An Introduction to the Quantum Hall Effect

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In this paper, the Hall Effect, Integer Quantum Hall (IQH) Effect, and Fractional Quantum Hall (FQH) Effect are discussed. The Hall Effect is explained in terms of the Drude Model of metals and IQH states through a band structure of non-interacting electrons. We introduce interactions between our electrons to explain FQH states of odd-integer fillings.

I. INTRODUCTION

The Hall Effect is a resistance in the transverse direction of the current in a 2D conductor that emerges when a magnetic field is applied perpendicular to the conductor. The Integer Quantum Hall (IQH) Effect shows that this resistivity is quantized to extraordinary precision in integer multiples of fundamental flux when the conductor is brought to very low temperatures. A satisfactory model of IQH states emerges from a non-interacting model of the electrons in the conductor and a band structure of these states can be found. The Fractional Quantum Hall (FQH) Effect, or the emergence of Hall Resistances quantized in fractional multiples of flux quantum, requires us to treat the interactions between electrons, which is tractable in this paper for a few simple cases.

II. HALL EFFECT

In a 2D conductor carrying current with density $\vec{j} = j\vec{x}$, we define the $\hat{x}$ direction as the direction of the electromotive force and $\hat{y}$ transverse to this.

An experiment conducted by Edwin Hall to determine whether a magnetic field acts on the whole of such a conductor or just on the current itself (this was 20 years prior to the discovery of the electron) gave rise to what we now call the Hall Effect. Hall supposed that a magnetic field perpendicular to a conductor would cause current to collect on the edges of the surface, leading to a transverse field E_y emerging [1]. This field after a large time has passed negates the transverse current emerging from the magnetic field and produces a transverse resistivity which depends on the strength of the field applied. This resistance is well motivated by the Drude theory of metals and well documented by experiment (following discussion based on [2]).

If we apply a magnetic field in the $\hat{z}$ direction, electrons undergo a force (working in units where c=1 and e is positive):

$$\vec{f}_B = -\frac{e}{m}(\vec{p} \times \vec{B}) \tag{1}$$

in addition to the electromotive force $\vec{f}_E = -e\vec{E} = -eE\hat{x}$, that drives the current. The first order Drude Model relation for electron momentum gives:

$$\frac{d}{dt}\vec{p} = \vec{f}_E + \vec{f}_B - \frac{m}{\rho_o ne^2}\vec{p} \tag{2}$$

with ρ_o being the resistivity of the material in the absence of magnetic fields and n the density of charge carriers. We desire a steady state solution (after enough time has passed, momentum becomes constant), which leads to the following set of coupled equations:

$$0 = -eE_x - \frac{e}{m}Bp_y - \frac{\rho_o ne^2}{m}p_x$$

$$0 = -eE_y + \frac{e}{m}Bp_x - \frac{\rho_o ne^2}{m}p_y$$

We make the substitution $\vec{j} = -\frac{ne}{m}\vec{p}$ and collect our terms in a matrix

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \rho_\text{o} & \frac{B}{ne} \\ -\frac{B}{ne} & \rho_\text{o} \end{bmatrix} \begin{bmatrix} j_x \\ j_y \end{bmatrix} \tag{3}$$

We can define then the resistivity tensor:

$$\begin{bmatrix} \rho_{xx} & \rho_{yx} \\ \rho_{xy} & \rho_{yy} \end{bmatrix} = \begin{bmatrix} \rho_\text{o} & \frac{B}{ne} \\ -\frac{B}{ne} & \rho_\text{o} \end{bmatrix} \tag{4}$$

Note in particular that $\rho_{xy} = -\rho_{yx}$ (ρ_{xy} is called the Hall resistivity). One reads easily that:

$$\rho_{xy} = \frac{-B}{ne} = R_H B \tag{5}$$

Where $R_H = \frac{-1}{ne}$, commonly known as the Hall Coefficient, is a function only of the density of charge carriers in the material. As B can be fixed and ρ_{xy} may be measured to high precision, whether values of R_H measured in this manner agree with theoretical expectations is an excellent test of our understanding of metals.

