Coherent States in Laser Physics

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1. Introduction

It is a common statement that the electromagnetic field established in the optical cavity of a single-mode laser (operated above threshold) can be represented by the coherent state. The main objective of this project is to verify this statement. For doing so, first optical correlation function and its relationship with coherent states are described. Next the semiclassical theory and fully quantum mechanical theory of lasers are introduced and at the end by incorporating the coherent states in quantum theory, the relationship of these two formalisms is clarified.

2. Optical Coherence and Coherent States

A convenient starting point for introducing the coherent states in optics comes from the quantization of the electromagnetic field. The free electromagnetic field obeys the source free Maxwell's equations. With no sources present Maxwell's equations are gauge invariant and a convenient choice of gauge for problems in quantum optics is the Coulomb gauge[1] ($\bigtriangledown . \mathbf{A} = 0$), then both $\mathbf{B}$ and $\mathbf{E}$ fields can be determined from the vector potential which satisfies the following equation

$$\bigtriangledown^2 \mathbf{A}(\mathbf{r}, t) = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}(\mathbf{r}, t)}{\partial t^2} \tag{1}$$

Then by applying an appropriate boundary condition for the field confined to a certain volume of space, the vector potential can be expressed as an expansion of a discrete set of orthogonal mode functions:

$$\mathbf{A}(\mathbf{r}, t) = \sum_k \left(\frac{\hbar}{2\omega_k \epsilon_o}\right)^{1/2} [a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} + a_k^\dagger \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t}] \tag{2}$$

and the corresponding form for the electric field becomes:

$$\mathbf{E}(\mathbf{r}, t) = i \sum_k \left(\frac{\hbar\omega_k}{2\epsilon_o}\right)^{1/2} [a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} - a_k^\dagger \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t}] \tag{3}$$

In classical electromagnetic theory the Fourier amplitudes in above field equation are complex numbers, but after the quantization of the electromagnetic field a_k and $a_k^\dagger$ considered to be adjoint operators with bosonic commutation relations. Finally dynamics of the field amplitudes can be described by an ensemble of independent harmonic oscillators.

Now that we know the field at each point in space or time, the next question is how they are related to each other. Generally the correlation between the field at the space-time point $x_1 = (\mathbf{r}_1, t_1)$ and the field at the space-time point $x_2 = (\mathbf{r}_2, t_2)$ is expressed as the correlation function[2]

$$G^{(1)}(x_1, x_2) = Tr\{\rho E^{(-)}(x_1)E^{(+)}(x_2)\} \tag{4}$$

where ρ is the density operator, and $E^{(+)}$ contains all amplitudes which vary as $\hat{a}e^{-i\omega t}$ for $\omega > 0$ and $E^{(-)}$ contains all amplitudes which vary as $\hat{a}^{\dagger}e^{i\omega t}$. The idea of coherence[3] is associated with the possibility of producing interference fringes when two fields are superposed. The highest degree of optical coherence correspond to establishing fringes with maximum visibility. In two-slits Young's experiment, for example, it can be shown that intensity of superposition of fields in the observation screen coming from two slits at x_1 and x_2 is:

$$I = G^{(1)}(x_1, x_1) + G^{(1)}(x_2, x_2) + 2Re\{G^{(1)}(x_1, x_2)\} \tag{5}$$

If $G^{(1)}(x_1, x_2) = 0$, there would be no fringes and the fields are described as incoherent, but larger $G^{(1)}(x_1, x_2)$ more contrast for fringes will be. The magnitude of $|G^{(1)}(x_1, x_2)|$ is limited by the relation:[2]

$$|G^{(1)}(x_1, x_2)| \leq [G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)]^{1/2} \tag{6}$$

If we define the normalized correlation function as

$$g^{(1)}(x_1, x_2) = \frac{G^{(1)}(x_1, x_2)}{[G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)]^{1/2}} \tag{7}$$

Therefore the maximum visibility corresponds to $|g^{(1)}(x_1, x_2)| = 1$. It is clear that for a field which is an eigenstate of the operator $E^{(+)}(x)$ this relation holds. The coherent states are an example of such a field. It is precisely this property of the coherent states which led to their names by Glauber in 1963.[4] The situation can be generalize to the n^{th}-order optical coherence in which the coherent states satisfy $g^{(n)}(x_1 ... x_n, x_{n+1} ... x_{2n}) = 1$ for n^{th}-order normalized correlation function. Although the first-order correlation function is sufficient to account for classical interference experiments, to describe experiments involving intensity correlations, or correlation at more than two space-time points it is necessary to use higher-order correlation functions.

