Landau's theory of Fermi liquids

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Abstract

In this paper we are going to briefly review the Fermi liquid theory, which was firstly introduced as a generalization of Fermi gas theory and to explain the behaviour of ^{3}He. Afterward, we are going through its application for a weakly-interacting metal. At the end we briefly increase the horizon by discussing breakdowns of the theory, and introducing non-Fermi liquid theories, as theories for strange metals.

Introduction

Historically speaking, The Fermi liquid theory was developed by Landau to explain the properties of liquid ^{3}He [1], but soon it was expanded to a theory explaining the behaviour of normal metals with a weak electron-electron interaction. The theory is a natural expansion of a non-interacting system of fermions, usually called a Fermi-gas, to a system in which we gradually introduce an interaction, namely, a Fermi liquid.

The study of ^{3}He has become popular, mostly because of its large fusion energy, making it a possibility for a future energy source [2], its strange behaviour at low temperatures, such as superfluidity [3], and the fact that this substance stays as a liquid, even at the lowest temperature, under standard pressure. The last mentioned characteristic of ^{3}He makes this substance very special since, at its condensation temperature which is about $3K$ [3], where we have already entered the quantum regime, we are dealing with a liquid, unlike the more common case of a solid. This will give rise to the phenomena of a quantum liquid.

^{3}He contains two neutrons, one proton, and two electrons, making the whole atom a fermion. At high enough temperatures in the gas state of ^{3}He, we would expect a negligible interaction between these fermionic atoms, hence making the whole system a Fermi gas. We can slowly introduce the interaction between the fermions, and following the adiabatic theorem, the ground state of the Fermi gas would transform into the ground state of the interacting system. In this process, the spin, charge and momentum of the fermions, remain unchanged, while their dynamical properties, such as their mass, magnetic moment etc. are renormalized to a new set of values [4].

This was the basic idea of Landau behind the theory of Fermi liquids. Landau introduced a quasiparticle excitations in the new interacting theory, corresponding to the Fermi particles of the Fermi gas. This idea makes the liquid theory qualitatively similar to the Fermi gas theory, which was something that was observed experimentally for the liquid state of ^{3}He.

This theory, even though it was initially aimed to explain the similarities and differences seen between the liquid phase of ^{3}He and the non-interacting gas, it was soon after realized that it could be a good model for conduction electrons inside a normal metal, since they can also be seen as weakly interacting fermions. This theory is usually preferred over some of the basic and famous single-electron theories such as Hartree-Fock approximation, since it is not trying to explain the interaction using Slater determinants.

In the following section we will go through this theory more profoundly. Evidently there are cases in which this Fermi liquid description of electrons breakdown[1], due to strongly-interacting electrons. Some of these materials can be explained by non-Fermi liquid theories, which are usually called a "strange metals," as opposed to a normal metals. This is an ongoing area of research in many body physics.

Free electron Fermi gas

We start with the foundation on which the Fermi liquid theory of Landau builds up, a gas of N free electrons which are confined in a cubic box with volume $V = L^3$. Free means that the electrons do neither interact with an external potential $\mathcal{V}$ nor with each other. The finite box allows us to use periodic boundary conditions, but of course one could make the box infinitely large in the end by applying the thermodynamic limit. This allows us to write the wave function of the whole system as a Slater determinant

$$\Psi(\vec{r}_1,\sigma_1;\ldots;\vec{r_N},\sigma_N) = \frac{1}{\sqrt{N!}}\begin{vmatrix} \phi_1(\vec{r}_1,\sigma_1) & \ldots & \phi_N(\vec{r}_1,\sigma_1) \\ \vdots & \ddots & \vdots \\ \phi_1(\vec{r}_N,\sigma_N) & \ldots & \phi_N(\vec{r}_N,\sigma_N) \end{vmatrix}, \tag{1}$$