One may see in Table 1 that the expected value of R_H coincides relatively well with experimental data for elements with a small number of valence electrons, but the relation given by Equation 5 breaks down for elements with a high density of charge carriers. This is somewhat expected, as the Drude Model no longer applies in this limit.

Largely however, one sees that in a metal the Drude

Element	R_H^{free}	R_H^{calc}
Li	−13.2	−12.8±0.1
Na	−24.5	−24.6±0.1
K	−44.6	−44.8±0.1
Rb	−54.7	−54.2±0.1
Cs	−68.6	−53.4±0.2
Ca	−54	−60±10
Sr	−70	?
Ba	−78	−110±20
Cu	−7.3	−5.2±0.2
Ag	−10.4	−8.5±0.2
Au	−10.5	−8.1±0.2
Al	−3.4	−1.7±0.3
Pb	−4.7	−2.4±0.3
Rh	−7.5	11±3
Ir	−8.8	5±2
Pd	−9.1	−17±3
Pt	−9.4	?

TABLE 1: Metals that form a cubic lattice have had their Hall Coefficients measured to high precision. Shown here are the values for different groups of the periodic table. Column 1 shows expected R_H in Drude Model, Column 2 the measured value (units of R_H in 10^{-11} m^3C^{-1} and $c = 1$) with a low applied field at room temperature. Note that Group 1 Metals, for which the Drude Model applies well at room temperature, have a very good experimental agreement with theory [3].

Model approximates decently, applying a magnetic field perpendicular to the conducting surface produces a transverse resistivity linear to the field strength.

III. INTEGER QUANTUM HALL EFFECT

This linearity breaks down in the very low temperature regime, giving way to the IQH Effect.

In 1980, a group of researchers lead by Klaus von Klitzing published their measurements of the Hall resistivity of an oxidized silicon transistor at very low temperatures (≈ 1 K, achieved using liquid Helium) and strong magnetic fields (>10 Tesla). They found that in this regime, the Hall resistivity (the original paper expressed it as the Hall conductivity, but we continue with our scheme from section II) is quantized according to:

$$\rho_{xy} = -\frac{2\pi\hbar}{e^2}\frac{1}{n} = -\frac{\Phi_o}{en} \tag{6}$$

where $n \in \mathbb{Z}$ and we define a fundamental quanta of magnetic flux Φ_o [4]. As one may observe in Figure 1, increasing the magnetic field strength still leads to higher Hall resistivity, but the increases are not smoothly linear and take on a discrete nature.

This quantization can be explained through an analyisis of the magnetic field's interaction with the conductor when we ignore electron-electron interactions (plausible in the low temperature limit) as we demonstrate in this section.

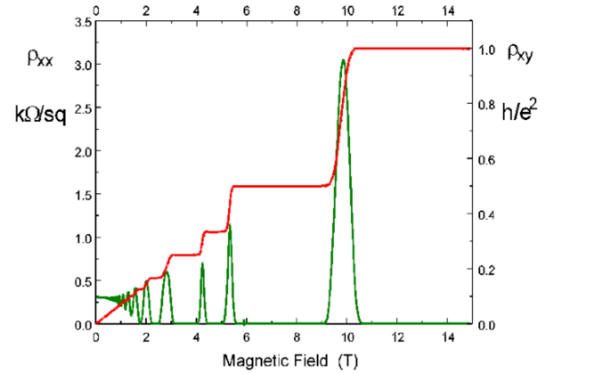

Figure 1: Response of ρ_{xy} (red) and ρ_{xx} (green) to increasing magnetic field in a GaAs material, sub-Helium temperatures [5]

A. Landau Levels and IQH Wavefunctions

If we ignore electron-electron interactions, we need only one electron wavefunction to solve the system (following discussion based on [6] and [7]). The Hamiltonian of a charged particle in a magnetic field with c=1 is known:

$$H = \frac{1}{2m}(\vec{p} + e\vec{A})^2 + U(\vec{r}) \tag{7}$$

We include some potential $U(\vec{r})$ since we know that near the edges of the material, some large potential must exist to confine our electrons [8]. We deceive ourselves for now and assume we are far from the edges of our material, neglecting $U(\vec{r})$ for now.