3. Semiclassical theory of laser

In the semiclassical theory the laser field is introduced in a self-consistent manner,[4] and an equation of motion for the field amplitude is obtained. It is assumed that a classical field **E** of frequency ω_o is in resonance with two levels atoms with energy gap of $E_2 - E_1 \simeq \hbar\omega_o$ and a dipole moment is induced at each excited atom (evaluated by quantum mechanics). Sum of all these induced dipole moments establishes a macroscopic polarization **P** at the gain medium. This polarization acts as a source term in Maxwell's equations and a new field **E** inside the active medium will be created. This process will continue until we reach to a self-consistent solution for the laser field. The whole process is shown in Fig.(1).

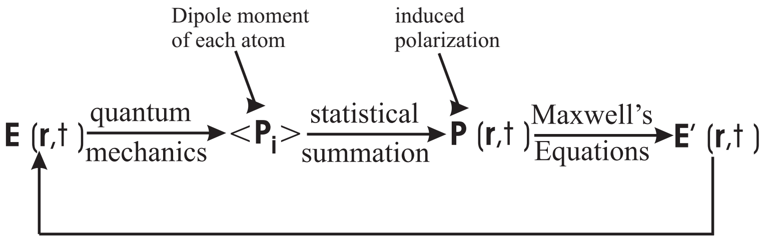

Fig. 1. A schematic diagram of semiclassical formalism of laser theory.

The losses of the electromagnetic field in the laser cavity are modeled by introducing a conductivity that causes attenuation of the field, then wave equation for a linearly polarized electric field becomes

$$-\frac{\partial^2 E}{\partial z^2}+\mu_o\sigma\frac{\partial E}{\partial t}+\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}+\mu_o\frac{\partial^2 P}{\partial t^2}=0 \tag{8}$$

Next expansion of $E(\mathbf{r},t)$ in terms of the normal modes of the optical cavity will be

$$E(\mathbf{r},t)=\sum_n \varepsilon_n(t)u_n(\mathbf{r})e^{i\omega_n t}+c.c. \tag{9}$$

where $\varepsilon(t)$ is a slowly varying envelope function of the field which satisfy the following relations:

$$|\frac{d\varepsilon(t)}{dt}|<<\omega|\varepsilon(t)|$$

$$|\frac{d^2\varepsilon(t)}{dt^2}|<<\omega|\frac{\varepsilon(t)}{dt}| \tag{10}$$

by assuming that the polarization $P(\mathbf{r},t)$ oscillate at the same frequency ω as the electric field, and multiply each term by $u^*(\mathbf{r})$ and integrate over the cavity volume, Eq.(8) reduces to

$$-i\omega(2\frac{d\varepsilon(t)}{dt}+\frac{\sigma}{\epsilon_o}\varepsilon)e^{i\omega t}=\frac{1}{V}\frac{\omega^2}{\epsilon_o}\int P^+(\mathbf{r},t)u^*(\mathbf{r})d^3r \tag{11}$$

in which $P^{(+)}(\mathbf{r},t)$ varies as $e^{-i\omega t}$. The macroscopic polarization $P(\mathbf{r},t)$ can be expressed by the relation

$$P(\mathbf{r},t)=\eta(\mathbf{r})<\hat{\mu}(\mathbf{r},t)> \tag{12}$$

where $\eta(\mathbf{r})$ is density of atoms in the cavity, and $<\hat{\mu}(\mathbf{r},t)>$ is the average dipole moment of an atom. Now the quantum mechanics enters into the calculation for evaluating the average atomic dipole moment, which after some calculation becoms

$$<|\hat{\mu}(\mathbf{r},t)|>=\frac{i(p_2-p_1)\mu_{12}^2T}{\hbar}|\varepsilon(t)|exp^{-i\omega t}u(\mathbf{r})[1-\frac{4\mu_{12}^2T^2}{\hbar^2}|\varepsilon(t)|^2|u(\mathbf{r})|^2+...]+c.c. \tag{13}$$