[1] Materials like $Y_{1-x}U_xPd_3$

where σ_i denotes the spin of the i'th particle, $\vec{r}_i$ the position of the i'th particle and $\phi_i(\vec{r}_j, \sigma_j)$ the plane wave of the i'th electron at $\vec{r}_j$ with spin σ_j $(i, j = 1, \ldots, N)$. This wave function contains the Pauli principle, since it vanishes identically if two particles are at the same position and have equal spin. Each one particle wave function is a plane wave of the form

$$\phi_{\vec{p}}(\vec{r}, \sigma) = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\vec{p}\vec{r}} \tag{2}$$

with momentum $\vec{p}$. By imposing the periodic boundary conditions in x, y, z-direction we find the ground state, which is called "Fermi sea". This means that all single particle states are filled up to the Fermi momentum p_F. In momentum space one can imagine this as a sphere with radius p_F, where all states within the sphere are occupied and all the others are empty. The energy is then given by

$$E(\tilde{n}_{\vec{p}\sigma}) = \sum_{\vec{p},\sigma} \frac{p^2}{2m} \tilde{n}_{\vec{p}\sigma}, \tag{3}$$

in which we introduced the distribution function $\tilde{n}_{\vec{p}\sigma}$, which is equal to 1 for $p \leq p_F$ and 0 otherwise. The energy at the Fermi surface is thus $\epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m}$. Furthermore, we can compute the number of particles by summing over all $p \leq p_F$, i.e.

$$N = 2 \sum_{p \leq p_F} 1 = 2 \frac{4\pi p_F^3/3}{(2\pi\hbar/L)^3} = \frac{V}{3\pi^2} \Big(\frac{2m\epsilon_F}{\hbar^2} \Big)^{\frac{3}{2}}, \tag{4}$$

where the factor of 2 accounts for the two possible spin states. This allows us to calculate the density of states

$$D(\epsilon) = \frac{dN}{d\epsilon} = \frac{V}{2\pi^2} \Big(\frac{2m}{\hbar^2} \Big)^{\frac{3}{2}} \sqrt{\epsilon} = \frac{mp}{\pi^2 \hbar^3}. \tag{5}$$

We obtain the specific heat (per volume) at low temperatures by first calculating the energy and then using

$$C = \left. \frac{\partial E}{\partial T} \right|_V \tag{6}$$

which eventually yields

$$C = \frac{\pi^2}{3} D(\epsilon_F) k_B^2 T = \frac{m p_F}{3\hbar^3} k_B^2 T, \tag{7}$$

where k_B is Boltzmann's constant. For a more detailed calculation please refer to [5].

Normal Fermi liquid

In the following we want to make the transition from the ideal Fermi gas to the real Fermi liquid.

Introduction of quasiparticles

Now we want to make the transition from the ideal free electron gas to the real Fermi liquid. The idea is to make a one-to-one correspondence between the eigenstates of the ideal free electron gas and the real Fermi liquid[2]. We assume that the interaction is turned on in an adiabatic way, i.e. infinitely slow. The eigenstates of the ideal system will progressively transform into certain eigenstates of the real system. The eigenstates of the real system can be written as