We are free to fix the gauge of our field so long as $\nabla \times \vec{A} = B\hat{z}$. We choose the Landau Gauge:

$$\vec{A} = Bx\hat{y} \tag{8}$$

This leads to the Hamiltonian:

$$H = \frac{1}{2m}p_x^2 + \frac{1}{2m}(p_y + eBx)^2 \tag{9}$$

We see that while $[p_x, H] \neq 0$, $[p_y, H] = 0$, motivating the guess for our wave function:

$$|\psi_{k_y}\rangle = e^{ik_y y} f(x) \tag{10}$$

Thus, $H|\psi_{k_y}\rangle = E|\psi_{k_y}\rangle$ depends only on k_y and x and in particular can be rearranged to be the Schröedinger Equation for a one dimensional quantum harmonic oscillator:

$$\frac{1}{2m}p_x^2 f(x) + \frac{e^2B^2}{2m}(x + \frac{\hbar}{eB}k_y)^2 f(x) = Ef(x) \tag{11}$$

with the frequency $\omega_L = \frac{eB}{m}$ centered around $x_o = -\frac{\hbar}{eB}k_y$. Thus, we can immediately say $f(x) = \phi_n(x-x_0)$, where ϕ_n is the nth excited wavefunction of the harmonic potential with frequency ω_L. Thus, our wavefuntions have the form:

$$\left|\psi_{k_y,n}\right\rangle = e^{ik_y y}\phi_n(x-x_0) \tag{12}$$

and energy spectrum $E_n = \hbar\omega_L(n+\frac{1}{2})$.

Our reader may be wondering why we have only considered the interaction of the magnetic field with electrons in our conductor, and not the electromotive force driving the current in the $\hat{x}$ direction. One sees that this term contributes a $-eEx$ term to the Hamiltonian. Note that p_y still commutes with this, and one can complete the square to create an analogous relation to Equation 11. All that this electromotive force amounts to in our wavefunction is a shift in the value of $x_o = -\frac{\hbar}{eB}k_y - \frac{mE}{eB^2}$. Thus, our wavefunctions take the form of Equation 12 whether this force is considered or not.

This energy spectrum and the associated wavefunctions are known as Landau Levels. Each Landau Level in the absence of an electromotive force may be interpreted as a flat band structure, meaning that changing the momentum does not affect the energy value. Thus, each k value is degenerate and the degeneracy of these Landau Levels is of particular importance to understanding the IQH Effect. We discuss this in the next subsection.

B. Degeneracy of E_n

The degeneracy of a Landau Level is determined by how many allowed values of k there are. We may count the number of allowed k values using $g = \sum_{k_{min}}^{k_{max}} 1$ in integral form:

$$g = \frac{1}{\Delta_k}\int_{k_{min}}^{k_{max}} dk \tag{13}$$

Calculating this integral requires us to examine the boundary conditions of the problem that influence k, namely $U(\vec{r})$ from Equation 7. $U(\vec{r})$ must satisfy two conditions (following discussion based on [9])):

1. As electrons must be confined to the material - thus we posit that at the edges of the material, $U(\vec{r})$ must increase rapidly and go towards infinity.

2. In the low temperature limit, internal interactions between electrons are weak. In our model of the IQH effect, we choose to neglect them altogether. Thus, $U(\vec{r})$ must vary slowly in the interior of the metal, else the interactions between electrons becomes too significant to ignore.

