) = -\frac{k}{r} + \frac{\beta}{r^2}$$

**a:** Show that the equation for the orbit now can be written on the form

$$r = p - r\epsilon\cos(\alpha\theta)$$

Are the formulas for $p$ and $\epsilon$ the same as for the Kepler problem? What is the expression for $\alpha$.

**b:** If $\beta$ is small $\alpha$ will be close to unity and the orbits will be precessing ellipses. Choose the unit of length so that $p = 1$ and plot the orbits for a suitable value of $\alpha$

**Problem 5.2.4**
(Question 3 of 1999 midterm)
The energy and angular momentum of a particle in a central potential V(r) can be written in polar coordinates as

$$E = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\theta}^2}{2} + V(r)$$

$$M = mr^2\dot{\theta}$$

Assume the potential is on the form $V(r) = -\frac{\alpha}{r^2}$ When will a particle starting with $\dot{r} < 0$ hit the origin? When will there be a closest approach $r_{min} \neq 0$

# 6 Rigid body dynamics

## 6.1 Kinetic energy of a rigid body.

LAST TIMES
discussed the Kepler problem.
TODAY
We wish to commence our discussion on rigid body dynamics. In lecture 4.3

we derived a general formula for the Lagrangian in a non inertial frame of reference.

$$\mathcal{L} = \frac{m}{2}(v^2 + 2\vec{v}\cdot(\vec{\omega}\times\vec{r}) + (\vec{\omega}\times\vec{r})^2) - m\vec{A}\cdot\vec{r} - \mathcal{U}(\vec{r})$$

The origin of our coordinate system accelerates with acceleration $\vec{A}$ with respect to an inertial frame of reference. In this system the position of a particle is represented by the vector $\vec{r}$, The non-inertial frame of reference was also rotating with angular velocity $\vec{\omega}$ about an axis through its origin. We also derived the equation of motion associated with this Lagrangian

$$\frac{md\vec{v}}{dt} = -\frac{\partial\mathcal{U}}{\partial\vec{r}} - m\vec{A} + 2m\vec{v}\times\vec{\omega} + m\vec{r}\times\frac{d\vec{\omega}}{dt} + m\vec{\omega}\times(\vec{r}\times\vec{\omega})$$

and identified the different terms. For this equation of motion to be valid the acceleration $\vec{A}$ and angular velocity $\vec{\omega}$ must be **externally imposed**. This assumption is reasonable enough when considering, say, a projectile on earth, that is too small to have any detectable effect on the motion of the earth.

TODAY
we wish to start examining the **dynamics** of a rigid body. Its motion will still be described in terms of translation and rotation, but these will now be the result of the forces and torques acting on the body.

We make the idealization of considering **rigid bodies**. Such an object can be considered to consist of a set of points $\{\alpha\}$ each with mass $m$. We impose the constraint that the distance between the different points **do not change in time**.

If we specify one point on a rigid body it may is still be rotated by an arbitrary angle about an axis through this point. If we give the three coordinates of the selected point, plus two angles specifying the orientation of the axis, plus the angle of rotation we can in principle locate any point on the body. Hence we need 6 coordinates to specify the position of a rigid body, which means that **a rigid body has 6 degrees of freedom** (see also lecture 7 3.1).

Another way of arriving at the same result is to note that if you specify the coordinates of a rigid body you need to specify the positions of three fixed points on it. If only one point is specified the body is free to rotate about any axis through that point. If two points are specified one is still free to rotate the body about an axis through the two points. If three points that do not lie on as straight line are specified the body is pinned down. However, the nine coordinates describing the three points are constrained by the fact that the distances $r_{12}, r_{13}, r_{23}$ are fixed by the rigid body conditions. This leaves 6 degrees of freedom.

Of the 6 degrees of freedom it is convenient to let three describe the position $\vec{R}$ of a chosen point $O$ on the body

$$\vec{R} = (R_x, R_y, R_z)$$

The other three degrees of freedom describes the orientation of the axes of a coordinate system fixed on the rigid body (we will later show how to do this through the Euler angles $\phi, \theta$ and $\psi$ defined in lecture 4.1).

Let $\vec{P}$ be some other point on the body. The position of this point in the inertial frame is

$$\vec{P} = \vec{R} + \vec{r}$$

where $\vec{r}$ is the vector $O \to P$. In some time interval $dt$ the point $O$ will move a distance

$$d\vec{R} = \vec{V} dt$$

where $\vec{V}$ is the velocity of the point $O$ with respect to an inertial frame.

The point $\vec{P}$ will in the same time interval rotate an angle $d\phi$ about some axis $\hat{n}$. We have

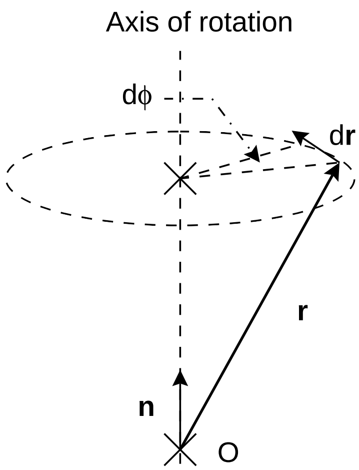
$$d\vec{r} = d\phi \; \hat{n} \times \vec{r}$$

The angular velocity vector is

$$\vec{\omega} = \frac{d\phi}{dt} \hat{n}$$

Hence

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$




and the velocity of the point $P$ in the inertial frame is

$$\vec{v} = \vec{V} + \vec{\omega} \times \vec{r} \tag{22}$$

Now, suppose some-one comes along and says that the most important point on the rigid body is not $O$ but $O'$ located a distance $\vec{a}$ from $O$. The velocity of the point $O'$ is

$$\vec{V}' = \vec{V} + \vec{\omega} \times \vec{a} \tag{23}$$

The vector $O' \to P$ is

$$\vec{r'} = \vec{r} - \vec{a}$$

Substituting (2) into (1) we find

$$\vec{v'} = \vec{V}' + \vec{\omega} \times \vec{r'}$$

**We conclude that the angular velocity vector does not depend on our choice of reference point ($O$ or $O'$).** However, the **translational velocity** will be different see (2).

We wish to apply the Lagrangian formalism to the dynamics of a rigid body. For this purpose we need a convenient description of its kinetic energy. The kinetic energy of the mass point $P$ is

$$T(P) = \frac{m}{2}(\vec{V} + \vec{\omega} \times \vec{r})^2 = \frac{m}{2}(V^2 + 2\vec{V} \cdot (\vec{\omega} \times \vec{r}) + (\omega \times \vec{r})^2)$$

The second term in the last expression can be rewritten

$$m\vec{V} \cdot (\vec{\omega} \times \vec{r}) = m\vec{r} \cdot (\vec{V} \times \vec{\omega})$$

Let $\phi$ be the angle between $\vec{\omega}$ and $\vec{r}$. Then

$$(\vec{\omega} \times \vec{r})^2 = \omega^2 r^2 \sin^2 \phi = \omega^2 r^2 - (\vec{\omega} \cdot \vec{r})^2$$

The kinetic energy of the point $P$ is thus

$$T(P) = \frac{m}{2} V^2 + m\vec{r} \cdot (\vec{V} \times \omega) + \frac{m}{2}[\omega^2 r^2 - (\vec{\omega} \cdot \vec{r})^2]$$

We next sum the kinetic energies of all the mass points $\alpha$ of the body noting that $\vec{V}$ and $\vec{\omega}$ is the same for all the points. Let

$$M = \sum_\alpha m$$

be the total mass. The position of the **center of mass**, $\vec{r}_{cm}$, is

$$\vec{r}_{cm} = \frac{1}{M} \sum_\alpha \vec{r}_\alpha m$$

We find

$$\boxed{T = \frac{M}{2} V^2 + M\vec{r}_{cm} \cdot (\vec{V} \times \omega) + \frac{1}{2} \sum_\alpha m[\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2]}$$

It is often convenient to choose the reference point $O$ to be the center of mass. The expression for the kinetic energy then simplifies to

$$\mathcal{T} = \frac{M}{2} V^2 + \frac{1}{2} \sum_\alpha m[\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2]$$

SUMMARY

- Of the 6 degrees of freedom of a rigid body 3 represent translational motion of a reference point $O$ on the body and 3 rotation of the body about the reference point.

- We showed that the angular velocity vector $\vec{\omega}$ is independent of the choice of reference point unlike the translational velocity $\vec{V}$.
- We have derived a general expression for the kinetic energy of a rigid body.
- This expression simplifies if the center of mass is chosen as the reference point.

## 6.2 Moment of inertia tensor.

LAST TIME

- Showed that of the 6 degrees of freedom of a rigid body, 3 represent translational motion of a reference point $O$ on the body, 3 describe rotation of the body about the reference point.
- Showed that the angular velocity vector $\vec{\omega}$ is independent of the choice of reference point, unlike the translational velocity $\vec{V}$.
- Derived a general expression for the kinetic energy of a rigid body.
- This expression simplifies if the center of mass is chosen as the reference point.

TODAY
We wish to introduce the concept of moment of inertia.

Recall the expression for the kinetic energy found last time

$$T = \frac{M}{2}V^2 + M\vec{r}_{cm} \cdot (\vec{V} \times \omega) + \frac{1}{2}\sum_{\alpha} m[\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2]$$

here
$O$ = fixed reference point on body
$\vec{r}_{cm}$ = center of mass relative to $O$
$m$ mass of point on body
$\vec{r}_\alpha$ position relative to $O$ of mass point
$\vec{\omega}$ = angular velocity of body

$\vec{V}$ = translational velocity of $O$
$M$ = mass of body.