$$|\Psi\rangle = |\Psi_0\rangle + \lambda |\Psi_1\rangle, \tag{8}$$

where $|\Psi_0\rangle$ is an eigenstate of the ideal gas, $|\Psi_1\rangle$ is a correction to the ideal eigenstate and λ is a parameter characterizing interactions, i.e. $\lambda = 0$ when we have no interactions and $\lambda \ll 1$ when we turned on a small interaction. In order to generate all real eigenstates, we must assume that the real ground state is generated by some eigenstate of the ideal system, which is described by the distribution $\tilde{n}_{\vec{p}\sigma}$. This statement defines a normal fermion system [7]. We imagine that once the interaction is fully turned on each ^{3}He atom has collected a cloud of other atoms around itself, through this interaction. We define the atom surrounded by its cloud as a quasiparticle [7, 3].
Actually, the adiabatic switching method is somewhat questionable, because when quasiparticle collisions excite the real system, the state describing this situation will decay exponentially over time, according to Fermi's golden rule. In that case, the time over which the interaction is turned on is greater than the lifetime of the excited state, this state has decayed before we reach it. Hence, the switching procedure is no longer reversible. Conversely, if the interaction is turned on too quickly, the process is no longer adiabatic and we would perturb the system. This problem does obviously not arise, if the state under consideration is the ground state, because it is stable and can therefore be precisely defined. The only region where this approach is valid is near the Fermi surface, because quasiparticle lifetimes become sufficiently large there. This can be seen by the $\propto \sqrt{\epsilon}$ behaviour of the density of states in Fermi's golden rule. That region can be imagined as a transitional zone, where quasiparticles get excited above the Fermi level, due to collisions. At very low temperatures, the excited states are very close to the Fermi momentum, which means that damping is negligible. Thus, we conclude that our approach is physically meaningful.

Energy of the quasiparticles

The total energy of the system is not the sum of the single particle energies anymore, since we have interactions. In this case the energy is a functional of the

[2] For a more sophisticated approach with Green's functions c.f. [6]

particle distribution $E[n_{\vec{p}\sigma}]$, what is a direct conclusion of the Hohenberg-Kohn theorem [8]. Remember a functional is an object which takes a function as an input and returns a number. Therefore, in order to get the total energy of the system, we need to know the particle distribution. If $n_{\vec{p}\sigma}$ is sufficiently close to the ground state distribution $n^0_{\vec{p}\sigma}$, we can Taylor expand the energy functional

$$E[n_{\vec{p}\sigma}] = E[n^0_{\vec{p}\sigma}] + \sum_{\vec{p}\sigma} \epsilon(\vec{p})\delta n_{\vec{p}\sigma} + \mathcal{O}(\delta n^2). \quad (9)$$

In the first order $\epsilon(\vec{p}) = \frac{\delta E[n_{\vec{p}\sigma}]}{\delta n_{\vec{p}\sigma}}$ denotes the functional derivative of the total energy with respect to the distribution function and $\delta n_{\vec{p}\sigma} = n_{\vec{p}\sigma} - n^0_{\vec{p}\sigma}$ is the deviation of $n_{\vec{p}\sigma}$ from the ground state distribution function $n^0_{\vec{p}\sigma}$. We can imagine $\epsilon(\vec{p})$ as the change of energy of the system when we add a single quasiparticle with momentum $\vec{p}$ to the system. Note that we actually should expand the expression to second order, in order to account for interactions between the particles, but for simplicity we neglect it for now and will include it in the next section.
If our distribution is close to a step function $\theta(x)$, i.e. close to a Fermi-Dirac distribution at $T = 0$, we can replace $n_{\vec{p}\sigma}$ in $\epsilon(\vec{p})$ by $\theta(p_F - |\vec{p}|)$. We can therefore approximate $\epsilon(\vec{p})$ in the vicinity of the Fermi surface in powers of $p - p_F$ [6]. This procedure yields

$$\epsilon(\vec{p}) - \epsilon(\vec{p_F}) \approx v_F(p - p_F) = \frac{p_F}{m^*}(p - p_F), \quad (10)$$

whereby v_F is the velocity of the quasiparticle on the Fermi surface. Furthermore, in analogy to the ideal gas we defined the effective mass $m^* = \frac{p_F}{v_F}$ of the quasiparticle. The effective mass thus determines the density of state and the specific heat of the real Fermi liquid. The expressions look the same as in the free Fermi gas, the only thing we have to change is replacing the mass m by the effective mass m^*, which gives us

$$D(\epsilon_F) = \frac{m^* p_F}{\pi^2 \hbar^3} \quad (11)$$

and

$$C = \frac{m^* p_F}{3\hbar^3} k_B^2 T. \quad (12)$$

Thus, we also see the the linear behaviour of the heat capacity at very low temperatures, but with a different slope than free electron gas.