where p_2 and p_1 are population of upper and lower states respectively, and T is the average natural lifetime, which is assumed to be similar for the the upper and lower states. Substituting Eqs.(12) and (13) into Eq.(11) gives us:

$$\frac{d\varepsilon}{dt} + \frac{\sigma}{2\epsilon_o} = \frac{(p_2 - p_1)\mu_{12}^2 \omega T}{2\epsilon_o \hbar V} \varepsilon(t)[\int \eta(\mathbf{r})|u(\mathbf{r})|^2 d^3r - \frac{4\mu_{12}^2 T^2}{\hbar^2}|\varepsilon(t)| \int \eta(\mathbf{r})|u(\mathbf{r})|^4 d^3r + ...] \tag{14}$$

By introducing the following relations for the mean atomic density

$$\bar{\eta} = \frac{1}{V} \int \eta(\mathbf{r})|u(\mathbf{r})|^2 d^3r \tag{15}$$

and the quantity

$$F = \frac{1}{V} \int \eta(\mathbf{r})|u(\mathbf{r})|^4 d^3r \tag{16}$$

We reach to the equation of motion for the electric field

$$\frac{d\varepsilon}{dt} = [\frac{\omega \bar{\eta}(p_2 - p_1)\mu_{12}^2 T}{2\epsilon_o \hbar} - \frac{\sigma}{2\epsilon_o}]\varepsilon - \frac{2\omega F \mu_{12}^4 (p_2 - p_1) T^3}{\epsilon_o \hbar^3}|\varepsilon|^2 \varepsilon \tag{17}$$

For sufficiently small amplitude, the cubic term in Eq.(17) can be ignored and this equation can account for the exponential growth or decay, according to:

$$\frac{\omega \bar{\eta}(p_2 - p_1)\mu_{12}^2 T}{2\epsilon_o \hbar} > or < \frac{\sigma}{2\epsilon_o} \tag{18}$$

The term on the left in above equation, which is proportional to the population inversion, clearly represents the gain of the system (provided $p_2 > p_1$), and the term on the right, which is proportional to the conductivity σ, clearly represents the loss. When exact equality holds, the gain just equals the loss, and we are at the threshold of oscillation. Above threshold, when the gain exceeds loss, the amplitude grows exponentially until the nonlinear or saturation term begins to assert itself and to inhibit further growth. However, it should be noted that the initial growth depends on the presence of a non-zero field at the beginning, and in its absence the field remains zero for all time, no matter how high the population inversion may be. This is a fundamental weakness of the semi-classical treatment.

4. Semiclassical Laser Theory with Spontaneous Emission Noise

Eq.(17) in the previous section, in a completely deterministic manner, describes the dynamical behavior of the laser field. The optical field is free from all fluctuations, and therefore any questions about the coherence properties of laser light are outside the domain of this theory. For example, Eq.(17)

can tell us nothing about the spectral width of the laser field, which is here regarded as completely monochromatic, although a finite spectral width is always observed. The key for a better understanding of the laser theory lies in the fluctuations of the optical field, which are brought about by random spontaneous atomic emissions. Spontaneous emission are of course included automatically in any fully quantized treatment of the laser problem, but they do not appear naturally in deterministic semiclassical theories.

The effects of spontaneous emission in a semiclassical theory can be considered by adding a noise term to the equation of motion, Eq.(17), which then takes the form

$$\frac{d\varepsilon}{dt} - [\frac{\omega\bar{\eta}(p_2 - p_1)\mu_{12}^2 T}{2\epsilon_o\hbar} - \frac{\sigma}{2\epsilon_o}]\varepsilon + \frac{2\omega F \mu_{12}^4 (p_2 - p_1) T^3}{\epsilon_o \hbar^3}|\varepsilon|^2\varepsilon = \zeta(t) \quad (19)$$

Usually the noise term, $\zeta(t)$, is considered as a complex Gaussian random function of zero mean, with a correlation time that is of the order of the lifetime for spontaneous atomic emission.[5] The equation of motion, Eq.(19), is a *Langevin* type of equation.[3] The field amplitude $\varepsilon(t)$ in Eq.(19) is no longer deterministic, instead it is a random process, with a certain probability density $p(\varepsilon, t)$ determined by the statistical properties of the noise $\zeta(t)$.