We wish to rewrite the last term in the expression for the kinetic energy

$$\frac{1}{2}\sum_{\alpha} m[\omega^2 r_\alpha^2 - (\vec{\omega}\cdot\vec{r}_\alpha)^2] \tag{24}$$

on component form, using the "dummy" indices $i, j, k$ to represent any of the three components $x, y, z$. In this notation

$$\omega^2 = \sum_i \omega_i \omega_i$$

$$r_\alpha^2 = \sum_k r_{\alpha k} r_{\alpha k}$$

$$(\vec{\omega}\cdot\vec{r}_\alpha)^2 = \sum_{ij} \omega_i r_{\alpha i} r_{\alpha j} \omega_j$$

Recalling the definition of the Kronecker $\delta$

$$\delta_{ij} = \begin{cases} 1 & , \quad i = j \\ 0 & , \quad i \neq j \end{cases}$$

we also have

$$\omega^2 = \sum_{ij} \omega_i \omega_j \delta_{ij}$$

This allows us to rewrite (1) as

$$\sum_{\alpha}\sum_{ij} \omega_i [\delta_{ij} \sum_k r_{\alpha k} r_{\alpha k} - r_{\alpha i} r_{\alpha j}] \omega_j \equiv \sum_{ij} \omega_i I_{ij} \omega_j$$

where we define the **moment of inertia tensor** (or matrix) as

$$I_{ij} = \sum_{\alpha} m \begin{pmatrix} y_\alpha^2 + z_\alpha^2 & -x_\alpha y_\alpha & -x_\alpha z_\alpha \\ -x_\alpha y_\alpha & x_\alpha^2 + z_\alpha^2 & -y_\alpha z_\alpha \\ -x_\alpha z_\alpha & -y_\alpha z_\alpha & x_\alpha^2 + y_\alpha^2 \end{pmatrix}$$

In vector notation we write

$$\sum_{ij} \omega_i I_{ij} \omega_j = \vec{\omega}\cdot I\vec{\omega}$$

The kinetic energy of a rigid body is then

$$T = \frac{M}{2}V^2 + M\vec{r}_{cm} \cdot (\vec{V} \times \vec{\omega}) + \frac{1}{2}\vec{\omega} \cdot I\vec{\omega}$$

In practice we will not be summing over discrete mass points but use a continuum description of the rigid body. Let $\rho(\vec{r})$ be the **mass density** of the body. The position of the center of mass of the body is then

$$\vec{r}_{cm} = \frac{1}{M}\int_{volume} d^3r\rho(\vec{r})\vec{r}$$

EXAMPLE
Center of mass of a solid hemisphere (half grapefruit)
For reasons of symmetry the c.m. must lie on an axis (we call this the $z-axis$) through the center of the sphere, perpendicular to the cut. Let $a$ be the radius of the sphere. The mass of a slab of thickness $dz$ parallel to the cut is

$$dM = dz\pi(a^2 - z^2)\rho$$

the total mass is

$$M = \frac{2\pi\rho a^3}{3}$$

we have

$$z_{cm} = \frac{\pi\rho}{M}\int_0^a z(a^2 - z^2)dz = \frac{3a}{8}$$

The moment of inertia tensor for a continuous system can be written

$$I = \int_{volume} \rho(\vec{r})d^3r \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$

EXAMPLE
Moment of inertia of rectangular box (parallelepiped) about the c.m.

Let $a, b, c$ be the sides of the box parallel to the $x, y, z$-axes, respectively. We have for a typical diagonal element of the moment of inertia tensor, calculated with the center of mass as the reference point:

$$I_{xx}(cm) = \rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz (y^2 + z^2) = \rho abc (\frac{b^2 + c^2}{12}) = \frac{M(b^2 + c^2)}{12}$$

Similarly we have

$$I_{yy}(cm) = \frac{M(a^2 + c^2)}{12}$$

$$I_{zz}(cm) = \frac{M(a^2 + b^2)}{12}$$

It is easy to see that the off-diagonal elements are zero e.g.

$$I_{xy} = -\rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz \; xy = 0$$

The moment of inertia matrix about the center of mass is thus

$$I(cm) = M \begin{pmatrix} \frac{b^2+c^2}{12} & 0 & 0 \\ 0 & \frac{a^2+c^2}{12} & 0 \\ 0 & 0 & \frac{a^2+b^2}{12} \end{pmatrix}$$

SHIFTING THE ORIGIN

We next consider the problem of calculating the moment of inertia matrix about an arbitrary point, assuming the moments of inertia about the center of mass are known.

Let the position of an arbitrary point $P$ relative to the center of mass $O$ be $\vec{r}$. Let $O'$ be our new reference point and let the vector $O \to O'$ be $\vec{a}$. The position of $P$ relative to $O'$ is thus

$$\vec{r'} = \vec{r} - \vec{a}$$

A typical diagonal matrix element of the moment of inertia matrix about $O'$ is thus

$$I'_{xx} = \sum_\alpha m[(y_\alpha - a_y)^2 + (z_\alpha - a_z)^2] = \sum_\alpha m[y_\alpha^2 + a_y^2 + z_\alpha^2 + a_z^2 - 2a_y y -_\alpha 2a_z z_\alpha]$$

The last two terms in the square bracket are zero because $y$ and $z$ are coordinates relative to the center of mass yielding for a typical diagonal matrix element

$$I'_{xx} = I_{xx} + M(a_y^2 + a_z^2)$$

Similarly for an off diagonal matrix element

$$I'_{xy} = -\sum_\alpha m(x_\alpha - a_x)(y_\alpha - a_y) = I_{xy} - M a_x a_y$$

We conclude that

$$I' = I + M \begin{pmatrix} a_y^2 + a_z^2 & -a_x a_y & -a_x a_z \\ -a_y a_x & a_x^2 + a_z^2 & -a_y a_z \\ -a_z a_x & -a_z a_y & a_x^2 + a_y^2 \end{pmatrix}$$

The moment of inertia about an arbitrary point is equal to the the moment of inertia about the center of mass, plus the moment of inertia about the point, if all the mass had been located at the center of mass. Some of you will recognize this as a generalization of the **parallel axis theorem** that you may have encountered in elementary mechanics.

EXAMPLE

Consider again the rectangular box we looked at earlier. The moment of inertia at a corner located at

$$a_x = \frac{a}{2};\ a_y = \frac{b}{2};\ a_z = \frac{c}{2}$$

relative to the center of mass is

$$I' = M \begin{pmatrix} \frac{b^2+c^2}{12} & 0 & 0 \\ 0 & \frac{a^2+c^2}{12} & 0 \\ 0 & 0 & \frac{a^2+b^2}{12} \end{pmatrix} + M \begin{pmatrix} \frac{b^2+c^2}{4} & -\frac{ab}{4} & -\frac{ac}{4} \\ -\frac{ab}{4} & \frac{a^2+c^2}{4} & -\frac{bc}{4} \\ -\frac{ac}{4} & -\frac{bc}{4} & \frac{a^2+b^2}{4} \end{pmatrix}$$

or

$$I' = M \begin{pmatrix} \frac{b^2+c^2}{3} & -\frac{ab}{4} & -\frac{ac}{4} \\ -\frac{ab}{4} & \frac{a^2+c^2}{3} & -\frac{bc}{4} \\ -\frac{ac}{4} & -\frac{bc}{4} & \frac{a^2+b^2}{3} \end{pmatrix}$$

SUMMARY

We have

- defined the moment of inertia tensor
- expressed the kinetic energy of a rigid body in terms of this quantity
- given some examples of how it can be calculated.

**Example problem Problem 6.2.1**
(Question 2 of 2001 problem set 7) Three equal masses $m$ are located at the points $(a, 0, 0), (0, a, 0), (0, 0, a)$ in a Cartesian coordinates system (figure 2)

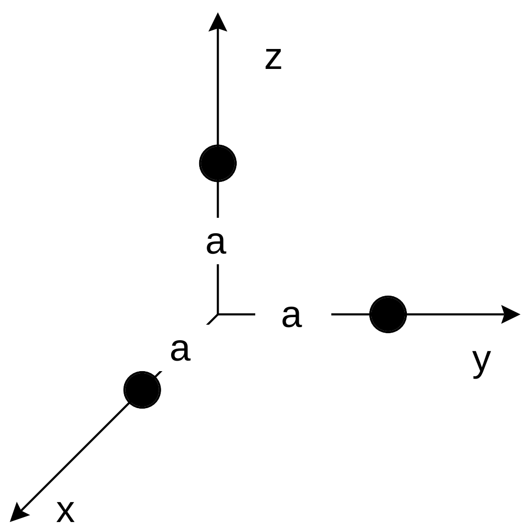



Figure 5:

**a:** Find the moment of inertia tensor about the origin.

**b:** Locate the center of mass.

**c:** Find the principal moments of inertia and directions of principal axes about the center of mass.

## 6.3 Some moment of inertia problems.

LAST TIME

- Defined the moment of inertia tensor
- Expressed the kinetic energy of a rigid body in terms of this quantity
- Gave some examples of how it can be calculated.

TODAY
Solve a few sample problems involving moments of inertia.

EXAMPLE
**Moment of inertia of a sphere about its center**
By symmetry the three principal moments of inertia are equal

$$I_{xx} = I_{yy} = I_{zz} = I$$

while all the off-diagonal products of inertia are zero. We have with $a$ the radius of the sphere

$$I_{xx} + I_{yy} + I_{zz} = 3I = 2\int \rho d^3r(x^2 + y^2 + z^2)$$

$$I = \frac{2}{3}\int_0^a dr\ 4\pi r^2\ r^2 = \frac{8\pi\rho a^5}{15}$$

The total mass is

$$M = \frac{4\pi\rho a^3}{3}$$

Hence

$$I = \frac{2Ma^2}{5}$$

EXAMPLE
**A massive cylinder and a cylindrical shell roll down an inclined plane. Which is fastest?**
The moment of inertia about its axis of a **cylindrical shell** of radius $a$ and thickness $\delta << a$ is

$$\rho 2\pi a^3 b\delta = Ma^2$$

where $b$ is the length of the cylinder, $\rho$ the mass density and $M$ the mass.

The moment of inertia of a solid cylinder is

$$I_{zz} = b\rho \int_0^a dr 2\pi r r^2 = \frac{\pi b a^4}{2} = \frac{Ma^2}{2}$$

If a cylinder has rolled down a vertical height drop $h$ the kinetic energy will be

$$Mgh = \frac{MV^2}{2} + \frac{I_{zz}\omega^2}{2}$$

where $V$ is the speed of the center of mass and $\omega$ is the angular velocity. The rolling constraint implies that $V = a\omega$. For the solid cylinder we find

$$Mgh = \frac{Ma^2\omega_{solid}^2}{2} + \frac{Ma^2\omega_{solid}^2}{4}$$

or

$$\omega_{solid}^2 = \frac{4gh}{3a^2}$$

while we find for the shell

$$Mgh = \frac{Ma^2\omega_{shell}^2}{2} + \frac{Ma^2\omega_{shell}^2}{2}$$

or

$$\omega_{shell}^2 = \frac{gh}{a^2}$$

i.e. the solid cylinder rolls faster!