Two examples of allowable $U(\vec{r})$ are shown in Figure 2 (one may even take a trigonometric function as [8] do in their analysis). It is an excellent approximation then to take $\Delta_k = \frac{2\pi}{L_y}$, as one may approximate $U(\vec{r})$ as an infinite square well potential of width L_y, for the purposes of calculating the degeneracy, given our conditions on $U(\vec{r})$.

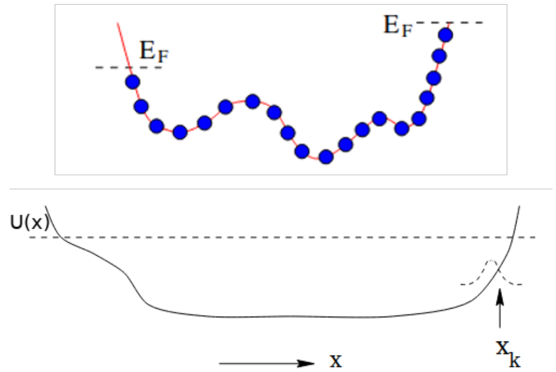

Figure 2: Top: An exaggerated example that $U(\vec{r})$ can vary between the end points, so long as its variations are small compared with the dimensions of the conductor. The difference in E_F on the left and right stems from an electromotive force driving current [7]. Bottom: A smoothly varying $U(\vec{r})$ and a $|\psi\rangle$ worked out in III.a shown near the edge - it is clear that a different wavefunction emerges on the edge states [9].

We define the dimensions of our conductor as having transverse length L_y and longitudinal length L_x. We discuss $\left|\psi_{k_y,n}\right\rangle$ in absence of an electromotive force for the moment, as it only adds more to the algebra while revealing no new physics.

We choose our origin $x = 0$ on the right side of the material and $x = -L_x$ on the left side. Note that $\left|\psi_{k_y,n}\right\rangle$ is highly localized in x, as it falls off exponentially away from $x_o = -\frac{\hbar}{eB}k_y$. Thus, any $\left|\psi_{k_y,n}\right\rangle$ with an x_o not on the interval $[-L_x, 0]$ has almost no probability to be in the conductor. This naturally defines $k_{min} = 0$ and $k_{max} = \frac{eB}{\hbar}L_x$ as the range of allowed k_y values.

We thus read off the degeneracy of one Landau Level from Equation 13:

$$g = \frac{L_y L_x eB}{2\pi\hbar} = \frac{L_y L_x B}{\Phi_o} \tag{14}$$

This result claims that the degeneracy of a Landau Level depends only on the external field applied and the area of the conductor. Understanding this result is key to our coming derivation of Equation 6.

C. The Quantization of IQH Flux from Landau Levels

On inspection, there are features of $\left|\psi_{k_y,n}\right\rangle$ and its energy spectrum that indicate a ρ_{xy} response to B like that shown in Figure 1. Firstly, there are large gaps in the band spectrum on the order of ω_L which would account for the spikes shown in Figure 1. Secondly, there are many allowed states at each energy level (one for each allowed k_y), which would account for the "plateaus" in ρ_{xy}. Intuition aside, we now show explicitly that this resistivity is quantized as per Equation 6 from our wavefunctions in Equation 12 (following discussion based on [7]).

To discuss the resistivity of this state, we must discuss the current density:

$$\vec{J} = I/A = -\frac{e}{L_x L_y}\sum_{n=1}^{\nu}\sum_{k_y}\left\langle\psi_{k_y,n}\right|\frac{d}{dt}\vec{x}\left|\psi_{k_y,n}\right\rangle \quad (15)$$

where we assume the first ν Landau Levels are occupied. Making the canonical substitution for velocity:

$$\vec{J} = -\frac{e}{mL_x L_y}\sum_{n=1}^{\nu}\sum_{k_y}\left\langle\psi_{k_y,n}\right|\vec{p}+e\vec{A}\left|\psi_{k_y,n}\right\rangle \quad (16)$$