For the *Langevin* equation of motion there exists an equation of motion for the probability density $p(\varepsilon, t)$, known as the *Fokker-Planck* equation.[3] It is convenient of replace the complex amplitude ε by a two-dimensional real vector $\mathbf{x}$ for real and imaginary parts of ε. Finally it can be shown that the *Fokker-Planck* equation corresponding to Eq.(19) takes the following form

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}[(a - \mathbf{x}^2)x_i p(\mathbf{x}, t)] + \sum_{i=1}^{2} \frac{\partial^2 p(\mathbf{x}, t)}{\partial x_i^2} \quad (20)$$

where a is so-called pumping parameter and it is expressed as

$$\frac{\omega\bar{\eta}(p_2 - p_1)\mu_{12}^2 T/2\epsilon_o\hbar - \sigma/2\epsilon_o}{\omega\bar{\eta}(p_2 - p_1)\mu_{12}^2 T/2\epsilon_o\hbar} \quad (21)$$

5. Quantum Theory of the Laser Field

As was shown in previous sections the semiclassical theory leads to a deterministic equation of motion for the field amplitude, and questions about linewidth and fluctuations are outside the domain of the theory. Fluctuations were incorporated at a later stage by the introduction of a noise source, but this procedure cannot be easily justified in a rigorous manner. The quantized field treatment provides a more consistent foundation for the laser theory. In addition, it is able to answer certain questions that are not meaningful at all within the framework of a semiclassical theory, such as how many photons are present in the laser cavity at threshold, and what is their probability distributions.

Once again we consider a set of identical two-level atoms interacting with the field of a single cavity mode. If the atomic population is inverted, the coupling between atoms and the field may cause the number of photons in the cavity mode to grow in time. We start by examining the effect of a single excited atom on the state of the laser field. The energy of the coupled system consisting of a single-mode electromagnetic field of frequency ω and a two-level atom of resonance frequency ω_o located at position $\mathbf{r}$, has the following form

$$\hat{H} = E_o + \frac{\hbar\omega_o}{2}(\hat{b}^\dagger\hat{b} - \hat{b}\hat{b}^\dagger) + \hbar\omega(\hat{n} + \frac{1}{2}) - \hat{\mu}_{12}.\hat{\mathbf{E}}(\mathbf{r},t)[\hat{b}(t) + \hat{b}^\dagger(t)] \tag{22}$$

Here E_o is the average energy of the two laser levels, μ_{12} is the transition dipole moment of the atom, and $\hat{n}$ is the photon number operator. It will be convenient to work in the interaction picture, in which the atomic lowering and raising operators evolve in time according to the relations

$$\hat{b}(t) = \hat{b}(t_o)e^{-i\omega_o(t-t_o)}$$
$$\hat{b}^\dagger(t) = \hat{b}^\dagger(t_o)e^{i\omega_o(t-t_o)}$$

Also the single-mode electric field $\mathbf{E}(\mathbf{r},t)$ in the interaction picture may then be expressed in the form

$$\mathbf{E}(\mathbf{r},t) = \frac{\hat{\mathbf{e}}}{V^{1/2}}(\frac{\hbar\omega}{2\epsilon_o})^{1/2}[\hat{a}(t_o)u(\mathbf{r})e^{-i\omega_o(t-t_o)} + h.c.] \tag{23}$$

where $\hat{\mathbf{e}}$ is a real unit polarization vector representing a linearly polarized wave. The interaction term in Eq.(22) becomes

$$\hat{H}_I(t) = \hbar g[\hat{a}\hat{b}^\dagger u(\mathbf{r})e^{-i(\omega_o-\omega)(t-t_o)} + \hat{a}\hat{b}u(\mathbf{r})e^{-i(\omega_o+\omega)(t-t_o)} + h.c.] \tag{24}$$

where g is the coupling constant characterizing the strength of the interaction

$$g = -(\frac{\omega}{2\hbar\epsilon_o V})^{1/2}\hat{\mu}_{12}.\hat{\mathbf{e}} \tag{25}$$