EXAMPLE

**The physical pendulum** A rigid body that swings under the influence of gravity about a **fixed** horizontal axis is called a physical pendulum. We label that axis the $x-$axis. The constraint that the axis is fixed means that

$$\vec{\omega} = (\omega_x, 0, 0)$$

The kinetic energy is then

$$T = \frac{I_{xx}\omega_x^2}{2}$$

The **radius of gyration** is defined as

$$k = \sqrt{\frac{I_{xx}}{M}}$$

where $M$ is the mass of the pendulum. We let $l$ be the perpendicular distance between the center of mass and the axis of rotation, and $\theta$ the angle whose angular velocity is $\omega_x$ with $\theta = 0$ corresponding to the case where the center of mass is directly below the axis of rotation.

The Lagrangian of the pendulum is then

$$\mathcal{L} = \frac{Mk^2\dot{\theta}^2}{2} + Mgl\cos\theta$$

with equation of motion

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}) - \frac{\partial\mathcal{L}}{\partial\theta} = M(k^2\ddot{\theta} + gl\sin\theta)$$

We conclude that the behavior of the physical pendulum is the same as that of a **mathematical pendulum** ( pendulum with a point mass $M$)with **effective length**

$$l_{eff} = \frac{k^2}{l}$$

FINDING THE PRINCIPAL AXES

In general the moment of inertia tensor is non-diagonal, but since it is a real symmetric matrix it can always be diagonalized with real and orthogonal eigenvalues. If $\hat{e}_i$ is an eigenvector of $I$ with eigenvalue $I_i$

$$I\hat{e}_i = I_i\hat{e}_i, \ i = 1, 2, 3$$

we refer to $\hat{e}_i$ as a **principal axis** of $I$, and $I_i$ as a **principal moment**. The Cartesian coordinate system with axes $\hat{e}_i$ is the **principal axes frame**. The orthogonal rotation matrix $R$ which diagonalizes the matrix $I$

$$RIR^{-1} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

is

$$R = \begin{pmatrix} e_{1x} & e_{1y} & e_{1z} \\ e_{2x} & e_{2y} & e_{2z} \\ e_{3x} & e_{3y} & e_{3z} \end{pmatrix}$$

Some general results:

- If all three principal moments are equal (as e.g. for the moments of inertia about the center of a sphere) the moment of inertia matrix is proportional to the unit matrix, and any Cartesian frame is a principal axis frame.

- If two principal moments are equal but different from the third we talk about a **symmetric top**. If e.g. $I_1 = I_2 \neq I_3$ we can choose the axes perpendicular to $e_3$ arbitrarily.

- If all three principal moments are different we talk about an **asymmetric top**.

- The principal moments satisfies the **triangle inequality** that the sum of two principal moments is always larger than or or equal to the third. If $I_1 + I_2 = I_3$ the body is a plane sheet with normal along $\hat{e}_3$.

Some comments about how to solve this numerically is given in the Maple worksheet at http://www.physics.ubc.ca/˜birger/p206l19.mws (or .html)

**Example problems**

**Problem 6.3.1**

(Question 3 of 1999 problem set 6) A sphere of radius $a$ has a spherical cavity of radius $a/2$ centered at a distance $a/2$ from the center of the center of the sphere. Except for the cavity the mass is uniformly distributed.

**a:** Find the center of mass of the object. Find the moments of inertia with respect to the center of the sphere.

**b:** Find the moments of inertia with respect to the center of mass.

Use coordinate systems in which the z-axis goes through the centers of the sphere and the cavity.

**Problem 6.3.2**

(Question 1 of 2001 problem set 7)

A cylinder of mass $m$ radius $a$ rolls down an inclined plane (Figure 1), starting from rest at height $h = 0$. The angle of inclination is $\theta$.

**a:** Find the velocity and angular velocity of the cylinder as a function of time.(The moment of inertia of the cylinder about its symmetry axis is $ma^2/2$).

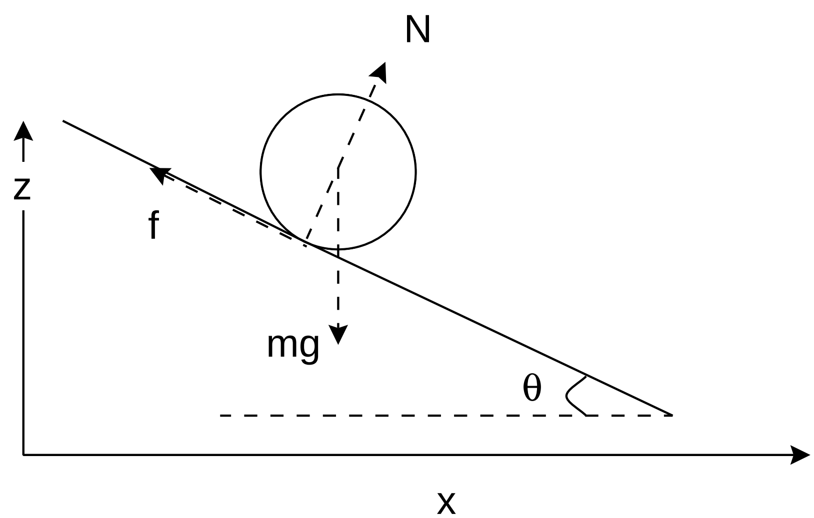



Figure 6:

**b:** Find the force of friction between the plane and the cylinder.

**c:** If the coefficient of static friction between the inclined plane and the cylinder is $\mu$, What is the maximum value of the angle for the cylinder to roll without slipping?

**Problem 6.3.3**
(Question 5 of 2002 final exam)
A sphere of radius $r$ rolls without slipping under the influence of gravity down a fixed larger sphere of radius R starting from rest near the top. Calculate where it will fall off.

**Problem 6.3.4**
(Question 5 of 2001 final exam)
A uniform rod of length $r\sqrt{(3)}$, mass $m$, slides under the influence of gravity with its ends on a smooth vertical circle of radius $r$.
**a:**
What is the equilibrium position of the rod?
**b:**

Find the equation of motion of the rod using its angle $\theta$ with the horizontal as a generalized coordinate.
**c:** how much energy does the rod need to rotate rather than oscillate?

**Problem 6.3.5**
(Question 1 of 2000 final exam)
A thin square of side $a$, mass $m$ is located so that the corners are at [0,0,0],[a,0,0],[a.a,0],[0,a,0].
**a:** Find the moment of inertia tensor with respect to the origin.
**b:** Find directions of a set of principal axes.
**c:** What are the principal moments of inertia?

**Problem 6.3.5**
(Question 4 of 1999 final exam)

**a:** A uniform solid ball of mass $m$ radius $a$ rolls without slipping down an inclined plane with angle of inclination $\theta$. What is the acceleration of the ball along the plane.

**b:** Another ball with the same mass and radius rolls down the same plane. The second ball is made of a denser material and contains a spherical cavity at its center. The radius of the cavity is $a/3$. What is now the acceleration of the second ball?

## 6.4 Lagrangian of a rigid body. Angular momenta and torques.

LAST TIME
We solved some problems involving the moment of inertia tensor.
TODAY
We wish to make some general comments about the equations of motion of a rigid body. In lecture 6.1 we found for the kinetic energy of a rigid body

$$\mathcal{T} = \frac{M}{2}V^2 + M\vec{r}_{cm} \cdot (\vec{V} \times \vec{\omega}) + \frac{1}{2}\vec{\omega} \cdot I\vec{\omega} \tag{25}$$

Here
$M =$ total mass of body

$\vec{V}$ = velocity of a reference point on the body.
$\vec{r}_{cm}$ = position of center of mass relative to reference point.
$\vec{\omega}$ = angular velocity of body.
$I$ = moment of inertia tensor.
We write for the Lagrangian of the body

$$\mathcal{L} = \mathcal{T} - \mathcal{U}$$

LINEAR MOMENTUM

In a Newtonian description the total linear momentum of the body is

$$\vec{P} = \sum_{\alpha} m\vec{v}_{\alpha}$$

where the velocity of the mass point $\alpha$ is

$$\vec{v}_{\alpha} = \vec{V} + \vec{\omega} \times \vec{r}_{\alpha}$$

so that

$$\vec{P} = M\vec{V} + M\vec{\omega} \times \vec{r}_{cm}$$

If we use the vector identity

$$\vec{r}_{cm} \cdot (\vec{V} \times \vec{\omega}) = \vec{V} \cdot (\vec{\omega} \times \vec{r}_{cm})$$

and differentiate, we find (as expected) that the expression for the linear momentum agrees with the Lagrangian definition of momentum

$$\vec{P} = \frac{\partial \mathcal{L}}{\partial \vec{V}} = M\vec{V} + M\vec{\omega} \times \vec{r}_{cm}$$

FORCES

We next assume that the total force $\vec{F}$ is the sum of forces distributed among the mass points

$$\vec{F} = \sum_{\alpha} \vec{f}_{\alpha}$$

Suppose one mass point is shifted by an amount $\delta\vec{r}$ then

$$\delta\mathcal{U} = \vec{f}_{\alpha} \cdot \delta\vec{r}_{\alpha}$$

If we express $\mathcal{U}$ as the potential energy of all the mass points

$$\vec{f}_\alpha = -\frac{\partial U}{\partial \vec{r}_\alpha}$$

Suppose all the mass points are shifted by the same amount $\delta\vec{R}$ then

$$\delta\mathcal{U} = \sum_\alpha \frac{\partial \mathcal{U}}{\partial \vec{r}_\alpha} \cdot \delta\vec{r}_\alpha = -\delta\vec{R} \cdot \sum_\alpha \vec{f}_\alpha = -\delta\vec{R} \cdot \vec{F}$$

We thus can write

$$\frac{\partial \mathcal{L}}{\partial \vec{R}} = -\frac{\partial \mathcal{U}}{\partial \vec{R}} = \vec{F}$$