Quasiparticle interaction

Another important part of the theory is an effective interaction between the quasiparticles. For simplicity we will only consider spin independent interactions in the following. A generalization to the spin dependent case can be found in [6] or [7]. In the previous section we neglected the interaction, but in order for the theory to be physically meaningful, i.e. satisfying Galilean invariance, we must include interactions [5]. Including the interaction part in the expansion, yields

$$E[n_{\vec{p}\sigma}] = E[n^0_{\vec{p}\sigma}] + \sum_{\vec{p}\sigma} \epsilon(\vec{p})\delta n_{\vec{p}\sigma} + \frac{1}{2} \sum_{\vec{p}\vec{p}';\sigma\sigma'} f(\vec{p},\vec{p}')\delta n_{\vec{p}\sigma}\delta n_{\vec{p}'\sigma'} + \mathcal{O}(\delta n^3), \quad (13)$$

where $f(\vec{p},\vec{p}')$ is called the interaction function of the quasiparticles and describes the interaction of two particles with momentum $\vec{p}$ and $\vec{p}'$. In the case of a Fermi gas we have $f(\vec{p},\vec{p}') = 0$. By differentiating (13) with respect to $n_{\vec{p}\sigma}$ we find

$$\tilde{\epsilon}(\vec{p}) = \epsilon(\vec{p}) + \sum_{\vec{p}';\sigma'} f(\vec{p},\vec{p}')\delta n_{\vec{p}',\sigma'} \quad (14)$$

what is called the "true" quasiparticle energy [3]. This allows us to rewrite (13), such that

$$E[n_{\vec{p}\sigma}] = E[n^0_{\vec{p}\sigma}] + \sum_{\vec{p}';\sigma'} \tilde{\epsilon}(\vec{p})\delta n_{\vec{p},\sigma} \quad (15)$$

and expanding this in the vicinity of the Fermi surface gives us

$$\epsilon(\vec{p}) - \epsilon(\vec{p_F}) = \frac{p_F}{m^*}(p - p_F) + \sum_{\vec{p};\sigma} f(\vec{p},\vec{p}')\delta n_{\vec{p}',\sigma'}. \quad (16)$$

It is conventional to define the dimensionless quantity

$$F(\vec{p},\vec{p}') \equiv \frac{dn}{d\epsilon} f(\vec{p},\vec{p}') = \sum_l F_l P_l(\cos\theta), \quad (17)$$

where $n = \frac{N}{V}$, $P_l(\cos\theta)$ are the Legendre polynomials and θ is the angle between the momenta $\vec{p}$ and $\vec{p}'$. This procedure is justified, because $\vec{p}$ and $\vec{p}'$ are momenta close to the Fermi surface. Hence, the system is invariant under rotations in momentum space and we may expand $F(\vec{p},\vec{p}')$ in Legendre polynomials. Thus, the two particle interaction is completely determined by the F_l, which are sometimes called Landau parameters. Now we want to find a relation between the real and the effective mass. For that we make a Galileo boost to a system I', which moves with respect to the original system I with velocity $\vec{u}$. Therefore, the distribution function becomes

$$n_{\vec{p}\sigma} \to n_{\vec{p}-m\vec{u},\sigma} = \theta(p_F - |\vec{p} - m\vec{u}|). \quad (18)$$