We are interested in the transverse current density, or the y-component of $\vec{J}$. Using our Landau Gauge once more:

$$J_y = -\frac{e}{mL_x L_y}\sum_{n=1}^{\nu}\sum_{k_y}\left\langle\psi_{k_y,n}\right|\hbar k_y + eBx\left|\psi_{k_y,n}\right\rangle \quad (17)$$

We know that the expectation value of x for $\left|\psi_{k_y,n}\right\rangle$ must be x_o:

$$eB\left\langle\psi_{k_y,n}\right|x\left|\psi_{k_y,n}\right\rangle = eBx_o = eB\left(\frac{-\hbar}{eB}k_y - \frac{mE_x}{eB^2}\right) \quad (18)$$

$$eB\left\langle\psi_{k_y,n}\right|x\left|\psi_{k_y,n}\right\rangle = -\hbar k_y - \frac{mE_x}{B} \quad (19)$$

Plugging Equation into Equation 17 gives:

$$J_y = e\sum_{n=1}^{\nu}\sum_{k_y}\frac{E_x}{B} \quad (20)$$

There is no dependence left in our sum over n or k_y, so the result is:

$$J_y = \frac{e}{L_x L_y}\nu g\frac{E}{B} = eE_x\nu\frac{1}{\Phi_o} \quad (21)$$

We now calculate J_x:

$$J_x = -\frac{e}{mL_x L_y}\sum_{n=1}^{\nu}\sum_{k_y}\left\langle\psi_{k_y,n}\right|p_x\left|\psi_{k_y,n}\right\rangle \quad (22)$$

Though $\left\langle\psi_{k_y,n}\right|p_x\left|\psi_{k_y,n}\right\rangle$ vanishes for all n. Thus, collecting our terms in a matrix:

$$\begin{bmatrix}E_x\\E_y\end{bmatrix} = \begin{bmatrix}\rho_{xx} & -\rho_{xy}\\ \rho_{xy} & \rho_{yy}\end{bmatrix}\begin{bmatrix}J_x\\J_y\end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\rho_{xx} & -\rho_{xy}\\ \rho_{xy} & \rho_{yy}\end{bmatrix}\begin{bmatrix}0\\ \frac{e\nu}{\Phi_0}\end{bmatrix} \quad (23)$$

We may thus read off immediately from Equation 23. that $\rho_{xy} = -\frac{\Phi_o}{e\nu}$ where ν is an integer. Thus, the non-interacting Hamiltonian given by Equation 7 is able to predict the proper degeneracies, proper energy levels, and even the proper quantized resistivities of IQH states.

IV. FRACTIONAL QUANTUM HALL EFFECT

Just a short two years after the IQH effect was established and explained through Landau Levels, research teams at Bell Laboratories and Princeton discovered the FQH Effect, which had not been predicted or expected by theorists. They measured a Hall resistivity across a semiconductor cooled to ≈ 0.1 K under an applied magnetic field of 30 Tesla of $\rho_{xy} = \frac{-\Phi_o}{e}\frac{1}{1/3}$ [10].

This does not mesh immediately with the scheme we developed to explain IQH states. To make matters worse, this value of $\nu = 1/3$ is not the only extension of our theory we need to make. Since its discovery in 1982, dozens of FQH plateaus have been found, five of which are plotted in Figure 3 [7, 11].

The scheme we have developed for IQH states relies on the absence of interactions between electrons. If this has proved inadequate, the natural next step is to include interactions between electrons. We follow the derivations of [7, 9, 12] that produce the ground state of any FQH state with $\rho_{xy} = \frac{\Phi_o}{e}\frac{1}{1/\nu}$, where ν is an odd integer.