The Lagrangian equation of motion associated with $\vec{R}$ is then nothing but

$$\frac{d\vec{P}}{dt} = \vec{F}$$

ANGULAR MOMENTUM

We write the total angular momentum $\vec{l}$ as the sum of the angular momenta of all the mass points, then

$$\vec{l} = \sum_\alpha m\vec{r}_\alpha \times \vec{v}_\alpha = \sum_\alpha m\vec{r}_\alpha \times (\vec{V} + \vec{\omega} \times \vec{r}_\alpha)$$

$$= M\vec{r}_{cm} \times \vec{V} + \sum_\alpha m\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) \tag{26}$$

In order to show that this expression is equivalent to the Lagrangian expression

$$\vec{l} = \frac{\partial \mathcal{L}}{\partial \vec{\omega}} \tag{27}$$

let us go back to the expression for the kinetic energy that we established before we introduced the momentum of inertia tensor (lecture 6.1)

$$\mathcal{T} = \frac{M}{2}V^2 + M\vec{r}_{cm} \cdot (\vec{V} \times \vec{\omega}) + \frac{1}{2}\sum_\alpha m(\vec{\omega} \times \vec{r}_\alpha)^2$$

Using

$$\vec{r}_{cm} \cdot (\vec{V} \times \vec{\omega}) = \vec{\omega} \cdot (\vec{r}_{cm} \times \vec{V})$$

and

$$(\vec{\omega} \times \vec{r}_\alpha)^2 = \vec{\omega} \cdot (\vec{r} \times (\vec{\omega} \times \vec{r}_\alpha))$$

we find

$$\frac{\partial \mathcal{T}}{\partial \vec{\omega}} = \frac{\partial \mathcal{L}}{\partial \vec{\omega}} = M\vec{r}_{cm} \times \vec{V} + \sum_\alpha (m\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha))$$

which shows that (2) and (3) agree.

TORQUES

The total torque $\vec{N}$ acting on a body is the sum of all the torques acting on the mass points

$$\vec{N} = \sum_\alpha \vec{r}_\alpha \times \vec{f}_\alpha$$

We write for the angular velocity

$$\vec{\omega} = \frac{d\vec{\phi}}{dt}$$

and consider a small rotation of the body in which

$$\delta\vec{r}_\alpha = \delta\vec{\phi} \times \vec{r}_\alpha$$

The change in potential energy is

$$\delta U = -\sum_\alpha \vec{f}_\alpha \cdot \delta\vec{r}_\alpha = -\sum_\alpha \vec{f}_\alpha \cdot (\delta\vec{\phi} \times \vec{r}_\alpha) = -\delta\vec{\phi} \cdot \sum_\alpha \vec{r}_\alpha \times \vec{f}_\alpha = -\delta\vec{\phi} \cdot \vec{N}$$

We thus have

$$\frac{\partial L}{\partial \vec{\phi}} = -\frac{\partial U}{\partial \vec{\phi}} = \vec{N}$$

The Lagrangian equation of motion associated with $\vec{\phi}$ is thus

$$\frac{d\vec{l}}{dt} = \vec{N}$$

Finally, let us consider a shift in the reference point from $O$ to $O'$ so that

$$\vec{r}'_\alpha = \vec{r}_\alpha - \vec{a}$$

The torque about the new reference point is

$$\vec{N}' = \vec{N} - \vec{a} \times \sum_{\alpha} f_{\alpha} = \vec{N} - \vec{a} \times \vec{F}$$

We conclude that if the net force on a rigid body is zero the torque is independent of the reference point!

SUMMARY

We have analyzed the Lagrangian equations of motion for a rigid body in terms of force $\vec{F}$, torque $\vec{N}$, momentum $\vec{P}$ and angular momentum $\vec{l}$. We found the equations of motion could be expressed as

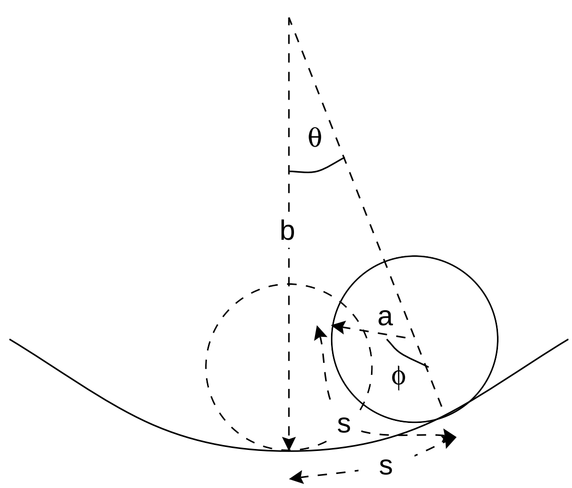
$$\frac{d\vec{P}}{dt} = \vec{F}$$

$$\frac{d\vec{l}}{dt} = \vec{N}$$

**Example problems**

**Problem 6.4.1**

(Question 3 of 2001 problem set 7) A small steel ball rolls, without slipping, back and forth in a vertical plane, inside a spherical bowl of radius $b$. The radius of the ball is $a$.



**a:** Write down the equation of motion using the angle $\theta$ as generalized coordinate (see figure)

**b:** Find the period of small amplitude oscillations ($\sin\theta \approx \theta$).

**Problem 6.4.2**
(Question 1 of 2002 final exam)
A car is started from rest with constant acceleration $a$, and with one of its doors initially at right angles with the side of the car. Approximate the door as a mass $m$ distributed uniformly over a rectangle of height $h$ and width $w$. The door can rotate about a vertical axis along a side of the rectangle.
**a:** Derive an expression for the moment of inertia of the door about the axis of rotation.
**b:** Find a formula for the time it takes for the door to close. (The formula may involve an integral that you don't need to solve).
**c:** Make a rough estimate for the time for the door to close if the acceleration is 1 $\mathrm{ms}^{-2}$. Assume reasonable values for the width and height of the door.

**Problem 6.4.3**
(Question 1 of 2001 final exam)
A uniform rod of mass $m$, length $a$, has one end attached to a smooth hinge and can swing in a vertical plane (as a physical pendulum). The rod start out in a horizontal position from rest and is allowed to fall.
**a:** Find the moment of inertia of the rod about a horizontal axis through the hinge perpendicular to the rod.
**b:** Find the horizontal and vertical force on the hinge as a function of the angle $\theta$ of the rod with the vertical.
**c:** At what angle is the horizontal force on the hinge a maximum? In which direction does it act?

**Problem 6.4.4**
(Question 3 of 2000 final exam)
A cylinder of radius $a$ is balanced on top of a larger fixed cylinder of radius $b > a$. The axes of the two cylinders are parallel. The balance is slightly disturbed so that the top cylinder starts to roll down the fixed cylinder. The coefficient of static friction is large enough that no slipping occurs as long

as there is a normal force between the two cylinders. At a critical angle $\theta_c$ between the line of centers and the vertical the rolling cylinder will separate from the fixed one.
**a:** Find the equations of motion assuming the motion is rolling without slipping.
**b:** Find the critical angle $\theta_c$.

The moment of inertia of a cylinder about its axis is $\frac{ma^2}{2}$. The formula for acceleration in plane polar coordinates is

$$\vec{a} = [\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2]\hat{e}_r + [r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}]\hat{e}_\theta$$

**Problem 6.4.4**
(Question 3 of 1999 final exam)
A Thin uniform rod of length $a$ and mass $m$ is constrained to rotate with constant angular velocity $\vec{\omega}$ about an axis through the center of the rod making an angle $\theta$ with the rod.

**a:** Find the magnitude and direction of the angular momentum.

**b:** Find the magnitude and direction of the torque.

## 6.5 Euler's equations for a rigid body.

LAST TIME
We analyzed the Lagrangian equations of motion for a rigid body in terms of force $\vec{F}$, torque $\vec{N}$, momentum $\vec{P}$ and angular momentum $\vec{l}$. We found the equations of motion could be expressed as

$$\frac{d\vec{P}}{dt} = \vec{F}$$

$$\frac{d\vec{l}}{dt} = \vec{N}$$

The angular momentum vector $\vec{l}$ and momentum vector $\vec{P}$ in the above equations refer to an inertial frame, which in general will not be one in which the moment of inertia tensor is diagonal. However, calculations are often much simpler in a principal axis frame of reference.

TODAY

We will consider a system in which the point of reference $O$ is fixed. At a given instant the body rotates with angular velocity $\vec{\omega}$ with respect to the inertial frame. As we shall see, Euler's equation provides a convenient framework for describing the time evolution of the components of the angular velocity in the principal axis frame.

Suppose $\vec{A}$ is a vector in a direction which is fixed with respect to the rigid body then in the inertial frame

$$\left.\frac{d\vec{A}}{dt}\right|_{inertial} = \vec{\omega} \times \vec{A}$$

Suppose instead the vector $\vec{A}$ is changing at the rate

$$\left.\frac{d\vec{A}}{dt}\right|_{body}$$

in the body centered reference frame. Then

$$\left.\frac{d\vec{A}}{dt}\right|_{inertial} = \vec{\omega} \times \vec{A} + \left.\frac{d\vec{A}}{dt}\right|_{body}$$

We now let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be a set of Cartesian unit vectors in the principal axis directions on the body and let $\vec{A} = \vec{l}$ be the angular momentum vector. On component form

$$\vec{\omega} \times \vec{l} = (\omega_2 l_3 - \omega_3 l_2)\hat{e}_1 + (\omega_3 l_1 - \omega_1 l_3)\hat{e}_2 + (\omega_1 l_2 - \omega_2 l_1)\hat{e}_3$$

$$\vec{l} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3$$

where $I_1, I_2, I_3$ are the eigenvalues of the moment of inertia tensor. Substituting into

$$\vec{N} = \left.\frac{d\vec{l}}{dt}\right|_{inertial} = \left.\frac{d\vec{l}}{dt}\right|_{body} + \vec{\omega} \times \vec{l}$$

gives us **Euler's equations**

$$N_1 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3$$

$$N_2 = I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_3\omega_1$$

$$N_3 = I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\omega_1\omega_2$$

EXAMPLE

**Free rotation of a symmetric top**

An object for which

$$I_1 = I_2 \neq I_3$$

is called a **symmetric top**. The 3-axis is the **symmetry axis**. The earth is a symmetric top to a very good approximation, with the symmetry axis extending from the south to the north pole. The earth is slightly oblate (fatter at the equator)

$$\frac{I_3}{I_1} \approx 1.003$$

The angular velocity vector is slightly off the N-S axis (by about $2''$ of arc). External torques from the sun and the moon are sufficiently small that the earth's rotation can be considered to be torque free. If we put $\vec{N} = 0$ in Euler's equation and $I_1 = I_2$ we get

$$0 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_1)\omega_2\omega_3$$

$$0 = I_2 \frac{d\omega_2}{dt} - (I_3 - I_1)\omega_3\omega_1$$

$$0 = I_3 \frac{d\omega_3}{dt}$$

We note that the component of the angular velocity about the symmetry axis is conserved.

