Some Notes on the Schmid Model

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1 Introduction

The Schmid model considers the behaviour of a particle on a 1d lattice that interacts with a thermal bath. Of key interest in such models is the response of the particle to an applied force. The *mobility* may be defined loosely [1] as the ratio of the terminal velocity of a particle to the applied driving force:

$$\mu = \frac{v_F}{F} = \lim_{t\to\infty} \frac{\langle x(t_0+t) - x(t_0)\rangle}{tF} \tag{1}$$

The system's response to a time dependent force can also be studied in which case the mobility in frequency space is

$$\mu(\omega) = |\omega| \langle xx(\omega)\rangle \tag{2}$$

where $\langle xx(\omega)\rangle$ is the Fourier transform of the two-point correlation function $\langle x(t)x(0)\rangle$.

The mobility of a classical particle subject to viscous resistance $\eta\dot{x}$ can be obtained from the steady-state condition for the equation of motion, $F = \eta v_F \Rightarrow \mu = 1/\eta$. Calderra and Leggett [2] were able to show how such classical behaviour arises from a quantum mechanical model by considering a particle that interacts with an infinite set of harmonic oscillators that represent the other degrees of freedom of the system. An otherwise free particle is confined to "localized" states once the coupling to the environment becomes sufficiently strong. The Schmid model investigates the modifications to this transition due to the presence of a periodic lattice.

In Schmid's first study of the model [3] he was able to demonstrate the existence of a transition from localized to delocalized behaviour as the strengths of the lattice potential and dissipation were varied. He also found a duality in the behaviour of the system relating the mobility in the two limits of strong and weak dissipation.

The Schmid model is instructive in that the analysis involves the use of path integrals and influence functionals, renormalization group arguments, and duality properties.

2 The Model

We will work in dimensionless units. The particle of interest has mass M and coordinate q, and moves in a periodic potential. Since we are interested in the particle's mobility, we also couple to a small driving force $F(t)$. The action for the particle (without coupling to the dissipative bath) is then

$$S_0(q(t);t) \quad = \quad \int_0^t dt' \left(\tfrac{M}{2}\dot{q}(t)^2 + g\cos(q(t)) + F(t)q(t)\right) \tag{3}$$

To introduce dissipation into the system, we couple the particle to a set of harmonic oscillators, indexed by α, that represent the other degrees of freedom of the lattice. For oscillator coordinates x_α, frequencies ω_α, and masses m_α, the contribution to the action that includes the interaction of the particle with the oscillators and the kinetic term for the oscillators is

$$S_1 = -\sum_\alpha \left(C_\alpha x_\alpha q + \tfrac{m_\alpha}{2}\dot{x}_\alpha^2\right). \tag{4}$$

This is the standard way to couple a particle to a heat bath. Once the oscillators are "integrated out," the effective action will remember the effects of the oscillators only through a weighted density of states function

$$J(\omega) = \tfrac{\pi}{2}\sum_\alpha \frac{C_\alpha^2}{m_\alpha\omega_\alpha}\delta(\omega-\omega_\alpha), \quad \omega > 0. \tag{5}$$

We will assume an ohmic bath, which means that we take $J(\omega) = \eta\omega$. This is a reasonable form for the spectrum, and greatly simplifies many integrals. In particular, it can be shown that integrating out the oscillators leaves an effective contribution to the action (corresponding to S_1)

$$S_{1,eff} = \frac{\eta}{4\pi}\int_0^t dt' \int_0^t dt'' \left(\frac{q(t')-q(t'')}{t'-t''}\right)^2 \tag{6}$$

This effective contribution to the action is quadratic in the particle coordinates and should in principle be easy to work with. In Fourier transformed form we have

$$\int_0^t dt' \tfrac{M}{2}\dot{q}(t')^2 + \tfrac{\eta}{4\pi}\int_0^t dt' \int_0^t dt'' \left(\frac{q(t')-q(t'')}{t'-t''}\right)^2 \quad = \quad \tfrac{1}{2}\int \tfrac{d\omega}{2\pi}(M\omega^2 + \eta|\omega|)|q(\omega)|^2. \tag{7}$$

3 The Coulomb Gas Expansion

We are interested in the evolution of the particle's position with time and thus with the propagator

$$\begin{aligned} K(q_2,t;q_1,0) &= \langle q_2|\exp\left(i\hat{H}t\right)|q_1\rangle \\ &= \int_{q_1}^{q_2}\mathcal{D}x\exp\left(iS[x(t);t]\right) \\ &= \int_{q_1}^{q_2}\mathcal{D}x\exp\left(i\tfrac{1}{2}\int\tfrac{d\omega}{2\pi}(M\omega^2+\eta|\omega|)|q(\omega)|^2+\int_0^t\left(g\cos q(t)+F(t)q(t)\right)\right. \\ &\qquad \left.+\frac{\eta}{4\pi}\int dt\int dt'\left(\frac{q(t)-q(t')}{t-t'}\right)^2\right). \end{aligned} \tag{8}$$

The most challenging term remaining in the action is the cosine from the periodic potential. To handle it, we exploit the fact that the cosine is itself a pair of exponentials. We first expand

$$\exp\left(ig\int_0^t\cos q\right)=\sum_{n=0}^{\infty}\frac{(ig)^n}{n!}\int_0^t dt_1\ldots\int_0^t dt_n\prod_{i=1}^{n}\cos q(t_i) \tag{9}$$

and then split each cosine into exponentials

$$\begin{aligned} \prod_{i=1}^{n}\cos q(t_i) &= \prod_{i=1}^{n}\tfrac{1}{2}\left(\exp(iq(t_i))+\exp(-iq(t_i))\right) \\ &= \tfrac{1}{2^n}\sum_{\{e_i=\pm1\}}\exp\left(i\sum_{i=1}^{n}e_iq(t_i)\right). \end{aligned} \tag{10}$$