A. Symmetric Gauge and Single Particle Wavefunctions

In order to incorporate Coulomb Interactions, a change in gauge will make calculations easier. We use the same symmetric gauge as Laughlin does in his original derivation: $\vec{A} = \frac{B}{2}(x\hat{y} - y\hat{x})$. The kinetic energy operator is the Hamiltonian given by Equation 7 if we neglect $U(\vec{r})$:

$$T = \frac{1}{2m}(\vec{p}+e\vec{A})^2 = \frac{1}{2m}\vec{p}^2 + \frac{e}{m}(\vec{p}\cdot\vec{A}) + \frac{e^2}{2m}\vec{A}^2 \quad (24)$$

Capitalizing on our choice of gauge:

$$T = \frac{1}{2m}(p_x^2+p_y^2)+\frac{Be}{2m}(xp_y-yp_x)+\frac{e^2B^2}{8m}(x^2+y^2) \quad (25)$$

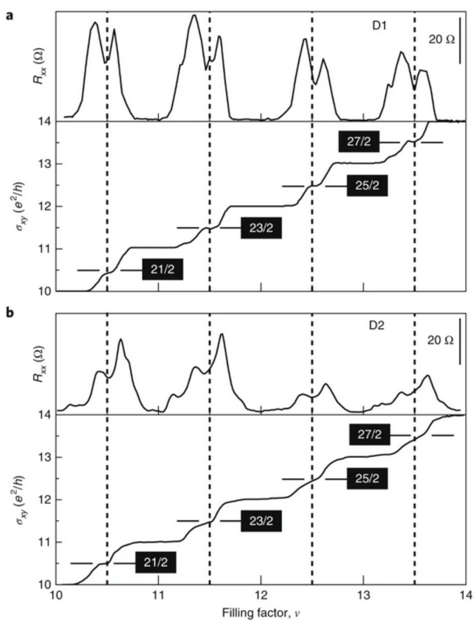

Figure 3: FQH plateaus can be observed in this graphene sample. Note that these are measured in conductivity, the inverse of resistivity (units $e^2/\hbar$). Sample was cooled to 0.30 mK., under field strength of a. 15 T, b. 21.5 T [11]

This first and last term form a two dimensional quantum harmonic oscillator, and the middle term can be written as $\frac{Be}{2m}\hat{L_z}$.

$$T = H_{HO} + \alpha L_z \tag{26}$$

We can thus write our single free-particle wavefunctions as $|\psi_{mn}\rangle$, with $m\hbar$ being the z projection of the angular momentum of the nth excited state of the harmonic potential given by $\omega_L = \frac{eB}{m}$. We are interested in FQH states with low fillings, so we assume n=0 and suppress this index.

It can be shown through some algebra that $|\psi_m\rangle$ takes the form:

$$|\psi_m\rangle \propto z^m \exp\left\{-\frac{|z|^2}{4l^2}\right\} \tag{27}$$

Where $z = x - iy$ and l is a characteristic length of the system $2\pi l^2 = \frac{\Phi_o}{B}$ (electron is relatively localized within a radius $\sqrt{2m}l$, m here the quantum number, not the mass). We see intuitively that the Gaussian corresponds to the HO ground state while z^m gives $|\psi_m\rangle$ the proper angular momentum.

It is worth noting that this is not a "new" wavefunction. If we desired, we could make the proper gauge transformations and return our single particle eigenstates to their form in Equation 12, but the form given in Equation 27 is much preferred for the coming analysis.

B. Laughlin's Trial Function

From our single, free-particle wavefunctions, adding interactions between our electrons gives a grand-ensemble wavefunction:

$$\Psi(z_1, ..., z_n) = f(z_1, ..., z_N) \exp\left\{-\frac{1}{4l^2}\sum_{i=1}^{N}|z_i|^2\right\} \tag{28}$$

where f is antisymmetric under particle exchange as electrons are fermions.