$$\omega_3 = const$$

Let us define

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

The equations of motion for the remaining components of the angular velocity are then

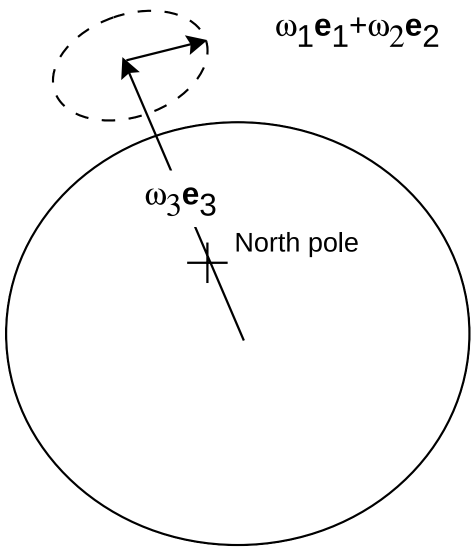
$$\frac{d\omega_1}{dt} = -\Omega\omega_2$$

$$\frac{d\omega_2}{dt} = \Omega\omega_1$$

with solution

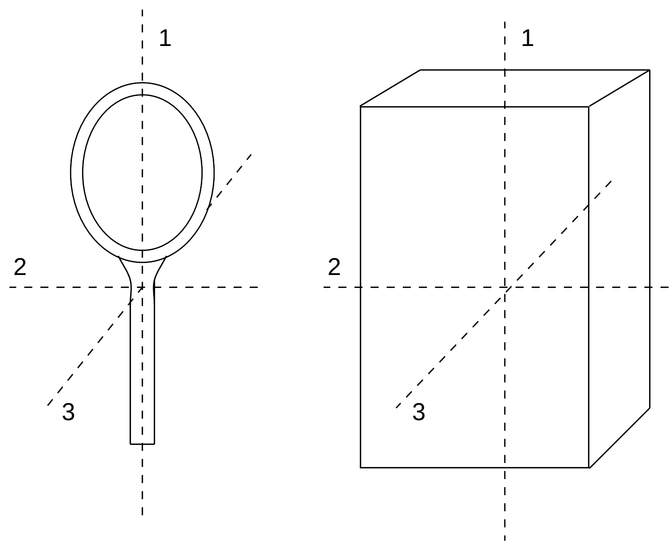
$$\omega_1 = A\cos(\Omega t + \phi)$$

$$\omega_2 = A\sin(\Omega t + \phi)$$




The picture that emerges is as follows:
In the principal axis frame of reference, fixed on the body, the body appears to be spinning about the symmetry axis with angular frequency $\omega_3$. In addition there is a component of the instantaneous axis of rotation perpendicular to the symmetry axis. This component is rotating (or precessing) about the symmetry axis with angular frequency $\Omega$. In the inertial frame the angular momentum vector is constant. The body appears to be spinning about its symmetry axis with angular velocity $\omega_3$. This axis is tilted with respect to the angular momentum direction, and it is precessing about it with angular velocity $\Omega$.

EXAMPLE




**Stability of rotation near a principal axis.**
Consider torque free rotation of an object (a tennis racket? a book?) for which the three principal axes of inertia are different. Let us assume that

$$I_3 > I_2 > I_1$$

¿From Euler's equations

$$0 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3$$

$$0 = I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_3\omega_1$$

$$0 = I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\omega_1\omega_2$$

Suppose the initial conditions are such that only one of the components of the angular velocity, say $\omega_1$, is nonzero. The object will then continue to spin about the 1-axis indefinitely. We now ask ourself the question is this situation **stable**? If a small perturbation is added to the angular velocity vector perpendicular to the axis of rotation, will this perturbation grow, stay the same or shrink? If the perturbation grows to become large the situation is unstable, if the amplitude stays the same the motion is neutrally stable, if the amplitude shrinks towards zero the motion is asymptotically stable. We have the first situation if we attempt to stand a pencil about its sharp point. The undamped pendulum which exhibits small oscillations about $\theta = 0$ is an example of a neutrally stable situation. In the case of damped oscillation around an equilibrium point this point is asymptotically stable.

*Case 1:*
**Rotation about 1-axis.**
Let us assume that initially the angular velocity vector $\vec{\omega}$ is close to the body 1-axis so that

$$\omega_2, \omega_3 << \omega_1$$

The Euler equation for $\omega_1$ allows us to make the approximation

$$\omega_1 \approx \Omega = const$$

Let us introduce the real constants (remember that $I_1 < I_2 < I_3$)

$$a^2 = \frac{(I_3 - I_1)\Omega}{I_2}$$

$$b^2 = \frac{(I_2 - I_1)\Omega}{I_3}$$

The Euler equation for $\omega_2$ and $\omega_3$ then become

$$\frac{d\omega_2}{dt} = a^2\omega_3$$

$$\frac{d\omega_3}{dt} = -b^2\omega_2$$

with solution

$$\omega_2 = Aa\sin(abt + \phi)$$

$$\omega_3 = Ab\cos(abt + \phi)$$

We conclude that the solution remains bounded and that rotation about the 1-axis is **neutrally stable**

*Case 2:*

**Rotation about 2-axis.**

Now, assume that initially the angular velocity vector $\vec{\omega}$ is close to the body 2-axis so that

$$\omega_3, \omega_1 << \omega_2$$

The Euler equation for $\omega_2$ now allows us to make the approximation

$$\omega_2 \approx \Omega = const$$

Define the real constants

$$c^2 = \frac{(I_3 - I_2)\Omega}{I_1}$$

$$b^2 = \frac{(I_2 - I_1)\Omega}{I_3}$$

The Euler equation for $\omega_1$ and $\omega_3$ then become

$$\frac{d\omega_1}{dt} = -c^2\omega_3$$

$$\frac{d\omega_3}{dt} = -b^2\omega_1$$

with solution

$$\omega_2 = c(Ae^{bct} + Be^{-bct})$$

$$\omega_3 = b(-Ae^{bct} + Be^{-bct})$$

If the boundary conditions are such that $A \neq 0$ the solution will grow exponentially. We conclude that rotation about the 2-axis is **unstable**.

*Case 3:*

**Rotation about 3-axis.**

Finally, assume that initially the angular velocity vector $\vec{\omega}$ is close to the body 3-axis so that

$$\omega_1, \omega_2 << \omega_3$$

The Euler equation for $\omega_3$ allows us to make the approximation

$$\omega_3 \approx \Omega = const$$

With

$$c^2 = \frac{(I_3 - I_2)\Omega}{I_1}$$

$$a^2 = \frac{(I_3 - I_1)\Omega}{I_2}$$

The Euler equation for $\omega_2$ and $\omega_3$ then become

$$\frac{d\omega_1}{dt} = -c^2\omega_3$$

$$\frac{d\omega_3}{dt} = a^2\omega_2$$

with solution

$$\omega_1 = Ac\sin(bct + \phi)$$

$$\omega_2 = -Ab\cos(bct + \phi)$$

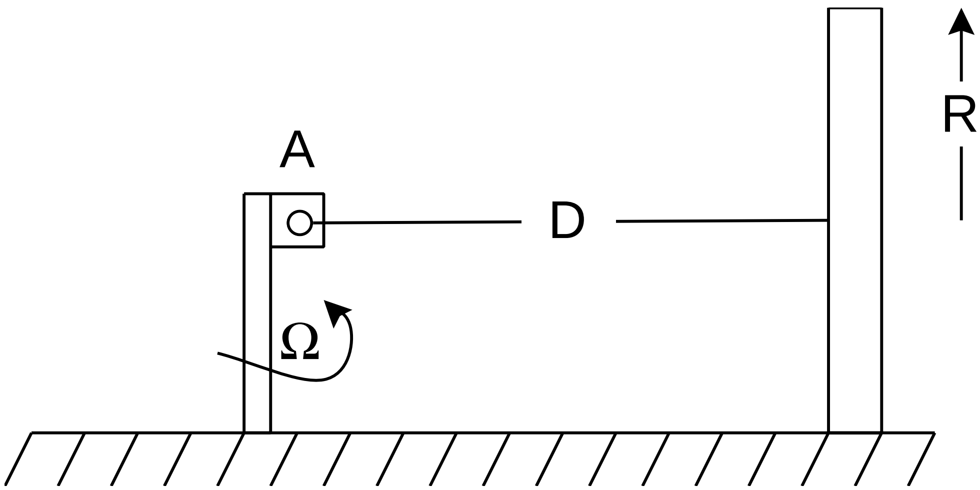
Hence, the rotation about the 3-axis is neutrally stable.

SUMMARY
We have derived Euler's equations for the rotation of a rigid body and given a few examples of their application.

**Example problems**
**Problem 6.5.1**
(Question 1 of 1999 problem set 7)
A uniform block of mass $m$ and dimensions $a$ by $2a$ by $3a$ spins about a long diagonal with angular velocity $\omega$ (as in Problem 2 of problem set 6). Find the torque that must be exerted on the block to keep $\omega$ fixed in magnitude and direction.

**Problem 6.5.2**
(Question 3 of 2001 problem set 8)

A cylindrical wheel of radius R rolls without slipping along a circular horizontal path of radius $a$ The jointed mechanism at A enables the wheel to be tilted up or down in the vertical plane. Find the normal force exerted by the floor on the wheel if the wheel is to keep to its path. The driving shaft is rotated with constant angular velocity $\Omega$.




**Problem 6.5.3**
(Question 2 of 2002 final exam)
**a:** A right cylinder has mass $m$, height $h$, radius $r$. Derive a formula for the moment of inertia tensor about the center of mass. (Hint: When calculating the moments of inertia about axes perpendicular to the axis of the cylinder you may start with their sum and use a symmetry argument.)
**b:** The cylinder is rotating with angular velocity $\omega$ about an axis through the center of mass. The axis of rotation reaches the top and bottom circular surfaces at the edges (a distance r from the center of the circles). Find the magnitude of the torque needed to maintain the rotation.
**c:** For what height to radius ratio will the torque in **b:** be zero.