Let us suppose that at time t_o, when the interaction is assumed to be turned on, the state of the coupled system is factorized, so that we can express the density operator as

$$\hat{\rho}(t_o) = \hat{\rho}_A(t_o) \otimes \hat{\rho}_F(t_o) \tag{26}$$

where $\hat{\rho}_A(t_o)$, $\hat{\rho}_F(t_o)$ are reduced density operators of the atom and the field, respectively. As a result of the interaction, $\hat{\rho}$ evolves in time in the interaction picture, and its form at a later time t can be given by the perturbation expansion. By taking the trace over the atomic variables, we can express the effect of the interaction on the state of the laser field at time t by

$$\hat{\rho}_F(t) = \hat{\rho}_F(t_o) + Tr_A \sum_{r=1}^{\infty} \frac{1}{i\hbar^r} \int_{t_o}^{t} dt_1 \int_{t_o}^{t_1} dt_2 ... \int_{t_o}^{t_{r-1}} dt_r \times [\hat{H}_I(t_1), [\hat{H}_I(t_2), [...[\hat{H}_I(t_r), \hat{\rho}(t_o)]]...]] \tag{27}$$

In order to determine the effect of a single excited atom on the laser field, we take the initial atomic state to be the upper state $\hat{\rho}_A(t_o) = |2><2|$ and we allow the interaction to proceed for a time that is of the order of the lifetime of the upper state. It is assumed that the two laser levels $|1>$ and $|2>$ decay to other states with average lifetime T_1 and T_2, respectively. The effect of an ensemble of such excited atoms on the field is therefore obtainable, at least approximately, by choosing $t - t_o = \Delta t$ in Eq.(27) to be the atomic lifetime, and averaging over the ensemble of Δt with an exponential probability distribution of average T_2. We shall limit ourselves to the first two non-vanishing contributions in the expansion in Eq.(27).

We now substitute Eq.(24) in Eq.(27). It is clear that if the lifetime $\Delta t = t - t_o$ is very long compared with the optical period $2\pi/\omega_o$, then the terms in $e^{\pm i(\omega+\omega_o)t}$ in Eq.(27) will integrate almost to zero and can be neglected. On the other hand, the contributions made by the terms in $e^{\pm i(\omega-\omega_o)t}$ depend strongly on the magnitude of the detuning $\omega - \omega_o$. For simplicity we shall assume that the detuning is so small that $|\omega - \omega_o|\Delta t << 1$, which allows us to neglect the oscillatory factors altogether, and to treat $\hat{H}_I$ as time-independent under the integral in Eq.(27), then we have

$$\hat{\rho}_F(t_o + \Delta t) = \hat{\rho}_F(t_o) + Tr_A\{\frac{\Delta t}{i\hbar}[\hat{H}_I, \hat{\rho}(t_o)] + (\frac{\Delta t}{i\hbar})^2 \frac{1}{2!}[\hat{H}_I, [\hat{H}_I \hat{\rho}(t_o)]] + ...\} \tag{28}$$

The problem therefore reduces to the evaluation of commutators of successively higher orders. If we only consider terms to 4th-order(after some calculations it can be shown that first and third-order terms will be vanished) and using the fact that $\hat{b}$ and $\hat{b}^\dagger$ are traceless after some calculation we find that

$$\begin{aligned} \Delta\hat{\rho}_F(t_o) &= -(gT_2)^2|u(\mathbf{r})|^2[\hat{a}\hat{a}^\dagger\hat{\rho}_F(t_o) - \hat{a}^\dagger\hat{\rho}_F(t_o)\hat{a} + h.c.] \\ &+ (gT_2)^4|u(\mathbf{r})|^4[\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t_o) + 3\hat{a}\hat{a}^\dagger\hat{\rho}_F(t_o)\hat{a}\hat{a}^\dagger - 4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t_o)\hat{a} + h.c.] \end{aligned} \tag{29}$$

where averaging over the lifetimes Δt with the following exponential probability distribution was taken.

$$P(\Delta t) = (1/T_2)e^{-\Delta t/T_2} \tag{30}$$

As in the semiclassical approach, the calculation has been carried through to the second non-vanishing contribution from the perturbation expansion. The first term in Eq.(29) corresponds to a single photon emission by the excited atom, whereas the second represents the contribution of two cycles of emission and re-excitation by the atom.