By using (18) while equating $\epsilon(\vec{p}) - \epsilon(\vec{p_F}) = \frac{p_F}{m}(p - p_F)$ and (16), we get

$$\frac{p_F}{m} m\vec{u}\hat{e}_{\vec{p}} = \frac{p_F}{m^*} m\vec{u}\hat{e}_{\vec{p}} + \sum_{\vec{p}\sigma} f(\vec{p},\vec{p}')(n_{\vec{p}-m\vec{u},\sigma} - n^0_{\vec{p}\sigma}), \quad (19)$$

where $\hat{e}_{\vec{p}}$ is the unit vector in $\vec{p}$ direction. Combining this result with (17) finally yields

$$\frac{m^*}{m} = 1 + \frac{F_1}{3}. \quad (20)$$

The first two coefficients F_0 and F_1 can be determined in experiments and we are able to calculate the effective mass. Another quantity which can be determined with these coefficients is the sound velocity

$$c = v_F \sqrt{\frac{1}{3}(1+F_0)(1+F_1)}. \tag{21}$$

In order to determine other quantities, such as the magnetic susceptibility, one has to take spin into account. This will introduce another set of coefficients, which can also be determined experimentally [5, 3].

For ^{3}He, the effective mass ratio and the Landau parameters are [9]:

	m^*/m	F_0	F_1
melting pressure[3]	6.2	94.0	15.7
zero pressure	3.0	10.1	6.0

Specific Heat

It is worth noting that even with quasiparticle interaction turned on, the specific heat would still vary linearly at low temperatures. Furthermore, the slope will not change from the first order calculations.

Since there is one-to-one mapping between non-interacting quasiparticle and interacting quasiparticle situations, equilibrium distribution still remains the Fermi distribution

$$\delta n_{\vec{p}\sigma}(T) = \frac{1}{e^{\beta(\tilde{\epsilon}(\vec{p})-\mu)}+1}$$

Thus, interaction term is $O(T^4)$ in energy and that will not affect heat capacity at second order [9]. See (6).

Predicted scaling of heat capacity was observed experimentally, see Figure 1.

Conclusion

We have extended the Fermi gas theory by adding weak interactions. By the virtue of adiabatic theorem, these can be though to lift the ground state of the Fermi gas to Fermi liquid, where we have a one-to-one mapping between the particles in both models. Due to the interactions, energy now depends on the relative position of the particles. To the first order, with no interactions between quasiparticles, this simply changes mass of the particles by the effective mass $m^* = \frac{p_F}{v_F}$. We then added interactions (spin independent) by considering second order terms in the energy functional. This allowed us to express the energy of the system in terms of Legendre polynomials with some free parameters which can be determined experimentally.

[3] 29 atm at 0.3 K

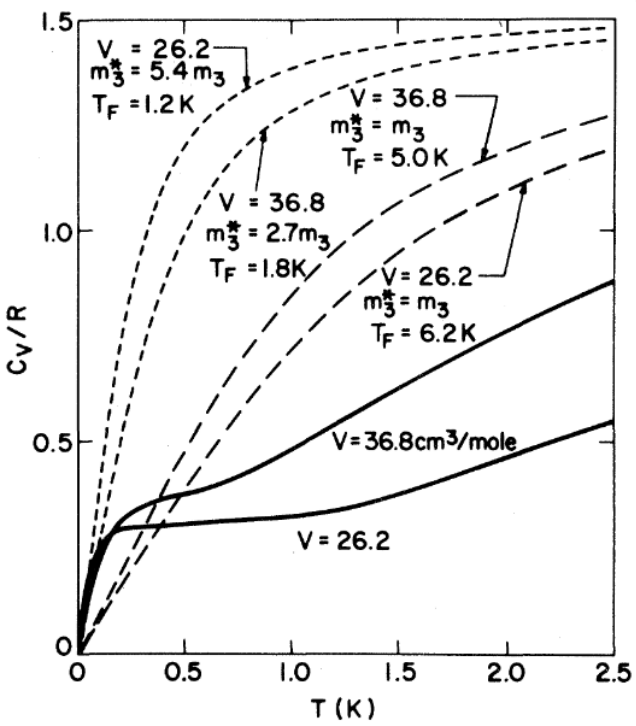

Figure 1: Specific heat of ^{3}He at two different densities [10]. At low temperatures heat capacity grows linearly.

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