**Problem 6.5.4**
(Question 4 of 2001 final exam)
A lamina is defined as a thin flat body. Consider a lamina shaped lake a rectangle of sides $a$ and $2a$, mass $m$
**a:**
Find the principal axes and moments of inertia of the body in a coordinate system centered at one of the corners, with the $x-$axis along the side of length $a$, the $y-$axis along a side of length $2a$ and the $z-$ axis perpendicular to the lamina.
**b:** The lamina rotates with constant angular velocity $\omega$ about the $x-$axis,

which is in the vertical direction. Find the torque needed to sustain the rotation.

**Problem 6.5.5**
(Question 4 of 2000 final exam)
A lamina is defined as a thin flat body. Let the direction of the axis perpendicular to the lamina be the body 3-axis.
**a:**
Show that for a lamina of arbitrary shape the principal moments of inertia satisfy

$$I_3 = I_1 + I_2$$

**b:**
Use Euler's equations of motion

$$N_1 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\omega_2\omega_3$$

$$N_2 = I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\omega_3\omega_1$$

$$N_3 = I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\omega_1\omega_2$$

to show that when the torque $\vec{N} = 0$

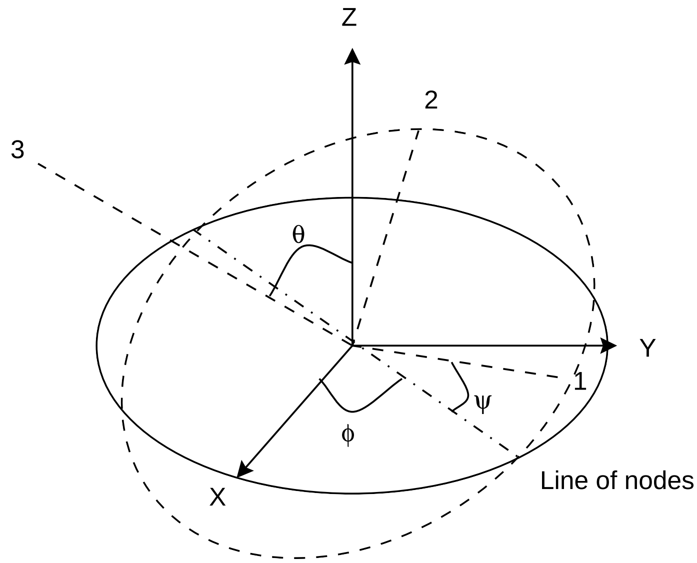
$$\omega_1^2 + \omega_2^2 = const$$

**c:** Under what condition will angular momentum component $\omega_3$ be constant as well.

## 6.6 Equation of motion for Euler angles. Symmetric top

LAST TIME
Derived Euler's equations for the rotation of a rigid body and gave a few examples of their application.

TODAY
We wish to present an alternative approach to the equations of motion in




which we describe the orientation of a body in terms of three orientational angles. A standard way to do this is through the Euler angles $\phi, \theta, \psi$.
In the figure above the three Cartesian coordinate axes labeled 1,2,3 represent the orientation of a set of axes, **fixed on the body**, typically in principal axes directions. The three axes labeled $X, Y, Z$ are **fixed in space**. The orientation of the body-centered coordinate system can be thought of as coming from three successive rotations:

1. by an angle $\phi$ about the $Z$-axis
2. by an angle $\theta$ about the new $x$-axis, which we will call the **line of nodes**
3. by an angle $\psi$ about new $z-$axis.

Our strategy is to express the three principal axis components $\omega_1, \omega_2, \omega_3$ of the angular velocity in terms of the rate of change, $\dot{\phi}, \dot{\theta}, \dot{\psi}$, of the Euler angles. Substituting into he kinetic energy of rotation

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{28}$$

then allows us to construct a Lagrangian from which we can derive equations of motion for the Euler angles.

We see from the figure that a unit vector in the direction of the line of nodes can be written

$$\hat{e}_N = \hat{e}_1 \cos\psi - \hat{e}_2 \sin\psi$$

The unit vector in the direction of the space $Z-$axis can be written

$$\hat{\mathbf{k}} = \hat{e}_3 \cos\theta + \hat{e}_2 \sin\theta \cos\psi + \hat{e}_1 \sin\theta \sin\psi$$

We find

$$\frac{d\vec{\psi}}{dt} = \dot{\psi}\hat{e}_3$$

$$\frac{d\vec{\theta}}{dt} = \dot{\theta}(\cos\psi\hat{e}_1 - \sin\psi\hat{e}_2)$$

$$\frac{d\vec{\phi}}{dt} = \dot{\phi}(\sin\theta\sin\psi\hat{e}_1 + \sin\theta\cos\psi\hat{e}_2 + \cos\theta\hat{e}_3)$$

Substitution into

$$\vec{\omega} = \frac{d\vec{\psi}}{dt} + \frac{d\vec{\theta}}{dt} + \frac{d\vec{\phi}}{dt}$$

then gives

$$\omega_1 = \vec{\omega}\cdot\hat{e}_1 = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$$

$$\omega_2 = \vec{\omega}\cdot\hat{e}_2 = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$

$$\omega_3 = \vec{\omega}\cdot\hat{e}_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

These expressions can then be substituted into (1) to get an expression for the kinetic energy which together with possible potential energy terms can be used to construct the Lagrangian.

The general case

$$I_1 \neq I_2 \neq I_3$$

is rather complicated and we will specialize to the case of the **symmetric top** for which

$$I_1 = I_2 \neq I_3$$

We find

$$T = \frac{I_3}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2 + \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

Consider a top in which a point on the symmetry axis a distance $a$ below the c.m. is fixed in space. Let us assume that the force of gravity acts in the spatial $Z$-direction. The potential energy is then

$$U = mga\cos\theta$$

and the Lagrangian is

$$\mathcal{L} = \frac{I_3}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2 + \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mga\cos\theta$$

The Lagrangian doesn't depend on the angles $\psi$ and $\phi$ so the corresponding components of the angular momentum will be conserved. We have

$$p_\psi \equiv l_3 = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = const$$

$$p_\phi \equiv l_Z = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta + I_1\sin^2\theta\dot{\phi} = const$$

We use these conservation laws to express $\dot{\phi}$ and $\dot{\psi}$ in terms of $l_3$ and $l_Z$:

$$\dot{\phi} = \frac{l_Z - l_3\cos\theta}{I_1\sin^2\theta}$$

$$\dot{\psi} = \frac{l_3 - I_3\dot{\phi}\cos\theta}{I_3} = \frac{l_3}{I_3} - \cos\theta\frac{l_Z - l_3\cos\theta}{I_1\sin^2\theta}$$

If we substitute these expressions into the formula for the energy

$$E = \frac{I_3}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2 + \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mga\cos\theta$$

we find an expression on the form

$$E = \frac{I_1}{2}\dot{\theta}^2 + U_{eff}(\theta)$$

i.e. we have **equivalent one degree of freedom problem** with an **effective potential**

$$U_{eff} = \frac{(l_Z - l_3 \cos\theta)^2}{2I_1 \sin^2\theta} + mga\cos\theta + \frac{l_3^2}{2I_3}$$

This expression is a bit easier to interprete if we make the substitution

$$u = \cos\theta$$

As $\theta$ increases from $\theta = 0$ to $\theta = \pi$ u decreases from $u = 1$ to $u = -1$.

$$U_{eff} = \frac{(l_Z - l_3 u)^2}{2I_1(1-u^2)} + mgau + \frac{l_3^2}{2I_3}$$

We note that $U_{eff} \to \infty$ as $u \to 1$ and $u \to -1$ and is finite in between. For any allowed value of the energy there must therefore be **turning points** $u_1$ and $u_2$ where $\dot{\theta} = 0$. The angle $\theta$ will then **nutate** between these turning points. The azimuthal angle of the symmetry axis of the top will be

$$\phi - \frac{\pi}{2}$$

Looking at the expression for $\dot{\phi}$

$$\dot{\phi} = \frac{l_Z - l_3 u}{I_1(1-u^2)}$$

we note that depending on the magnitude of $l_3$ and $l_Z$ $\dot{\phi}$ may or may not change sign in the interval $u_1 < u < u_2$. In the latter case the **precession** of the top will be **monotonic** in the former case the **direction of precession** will not be constant. We will discuss the problem of the heavy symmetric next time using Maple.

SUMMARY
We have

- defined the Euler angles, $\phi, \theta, \psi$ describing the orientation of a rigid body
- expressed the principal axis components of the angular velocity in terms of the time derivatives of the Euler angles.

- in the special case of the heavy symmetric top found an explicit expression for the kinetic energy of a rigid body in terms of the Euler angles.

- commenced an analysis of the behavior of the heavy symmetric top.

# 7 Hamiltonian mechanics

## 7.1 Hamiltonian equations of motion. Legendre transform.

LAST TIME
Finished our discussion of rigid body motion
TODAY
Start on our last major topic: **Hamiltonian mechanics**. The Hamiltonian approach offers a gateway to generalizations of classical mechanics to new areas of science ranging from quantum mechanics to thermodynamics and statistical mechanics and beyond. Unfortunately, since there is not much time left of the term we can only touch on a few highlights.

In the Lagrangian approach the basic dependent variables are the generalized coordinates and velocities

$$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t)$$

In the Lagrangian equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0; \quad i = 1, 2...N$$

the generalized coordinates and velocities are treated differently. For this reason we have to exercise some care when making use of constraints on the generalized velocities arising e.g. from conservation laws. We are not allowed to substitute such nonholonomic constraints directly into the Lagrangian, although we can substitute holonomic constraints. Indeed with the Lagrangian approach the actual value of the function does not seem to matter much, but getting the correct functional dependence does matter. For example we are free to multiply the Lagrangian by a constant or add a constant and we can add an arbitrary total time derivative. But, we are not allowed to substitute into the Lagrangian the value of conserved momenta before taking the derivatives to get the equations of motion. Only after the differential equations have been established are we allowed to do this substitution. So, to extend the powerful computational methods of classical mechanics to other areas of science it is useful to have some more flexibility in the treatment of variables. This is the main advantage of the Hamiltonian approach, while it may not offer that much of an advantage when it comes to solving practical problems in mechanics.