We may go further and define, for each configuration of $\{e_i\}$, a "charge density" $\sigma(t)=\sum_{i=1}^n e_i\delta(t-t_i)$. This allows us to replace $\sum_{i=1}^n e_iq(t_i)\to\int_0^t dt'\sigma(t')q(t')$. Then

$$\exp\left(ig\int_0^t\cos q\right)=\sum_{n=0}^{\infty}\left(\frac{ig}{2}\right)^n\int_0^t dt_1\ldots\int_0^t dt_n\sum_{\{e_i=\pm1\}}\exp\left(i\int_0^t dt'\sigma(t')q(t')\right). \tag{11}$$

With this replacement, the complete expression for the amplitude (8) is now expressed as a sum over integrals of amplitudes:

$$K(q_2,t;q_1,0) = \sum_{n=0}^{\infty}\left(\frac{ig}{2}\right)^n\int_0^t dt_1\ldots\int_0^t dt_n\sum_{\{e_i=\pm1\}}G(q_2,t;q_1,0;\sigma) \tag{12}$$

where

$$G(q_2, t; q_1, 0; \sigma) = \int_{q_1}^{q_2} \mathcal{D}q \exp i \left(\tfrac{1}{2} \int \tfrac{d\omega}{2\pi} (M\omega^2 + \eta|\omega|)|q(\omega)|^2 \right.$$
$$\left. \int_0^t dt'(\sigma(t') + F(t'))q(t') \right) \tag{13}$$

Rather than a cosine potential we have a new driving force $\sigma(t)$ that consists of a series of "kicks" at particular moments in the trajectory.

4 A few more details

If we want a finite mobility, we must impose the restriction that the time average of $\sigma(t)$ be zero, i.e. that the overall "charge" is neutral. The path integrals then give

$$K(x_2, t; x_1, 0) = \frac{1}{\sqrt{\det D}} \sum_{n=0}^{\infty} \left(\frac{ig}{2}\right)^{2n} \int_0^t dt_1 \ldots \int_0^t dt_{2n} \sum_{\{e_i\}} \exp \left(\tfrac{1}{2} \int_0^t dt' \int_0^{t'} dt'' \right.$$
$$\left. [iF(t') + \sigma(t')] D^{-1}(t'' - t')[iF(t'') + \sigma(t'')] \right) \tag{14}$$

where $D^{-1}(t)$ is the inverse Fourier transform of $D^{-1}(\omega) = (m\omega^2 + \eta|\omega|)^{-1}$.

The argument of the exponential in (14) looks like the action for a particle of coordinate $\sigma(t)$ interacting with the potential described by D. Using the expression for K as a generating function we can get the correlator $\langle q(t)q(0)\rangle$ to second order:

$$\begin{aligned} \langle q(t)q(0)\rangle &= K^{-1} \left. \frac{\delta^2 K}{\delta F(t) \delta F(0)} \right|_F = 0 \\ &= D(t) - \int_0^t dt' \int_0^t dt'' D(t - t') \langle \sigma(t)\sigma(t')\rangle D(t') \end{aligned} \tag{15}$$

where $\langle \sigma(\tau)\sigma(0)\rangle$ represents the correlation function for a pair of charges moving in the potential described by $D(t)$. Taking Fourier transforms, and using the expression (2) we have

$$\mu(\omega) = \tfrac{1}{\eta}(1 - D(\omega)S(\omega)) \tag{16}$$

where $S(\omega)$ is the Fourier transform of $\langle \sigma(\tau)\sigma(0)\rangle$.

This alone is not particularly enlightening, except to show that indeed the effect of the cosine potential is to reduce the mobility below the classical limit of $1/\eta$. The result above was obtained by keeping the lowest order in g^2; this is the "weak corrugation" limit.

5 Duality

It is interesting to examine briefly the opposite limit of approximation, where the cosine potential is taken to be very strong relative to the dissipation. Specifically, if $s \equiv 8(Mg) >> 1$ it can then be shown using instanton arguments that the mobility has the form

$$\mu(\omega) = \eta\Delta(\omega)\Sigma(\omega) \tag{17}$$

where $\Delta(\omega)$ and $\Sigma(\omega)$ are formally almost identical to $D(\omega)$ and $S(\omega)$ respectively except with the replacements $\eta \rightarrow 1/4\pi^2\eta$ and $Mg/\eta \rightarrow \sqrt{8s/\pi}e^{-s}$. The behaviour of $1 - \mu(\omega)$ in one regime looks much like $\mu(\omega)$ in the other regime [4].

Renormalization group arguments can be applied to the model to suggest that there are two phases with zero frequency mobilities $\mu = \eta$ and $\mu = 1$, and that a cross over occurs at $\eta = 1/2\pi$ in both of the limits studied. As usual, the behaviour in the intermediate regime cannot easily be studied.

References

[1] M.P.A. Fisher and W. Zwerger, "Quantum Brownian motion in a periodic potential," Phys.Rev. B32, 4190 (1985).

[2] A. O. Caldera and A. J. Leggett, "Path Integral Approach to Quantum Brownian Motion," Physica **121A**, 587 (1983).

[3] A. Schmid, "Diffusion and Localization in a Dissipative Quantum System" Phys. Rev. Lett. **51**, 1506 (1983).

[4] F. Guinea, V. Hakim and A. Muramatsu, "Diffusion and localization of a particle in a periodic potential coupled to a dissipative environment," Phys.Lett. 54, 263 (1985).