A real laser of course has many excited atoms and molecules distributed throughout the active medium, which are excited by some pumping mechanism at a certain rate R_2. The contribution to the change of $\hat{\rho}_F$ made by any one of them is therefore small, and if the laser field evolves slowly we may look on the change $\hat{\rho}_F(t_o)$ given by Eq.(29) as a good approximation. When $\hat{\rho}_F(t_o)$ is multiplied by R_2, and averaged over the different atomic positions $\mathbf{r}$, it gives the average rate of change of the density operator of the laser field brought about by gain mechanism, the result is

$$\begin{aligned}(\frac{\partial\hat{\rho}_F}{\partial t})_{gain} &= -\frac{1}{N}R_2(gT_2)^2\int\eta(\mathbf{r})|u(\mathbf{r})|^2[\hat{a}\hat{a}^\dagger\hat{\rho}_F-\hat{a}^\dagger\hat{\rho}_F\hat{a}+h.c.]d^3r\\ &+ \frac{1}{N}R_2(gT_2)^4\int\eta(\mathbf{r})|u(\mathbf{r})|^4[\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F+3\hat{a}\hat{a}^\dagger\hat{\rho}_F\hat{a}\hat{a}^\dagger-4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F\hat{a}+h.c.]d^3r\end{aligned} \tag{31}$$

also by using the same procedure, we can calculate the effect of loss on field density operator induced by absorption of an atom in lower state, the result is

$$(\frac{\partial\hat{\rho}_F}{\partial t})_{loss} = -\frac{1}{N_1}R_1(gT_1)^2\int\eta_1(\mathbf{r})|u(\mathbf{r})|^2[\hat{a}\hat{a}^\dagger\hat{\rho}_F(t_o)-\hat{a}^\dagger\hat{\rho}_F(t_o)\hat{a}+h.c.]d^3r \tag{32}$$

where it was assumed that N_1 atoms in lower states with some density $\eta_1(\mathbf{r})$ are present in the laser cavity. Finally, we combine the effects of the gain and the loss by adding the contributions given by Eqs. (31) and (32). This leads to the following master equation for the density operator $\hat{\rho}_F$ of the laser field

$$\begin{aligned}\frac{\partial\hat{\rho}_F}{\partial t} &= -\frac{1}{2}A[\hat{a}\hat{a}^\dagger\hat{\rho}_F-\hat{a}^\dagger\hat{\rho}_F\hat{a}+h.c.]-\frac{1}{2}C[\hat{a}^\dagger\hat{a}\hat{\rho}_F-\hat{a}\hat{\rho}_F\hat{a}^\dagger+h.c.]+\frac{1}{8}B[\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F\\ &+ 3\hat{a}\hat{a}^\dagger\hat{\rho}_F\hat{a}\hat{a}^\dagger-4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F\hat{a}+h.c.]\end{aligned} \tag{33}$$

Where the following abbreviations were used:

$$A = 2(\frac{R_2}{N})(gT_2)^2\int\eta(\mathbf{r})|u(\mathbf{r})|^2d^3r \tag{34}$$

$$C = 2(\frac{R_1}{N_1})(gT_1)^2\int\eta_1(\mathbf{r})|u(\mathbf{r})|^2d^3r \tag{35}$$

$$B = 8(\frac{R_2}{N})(gT_2)^4\int\eta(\mathbf{r})|u(\mathbf{r})|^4d^3r \tag{36}$$

for the coefficients characterizing the gain, loss, and nonlinearity respectively. In practice, the loss rate would be identified with the actual rate at which photons are lost from the laser cavity, which is largely determined by the reflectivity of the mirrors. Moreover, in practice the gain and loss rates A and C are usually comparable, even well above and well below threshold. On the other hand, the ratio of B to A is usually extremely small. It is given by the relation

$$\frac{B}{A} = 4(gT_2)^2 \frac{\int \eta(\mathbf{r})|u(\mathbf{r})|^4 d^3r}{\int \eta(\mathbf{r})|u(\mathbf{r})|^2 d^3r} \sim (gT_2)^2 F \tag{37}$$

where F is a dimensionless factor of order unity. We can make a rough numerical estimate of B/A

$$\frac{B}{A} \sim (gT_2)^2 \approx \frac{3}{8\pi} \frac{cT_2^2 \lambda^2}{TV}$$

If we take $T_2 \sim T \sim 10^{-8}$ s, $\lambda \sim 6 \times 10^{-5}$ cm, $V \sim 10^{-1}$ to $1\ cm^3$, which are typical values for some lasers, we find that

$$B/A \sim 10^{-6}\ \ to\ \ 10^{-7}$$

In general, the smaller the saturation coefficient B, the stronger the laser field will be in the cavity to establish a steady-state.