Solving for f analytically or numerically is not possible. However, Laugliln was able to guess a trial function that gives the proper $\rho_{xy} \propto \nu$, ν odd:

$$\psi(z_i) = \exp\left\{-\frac{1}{4l^2}\sum_{i=1}^{N}|z_i|^2\right\}\prod_{i<j}(z_i - z_j)^\nu \tag{29}$$

One sees that this form of $f(z_1, ...z_n)$ is perhaps the simplest guess Laughlin could make that could reasonably approximate interactions. We understand that the interactions are dependent on the distances between particles, and the displacement between each pair of particles appears in Equation 29 exactly once. Furthermore, it is antisymmetric under particle exchange as ν is odd. The power law dependency on ν is the only feature that defies an intuitive explanation. However, this relation can be shown to have to the properties we expect of FQH states.

It has been calculated that Laughlin's trial function very closely matches the true ground state energy for the odd ν case in computational tests at low N (as N becomes large, computational tests become impossible) [12]. We illustrate briefly how it produces the proper Hall resistivity.

Consider z_1 - the leading power of z_1 is approximately νN, which is thus the largest projection of L_z for z_1. In Section IV.A., we discussed that a particle is localized to a radius $r = \sqrt{2L_z}l = \sqrt{2\nu N}l$, meaning it occupies an area $A = \pi r^2 = 2\pi\nu N l^2 = \nu N \frac{\Phi_o}{B}$.

Given the area that it occupies, we can calculate the number of states in this Landau Level from Equation 15:

$$g = \frac{AB}{\Phi_o} = \nu N \tag{30}$$

If we were to rework our integral from section II.B., the only value that should change is the value of Δ_k picking up a scaling by $1/\nu$. Thus, the degeneracy matches a Landau Level of a state with $\rho_{xy} = \frac{\Phi_o}{e}\frac{1}{1/\nu}$.

V. CONCLUSION

The Hall Effect, IQH Effect, and FQH Effect are fascinating examples of how edges can lead to interesting states in condensed matter. It is excellent that our mechanism of band structures does an excellent job of describing IQH states, but they can only get you so far in FQH States. Much of the research done in FQH materials involves topological orders, which are still not fully understood. Thus, while a comprehensive theory of FQH states does not yet exist, that is why it is an exciting field of study.

[1] E. H. Hall, On a new action of the magnet on electric currents, American Journal of Mathematics **2**, 287 (1879).

[2] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Holt, Rinehart and Winston, New York, 1976).

[3] W. W. Schulz, P. B. Allen, and N. Trivedi, Hall coefficient of cubic metals, Phys. Rev. B **45**, 10886 (1992).

[4] K. v. Klitzing, G. Dorda, and M. Pepper, New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance, Phys. Rev. Lett. **45**, 494 (1980).

[5] W. Bao, Electrical and mechanical properties of graphene (2012).

[6] K. Sun, Topological insulators part i: Phenomena, `http://www-personal.umich.edu/~sunkai/teaching/Fall_2013/chapter3.pdf` (2013), lecture notes on topological insulators, UMich grad condensed matter class.

[7] D. Tong, The quantum hall effect, `http://www.damtp.cam.ac.uk/user/tong/qhe/one.pdf` (2016), lecture notes on QHE, Cambridge grad QHE class.

[8] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized hall conductance in a two-dimensional periodic potential, Phys. Rev. Lett. **49**, 405 (1982).

[9] S. M. Girvin, The quantum hall effect: Novel excitations and broken symmetries, `https://arxiv.org/pdf/cond-mat/9907002.pdf` (1998), lectures delivered at Ecole d'Et e Les Houches, July 1998.

[10] P. Ball, Physicists rewarded for 'fractional electrons', Nature **395** (1998).

[11] Y. Kim et. al, Even denominator fractional quantum hall states in higher landau levels of graphene, Nature Physics **15**, 154 (2019).

[12] R. B. Laughlin, Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations, Physical Review Letters **50**, 1395 (1983).