THE HAMILTONIAN

In lecture 12 we defined the **energy** as

$$E = \sum_i p_i \dot{q}_i - \mathcal{L}$$

If we change the value of the variables by infinitesimal amounts the energy changes by

$$dE = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i) - \frac{\partial \mathcal{L}}{\partial t} dt$$

Recalling that

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

and making use of the Lagrangian equations of motion we find

$$dE = \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial \mathcal{L}}{\partial t} dt \tag{29}$$

Now imagine that we express $\dot{q}_i$ in terms of $p_i$ and $q_i$ and eliminate $\dot{q}_i$ from the expression for energy. The resulting function

$$H(p_i, q_i, t) \equiv E(\dot{q}_i, q_i, p_i, t)$$

is called the **Hamiltonian** of the system. If the variables change by an infinitesimal amount the Hamiltonian changes by

$$dH = \sum_i (\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i) - \frac{\partial \mathcal{L}}{\partial t} \tag{30}$$

Comparing (1) and (2) we obtain the **Hamiltonian equations of motion**

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \frac{dH}{dt} &= \frac{\partial H}{\partial t} \end{aligned}}$$

The last equation follows from dividing (1) by $dt$ and noting that $\dot{p} \equiv dp/dt,\ \dot{q} \equiv dq/dt$.

EXAMPLE: **Harmonic oscillator**

$$\mathcal{L} = \frac{1}{2}(m\dot{q}^2 - kq^2)$$

where $m$ is the mass and $k$ the spring constant.

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$$

hence

$$H = \frac{p^2}{2m} + \frac{kq^2}{2}$$

The Hamiltonian equation of motion are then

$$\dot{p} = -\frac{\partial H}{\partial q} = -kx$$

$$\dot{q} = \frac{\partial H}{\partial q} = \frac{p}{m}$$

$$\dot{H} = 0$$

We see that these equation are, as could be expected, completely equivalent to the Lagrangian equation of motion for the system. The last equation is just a restatement of the law of conservation of energy. In effect, what we have done is replacing the second order differential equation, which constitutes the Lagrangian equation of motion, by two first order equations. In the general case of $N$ degrees of freedom we replace $N$ second order differential equations by $2N$ first order ones.

EXAMPLE: **Simple pendulum**

$$\mathcal{L} = \frac{ma^2\dot{\theta}^2}{2} + mga\cos\theta$$
$$p = ma^2\dot{\theta}$$
$$H = \frac{p^2}{2ma^2} - mga\cos\theta$$

The Hamiltonian equations of motion are then

$$\dot{\theta} = \frac{p}{ma^2}$$
$$\dot{p} = -mga\sin\theta$$

Again we get results that are completely equivalent to the results of the Lagrangian approach.

The most important result effected by going from a Lagrangian to a Hamiltonian description is the **change of variable** $\dot{q}_i \Rightarrow p_i$. This transformation, commonly called **Legendre transformation**, plays an important rôle in other areas of knowledge. At the turn of the last century Josiah Willard Gibbs revolutionized thermodynamics by pushing an analogy with mechanics. He divided the thermodynamic variables into pairs such as

entropy $S$/temperature $T$

volume $V$ /pressure $P$

surface area $A$/surface tension $\sigma$

length of spring $x$/ tension in spring $\mathcal{T}$

number of molecules $N$/chemical potential $\mu$

electric field $\vec{\mathcal{E}}$/ electric polarization $\vec{\mathcal{P}}$

etc.../etc

The common feature is that the products $ST$, $Vp$, $A\sigma$, $x\mathcal{T}$, $N\mu$, $\vec{\mathcal{E}}\vec{\mathcal{P}}\cdots$ all have dimension energy in analogy with $p_i\dot{q}$. The left member of the pairs can all be classified as **generalized displacements** while the right member as a **generalized force**. The analogy is here a bit flawed since it is $\dot{p}_i$ not $p_i$ which is the force and $q_i$ not $\dot{q}_i$ that is the displacement. The thermodynamic interpretation is that e.g $PdV$ is the **work** done by the gas associated with an infinitesimal volume displacement. The second law of thermodynamics says that for a **reversible** infinitesimal change

$$dE = TdS - PdV + \mu dN \cdots$$

Since the energy is a **property** of the **state of the system** (and not its history) we can put the energy as a function of the generalized displacements

$$E = E(S, V, A, x, N, \mathcal{P}..)$$

while the generalized forces can be defined as partial derivatives

$$T = \frac{\partial E}{\partial S},\ P = -\frac{\partial E}{\partial V},\ \mu = \frac{\partial E}{\partial N}, \cdots$$

The same analogy with mechanics occurs in microeconomics where the **marginal utility** or **demand** $D$ is the derivative

$$D = \frac{\partial U}{\partial Q}$$

where $Q$ is the quantity available or **supply**. In mechanics equilibrium requires forces to be balanced, in thermodynamic equilibrium generalized forces $T,\ P,\ \mu\cdots$ must be balanced while in economic equilibrium price balances marginal utility making price $\times$ supply = money. Apparently Gibbs objected to the latter analogy that "utility" is not a state variable analogous to energy but analogous to "heat" or "work" which depend on history, or to a Lagrangian which depend on how the variables are classified (kinetic versus potential energy). This would put classical economics at par with the discredited caloric theory of heat, in which heat was considered as a substance that could flow. (This issue is discussed at length in a somewhat controversial book by Philip Mirowski "More heat than light; Economics as social physics, physics as nature's economics", which students interested in history of ideas may wish to look at).

In thermodynamics it is sometimes awkward that in the expression for the energy the entropy is an independent variable, we don't generally have "entropostats" that allow us to control the entropy of a system. Thermostats are common though, we often put systems in contact with heat baths with fixed temperature. We make a change in independent variable from $S$ to $T$ by introducing the Helmholtz free energy $F = E - TS$. For an infinitesimal change in state

$$dF = dE - TdS - SdT = -SdT - PdV + \mu dN \cdots$$

and we see that we must consider the free energy as a function of the temperature rather than the entropy. Instead

$$S = \frac{\partial F}{\partial T}$$

Similarly, as we shall see next time: in mechanics it is often useful to have the freedom offered by the Hamiltonian approach in choosing variables.

In the Hamiltonian approach the dynamical variables, the coordinates and momenta, are treated in a symmetric fashion. E.g. if we introduce new variables

$$P_i = q_i, \;\; Q_i = -p_i, \;\; \tilde{H}(P_i, Q_i) = H(p_i, q_i)$$

then

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} = \frac{\partial \tilde{H}}{\partial P_i} = \dot{Q}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{\partial \tilde{H}}{\partial Q_i} = -P_i$$

i.e. the Hamiltonian $\tilde{H}$ describes a dynamical system in which the rôles of coordinates and momenta are reversed. Thus there is no longer a need to worry about treating kinetic and potential energy differently. Similarly in Gibbsian thermodynamics we don't need to worry about the difference between heat and work, they are treated symmetrically.

SUMMARY

- We carried out a transformation of variables from $\mathcal{L}(q_i, p_i, t)$ to $H(p_i, q_i, t)$. That is the emphasis is shifted from generalized velocities to generalized momenta.
- The transformation had strong analogies in other fields of science notably thermodynamics.
- In the Hamiltonian formalism we treat coordinates and momenta in a symmetric fashion and there is no need to maintain a conceptual distinction between kinetic and potential energy.

## 7.2 Poisson Brackets. Liouville's theorem.

LAST TIME
Started our discussion of Hamiltonian systems. In particular

- we carried out a **Legendre transformation** from $\mathcal{L}(q_i, p_i, t)$ to $H(p_i, q_i, t)$. The emphasis thus shifted from generalized velocities to generalized momenta.
- The transformation had strong analogies in other fields of science, notably thermodynamics.
- In the Hamiltonian formalism we treat coordinates and momenta in a symmetric fashion, and there is no need to maintain a sharp conceptual distinction between kinetic and potential energy.
- In Hamiltonian mechanics the second order Lagrangian equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

  are replaced but the **pair of first order equations**

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

TODAY
we wish to pursue the Hamiltonian formalism a bit further emphasizing connections to other fields of physics, particularly statistical and quantum mechanics.

LIOUVILLE'S THEOREM
Consider a system governed by Hamiltonian mechanics. For the time being let us limit ourselves to one degree of freedom. As we shall see the generalization to several degrees of freedom is relatively straightforward. Suppose that at time $t = 0$ the initial condition is $p = p_0$, $q = q_0$. At time $t$ thereafter the system will be in a new state

$$p = p_t(p_0, q_0), \quad q = q_t(p_0, q_0)$$

The mathematical object above is generally called a **mapping**. In practice, one cannot prepare a system **exactly** in the initial state, so we would like to know what happens if the system is prepared inside an **area**

$$A = dp\, dq$$

surrounding the point $p_0 q_0$ in phase space. After some time $t$ this area will be deformed into a new area

$$A' = |J| dp_0\, dq_0$$

where $|J|$ is the determinant of the **Jacobian** matrix

$$J = \begin{pmatrix} \frac{\partial p}{\partial p_0} & \frac{\partial p}{\partial q_0} \\ \frac{\partial q}{\partial p_0} & \frac{\partial q}{\partial q_0} \end{pmatrix}$$

We wish to find a differential equation for the time evolution of the Jacobian. and therefore let $t \Rightarrow dt$ be infinitesimal. In this case we let

$$p = p_0 + \dot{p} dt = p_0 - \frac{\partial H}{\partial q} dt$$

$$q = q_0 + \dot{q} dt = q_0 + \frac{\partial H}{\partial p} dt$$

and we find for the Jacobian

$$J = \begin{pmatrix} 1 - dt\frac{\partial^2 H}{\partial p \partial q} & -dt\frac{\partial^2 H}{\partial q^2} \\ dt\frac{\partial^2 H}{\partial p^2} & 1 + dt\frac{\partial^2 H}{\partial q \partial p} \end{pmatrix}$$

If we multiply out the Jacobian determinant

$$|J| = 1 + dt\left(\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q}\right) + (dt)^2\left(\frac{\partial^2 H}{\partial q^2}\frac{\partial^2 H}{\partial p^2} - \left(\frac{\partial^2 H}{\partial q \partial p}\right)^2\right)$$

In the limit that $dt$ is infinitesimal the last term can be neglected. From the theory of partial derivatives we know that the term linear in $dt$ is identically zero and we find the remarkable result that the area $A$ satisfies the differential equation

$$\frac{dA}{dt} = 0$$

or $A =$ constant. Consider next several degrees of freedom. The diagonal part of the Jacobian can be written

$$J = \begin{pmatrix} 1 - dt\frac{\partial^2 H}{\partial p_1 \partial q_1} & \cdots & \cdots & \cdots \\ \cdots & 1 + dt\frac{\partial^2 H}{\partial q_1 \partial p_1} & cdots & \cdots \\ \cdots & \cdots & 1 - dt\frac{\partial^2 H}{\partial p_2 \partial q_2} & \cdots \\ \cdots & \cdots & \cdots & 1 + dt\frac{\partial^2 H}{\partial q_2 \partial p_2} \end{pmatrix}$$

Again, it is easy to see that when multiplying out the determinant, terms linear in $dt$ will cancel, and we get the same result as before.