We have seen that the quantum theory of the laser provides information on the absolute strength or the absolute number of photons in the laser field. However, the distribution of the light intensity in the steady state is no different from that given by semiclassical theory with additive noise. This suggest a deeper connection between the two different treatments of the laser theory. It will be shown in the next section that with the help of the coherent state representation of the laser field, the operator master equation, Eq.(33), can be modified into the form of a *Fokker-Planck* equation.[3] This simply shows the essential equivalence of the two approaches.

6. Coherent State Representation of the Laser Field

In the coherent state representation it is possible to represent the density operator $\hat{\rho}_F(t)$ of the single-mode laser field in a diagonal form

$$\hat{\rho}_F = \int \phi(\nu, t)|\nu><\nu| d^2\nu \tag{38}$$

in which the integral is to be taken over the entire complex ν-plane. Here $|\nu>$ is the coherent state with complex amplitude ν, and $\phi(\nu, t)$ is a real weight function or phase space density. By replacing $\hat{a}$ and $\hat{a}^\dagger$ by differential operators

$$\hat{a}^\dagger|\nu><\nu| = (\nu^* + \frac{\partial}{\partial\nu})|\nu><\nu|$$
$$|\nu><\nu|\hat{a} = (\nu + \frac{\partial}{\partial\nu^*})|\nu><\nu|$$

and inserting the representation of Eq.(37) of the density operator $\hat{\rho}_F(t)$ into Eq.(33), and after some rearrangement of terms, we arrive at the following equation of motion

$$\int \frac{\partial\phi(\nu,t)}{\partial t}|\nu><\nu|d^2\nu = \int \phi(\nu,t)\{\frac{1}{2}A[\nu\frac{\partial}{\partial\nu}+\nu^*\frac{\partial}{\partial\nu^*}+\frac{2\partial^2}{\partial\nu\nu^*}]-\frac{1}{2}C[\nu\frac{\partial}{\partial\nu}+\nu^*\frac{\partial}{\partial\nu^*}]$$
$$B[-\frac{1}{2}|\nu|^2(\nu\frac{\partial}{\partial\nu}+\nu^*\frac{\partial}{\partial\nu^*})-\frac{3}{8}(\nu^2\frac{\partial^2}{\partial\nu^2}+\nu^{*2}\frac{\partial^2}{\partial\nu^{*2}})-\frac{5}{4}|\nu|^2\frac{\partial^2}{\partial\nu\partial\nu^*}$$
$$-\frac{7}{8}(\nu\frac{\partial}{\partial\nu}+\nu^*\frac{\partial}{\partial\nu^*})-\frac{\partial^2}{\partial\nu\partial\nu^*}-\frac{1}{2}(\nu\frac{\partial^3}{\partial\nu^*\partial\nu^2}+\nu^*\frac{\partial^3}{\partial\nu\partial\nu^{*2}})]\}|\nu><\nu|d^2\nu \tag{39}$$

in which the differential operators under the integral on the right operate on the coherent-state projector $|\nu><\nu|$. By formally integrating by parts, with the assumption that $\phi(\nu,t)$ vanishes at infinity faster than any power of ν,ν^*, we can convert the integrand on the right into a product of $|\nu><\nu|$ and a c-number function of ν,ν^*, and we obtain the formula

$$\int |\nu><\nu|\frac{\partial\phi(\nu,t)}{\partial t}d^2\nu = \int |\nu><\nu|\{-\frac{1}{2}(A-C)(\nu\frac{\partial}{\partial\nu}+\nu^*\frac{\partial}{\partial\nu^*})+A\frac{\partial^2}{\partial\nu\partial\nu^*}$$
$$+B[\frac{1}{2}(\nu\frac{\partial}{\partial\nu}+\nu^*\frac{\partial}{\partial\nu^*})|\nu|^2-\frac{3}{8}(\frac{\partial^2}{\partial\nu^2}\nu^2+\frac{\partial^2}{\partial\nu^{*2}}\nu^{*2})-\frac{5}{4}\frac{\partial^2}{\partial\nu\partial\nu^*}|\nu|^2$$
$$+\frac{7}{8}(\nu\frac{\partial}{\partial\nu}+\nu^*\frac{\partial}{\partial\nu^*})-\frac{\partial^2}{\partial\nu\partial\nu^*}+\frac{1}{2}(\frac{\partial^3}{\partial\nu^*\partial\nu^2}\nu+\frac{\partial^3}{\partial\nu\partial\nu^{*2}}\nu^*)]\}\phi(\nu,t)d^2\nu \tag{40}$$

Comparison of the coefficients of $|\nu><\nu|$ on both sides of the equation leads to a partial differential equation for $\phi(\nu,t)$. We shall simplify this equation by discarding certain terms. We first recall that B is a very small coefficient compared with A or C. We therefore retain only the most important terms in B. On the other hand, when the laser is operating anywhere near the steady state, $|\nu|$ is a large number, and $|\nu|^2$ is of the order of the average number of photons. It follows that among the terms in B containing first derivatives,

$$\frac{1}{2}(\frac{\partial}{\partial\nu}\nu+\frac{\partial}{\partial\nu^*}\nu^*)|\nu|^2\phi(\nu,t)$$

is the most important, and that among the second derivative terms in B

$$-\frac{3}{8}(\frac{\partial^2}{\partial\nu^2}\nu^2+\frac{\partial^2}{\partial\nu^{*2}})\nu^{*2})|\phi(\nu,t)-\frac{5}{4}\frac{\partial^2}{\partial\nu\partial\nu^*}|\nu|^2\phi(\nu,t)$$

are the most important terms. But unless the laser is operating so far above threshold that $|\nu|^2$ is 1000 times, say, greater than its threshold value(which implies that the pump parameter is of the order of 1000 or higher), these last terms are still small compared with the second derivative term in A. Similarly, it can be argued that the terms in B involving third derivatives are small compared with those involving first derivatives. With the understanding that the laser is not operating too far above threshold, we shall therefore retain only the first term in B. The differential equation for $\phi(\nu, t)$ then takes the much simpler form

$$\frac{\partial \phi(\nu,t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial \nu}[B|\nu|^2 - (A-C)]\nu\phi(\nu,t) + \frac{1}{2}\frac{\partial}{\partial \nu^*}[B|\nu|^2 - (A-C)]\nu^*\phi(\nu,t) + A\frac{\partial^2\phi(\nu,t)}{\partial\nu\partial\nu^*} \tag{41}$$

which is a *Fokker-Planck* equation for the phase space density $\phi(\nu, t)$. It is often convenient to replace the complex amplitude ν by a real two-dimensional vector $\mathbf{x}$, such that $\nu = x_1 + ix_2$ and we have

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{2}\frac{1}{2}\frac{\partial}{\partial x_i}(A - C - B\mathbf{x}^2)x_i\phi(\mathbf{x},t) + \frac{1}{4}A\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}\phi(\mathbf{x},t) \tag{42}$$

Finally, we may re-scale the equation by putting

$$(\frac{1}{8}AB)^{1/2}t = t'$$
$$(2B/A)^{1/4}\mathbf{x} = \mathbf{x}'$$

and arrive at the following equation

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{2}\frac{\partial}{\partial x_i}[(a - \mathbf{x}^2)x_i\phi(\mathbf{x},t)] + \sum_{i=1}^{2}\frac{\partial^2\phi(\mathbf{x},t)}{\partial x_i^2} \tag{43}$$

where

$$\frac{A-C}{(AB/2)^{1/2}} = a$$

This equation of motion is identical to the scaled Eq.(20) that was obtained from the semiclassical theory with additive noise, in which a is the dimensionless pump parameter. It follows that within the limits of the approximations that were made, we may identify $\mathbf{x}$ with the scaled amplitude of the laser electric field, and $\phi(\mathbf{x}, t)$ with its probability density. However, the quantized field treatment has the advantage over the semiclassical one that no assumption about the strength of the quantum fluctuations is needed. All scaling factors are well-defined in terms of the fundamental gain, loss and saturation coefficients A,B,C of the laser.

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