We form a mental picture of this situation by imagining the time evolution of the system trajectory as the motion of an **incompressible fluid** e.g. a cup of coffee is to a good approximation incompressible under normal conditions. The initial "area" $dp_0 dq_0$ is then analogous a drop of milk. If the cup is stirred the milk will spread. Nevertheless, the volume of milk stays constant.

We have briefly encountered **dissipative systems** in connection with drag, friction and the damped forced pendulum. In such systems the volume of phase space will **shrink** to a point, if the system approaches a unique equilibrium. Alternatively, if a dissipative system doesn't come to complete rest, it will approach an **attractor**. We saw in the case of the forced pendulum that this attractor does not necessary have integer dimension- it could be a "strange attractor" of fractal dimension. We later show how Liouville's theorem can be interpreted in a "semiclassical" approximation to quantum mechanics.

POISSON BRACKETS

Let $f(q_i, p_i, t)$ and $g(q_i, p_i, t)$ be two functions of the generalized coordinates and momenta. The Poisson bracket $[f, g]$ is defined as

$$[f,g] \equiv \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right)$$

One use of this quantity is to compute the time evolution of arbitrary functions of the generalized coordinates and momenta. We have, using the equations of motion

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\dot{q}_i \frac{\partial f}{\partial q_i} + \dot{p}_i \frac{\partial f}{\partial p_i}\right) = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial p_i}\right)$$

or

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f]}$$

If we specialize to the case of functions that do not depend explicitly on time we find

$$\frac{df}{dt} = [H, f]$$

We have previously shown that if $H$ does not depend on a particular coordinate the corresponding momentum is conserved. If the Poisson bracket with the Hamiltonian of a certain quantity vanishes this quantity is conserved. This result suggests another use of the Poisson bracket: to establish new conservation laws.

Some properties of Poisson brackets that follow more or less immediately from the definition:

- it is **antisymmetric**

$$[f, g] = -[g, f]$$

- the Poisson bracket with a **constant** $c$ is always zero

$$[f, c] = 0$$

- it is **linear**

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g]$$

- it satisfies the chain rule

$$[f_1 f_2, g] = f_1[f_2, g] + f_2[f_1, g]$$

- Some special Poisson brackets: Suppose $q_i$ is a generalized coordinate and $p_i$ a generalized momentum and $f$ a function of coordinates and momenta

$$[f, q_i] = \frac{\partial f}{\partial p_i}$$

$$[f, p_i] = -\frac{\partial f}{\partial q_i}$$

$$[q_i, q_k] = [p_i, p_k] = 0; \quad [p_i, q_k] = \delta_{ik}$$

  where $\delta_{i,k}$ is the Kronecker delta.

An important conceptual aspect of Poisson brackets is that they have a quantum analog

$$[f, g] \Rightarrow \frac{1}{i\hbar}[fg - gf]$$

where on the right hand side $f, g$ are the quantum **operators** corresponding to the classical functions on the left. If two operators commute it is in principle possible to prepare a system in an eigenstate of both. If they do not commute they satisfy **uncertainty relations**.

# 8 Suggestions for optional end of term projects.

**Problem 8.1: Sensitivity to initial conditions.**
(See section 1. of "The Nature of Deterministic Chaos" of the pendulum lab
http://monet.physik.unibas.ch/~elmer/pendulum/chaos.htm
and the Maple worksheet
http://physics.ubc.ca/~birger/n206l16c.mws (or .html)
for the stroboscopic plots in lecture 3.9). Consider a horizontally driven damped pendulum and pick two solutions with almost identical initial conditions. Discuss how the difference in angle grows or shrinks in time using parameter values so that
**a:** the solution approaches a limit cycle
**b:** the solutions move on a strange attractor
**c:** When the two solutions diverge wildly, will the stroboscopic plot of the strange attractor look different, depending on the initial condition?

**Problem 8.2: The foldover effect** (See section 1: of "Nonlinear Resonance" of the pendulum lab
http://monet.physik.unibas.ch/~elmer/pendulum/nonres.htm and the Maple worksheet
http://physics.ubc.ca/~birger/n206l16c.mws (or .html)
of lecture 3.9). Consider the horizontally driven pendulum and monitor the root mean square amplitude of the oscillations when the frequency is
**slowly increased** past the main resonance starting with a value below the resonance frequency.
**slowly decreased** past the main resonance starting with a value above the resonance frequency. Demonstrate the existence of **hysteresis** when the amplitude of forcing is large enough.

**Problem 8.3:**(Continuation of Problem 1 of problem set 6 2000.) It is convenient to use the dimensionless quantities defined in the model solution:
http://physics.ubc.ca/~birger/n206ts6/index.html
According to this parameterization the dimensionless length

$$\rho = \frac{r}{a}; \quad a = \left(\frac{l^2}{mMg}\right)^{1/3}$$

in terms of dimensionless time

$$\tau = \frac{t}{T}; \;\; T = \frac{ma^2}{l}$$

are solutions to the equations

$$\frac{d\theta}{d\tau} = \frac{1}{\rho^2}$$

$$\epsilon = \frac{\alpha}{2}(\frac{d\rho}{d\tau})^2 + u(\rho)$$

where

$$u(\rho) = \frac{1}{2\rho^2} + \rho$$

$$\alpha = \frac{M+m}{T^2}$$

$$\epsilon = \frac{E}{U_0}$$

with

$$U_0 = Mga$$

and $E$, $l$ the energy and angular momentum respectively.
**a:** Write a routine that computes the change $\Delta$ in the angle $\theta$ when $\rho$ changes from $\rho_1$ to $\rho_2$ and back, where $\rho_1, \rho_2$ are solutions to $\epsilon - u(\rho) = 0$.
**b:** Pick a convenient value of $\epsilon$ (e.g. $\epsilon = 3$) and find by trial and error values of $\alpha$ for which

$$\Delta = \frac{N_1}{N_2}\pi$$

and $N_1$ and $N_2$ are small integers.
**c:** Plot the orbits for some of the parameter values found under **b:**.

**Problem 8.4: Schwarzschild geodesics.**
In the general theory of relativity the space time geometry surrounding a massive star is described by the Schwarzschild metric

$$ds^2 = (1 - \frac{2GM}{c^2 r})c^2 d\tau^2 - \frac{dr^2}{1 - \frac{2GM}{c^2 r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Here $M$ is the mass of the star, $G$ is the gravitational constant, $c$ the speed of light, $\tau, r, \theta, \phi$ coordinates of a space-time point using spherical polar coordinates for the spatial part. The time $\tau$ is the time as seen by an observer far away from the star. An object such as the light planet Mercury will follow a geodesic path governed by the variational principle

$$\delta \int ds = 0$$

and $s$ is the "proper time" of an observer moving with the object. In general relativity it is conventional to let "time" and "mass" have dimension length and work with

$$m = \frac{GM}{c^2}, \quad t = c\tau$$

The variational principle then becomes

$$0 = \delta \int ds \sqrt{(1 - \frac{2m}{r})\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2m}{r}} - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)}$$

where "dot" indicates differentiation with respect to $s$
It can be shown that we may replace the variational principle by

$$0 = \delta \int ds((1 - \frac{2m}{r})\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2m}{r}} - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2))$$

Unless you are close to a black -hole (and have other things to worry about)

$$\frac{2m}{r} << 1$$

**a:** By manipulating the conservation laws in much the same manner as we did for the Kepler problem, obtain an equation for the orbit in terms of an effective potential.
**b:** Show that to lowest order in $2m/r$ we recover the Kepler result.
**c:** Show that to next order in $2m/r$ we recover an effective potential of similar to the one found in problem 2 of problem set 6 2000. One difference is that the correction term is proportional to $1/r^3$ not $1/r^2$
**d:** Integrate the equation of motion numerically ( Lecture **??** has an associated Maple worksheet

http://physics.ubc.ca/~birger/p206l20.html
which does a similar integration for the symmetric top) to estimate the precession for a special case. (E.g., you may dig out astronomical data for Mercury. Since the precession in this case is very small you may run into problems with numerical accuracy. You may instead wish to replace Mercury by a virtual planet for which the correction is larger. If you visit the web site of Kristin Schleich and Don Witt
http://noether.physics.ubc.ca/Sims/
you will find a Java simulation of Schwarzschild orbits and some more hints for this problem.

**Problem 8.5: From flutter to tumble.**
This problem is based on a paper by Andrew Belmonte, Hagai Eisenberg and Elisha Moses in Physical Review Letters Vol. 81 page 345-48. The paper describes the behavior of thin flat strip such as a falling paper strip which is constrained to move in two dimensions and is falling under the influence of gravity and drag forces. The paper is a very elegant example of the usefulness of dimensional analysis of the type we encountered when discussing drag in lecture 4. However, this time it is the Froude number not the Reynolds number which is important. You may simulate the model described by the equations for $\dot{V}_x, \dot{V}_y, \dot{\omega}$ on p. 347, by plotting $\theta(t)$ and trajectories of the center of mass in the $x-y-$plane (or by visualizing the results by some other means e.g. animation, but beware, being ambitious here may be quite time consuming!). The numerical differential equation solver of Maple should be adequate. Articles in the Physical Review Letters can be down-loaded from computers in the UBC domain from the e-journals at the UBC Library web-site.

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