Review of Measurement-Induced Criticality

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The paper presented here today hopes to give the reader some insight into a novel method of error detection in quantum information. Quantum information has been plagued by the process of quantum decoherence which can cause information loss in quantum circuits. As such, a method of protecting this information, or at the very least detecting information loss, is a necessity in quantum computing more so than classical computing. The methodology proposed by researchers from the United States makes use of the fact that measurements can be interspersed in a circuit of random unitaries to introduce two distinct phases of the entanglement entropy[1, 2]. At the critical point between the two phases, there is expected to be an entanglement that gives two regions a large correspondence of information between said regions, known as mutual information. This mutual information provides a duplicate of our encoded information, acting as a error detection should it be needed. We do not seek to make any headway into this field here, but merely present the information in a moderately digestible form for those who are interested.

I. INTRODUCTION

Quantum information is a new and impressively growing field. Just as we have done with classical computers before, the field of quantum information aims to understand how information is transmitted and stored on quantum computers. Due to the nature and direction that quantum computing has been taking, the topic combines a variety of topics from condensed matter, statistical mechanics, and quantum mechanics.

One of the many subtopics of quantum information, or even information in general, investigates how to efficiently "protect" encoded information. In classical computers, a bit can be altered unintentionally, called a soft error, through a number of processes, such as a cosmic ray interactions. As such, modern classical computers have a variety of options for error correction and error detection, such as Hamming codes. This gets more complicated when we look at quantum computers. Simply speaking, the nature of quantum computers, namely the fact that measurements and unitary operators, or unitaries, must be applied, generates a phenomenon known as quantum decoherence. Typically, coherent states are used to encode information and are defined by their minimal uncertainty for all time. However, Quantum decoherence strips coherent states of their necessary definite phase relationship that is required to maintain the encoded information. This puts a strain on quantum information theory and requires us to find a more complex solution to error correction and detection.

Over the years, a number of ways to prevent, slow, or mitigate quantum decoherence have been proposed. For example, researchers at both the University of British Columbia and the University of California, Santa Barbara were able to reduce the environmental decoherence rate by applying a high magnetic field[3]. Recently, the researchers at the University of California, Santa Barbara; the University of California, San Diego; and the University of Massachusetts, Amherst have published papers relating to measurement-induced criticality (or measurement-derived entanglement transition)[1, 2]. More precisely, they found that, in a quantum system consisting of measurements and unitary gates, there is a critical point in the ratio of measurements to unitary gates. At this critical point, it appears that there is a symmetry of mutual information that is shared between small regions, acting as a form of error correcting by preserving this quantum information. This novel method in quantum information is the main investigation of this review.

However, and rather unfortunately, this concept may not be as related to the topics discussed in the class as some of the others might be. Regardless, the discussions of many-body physics and second quantization that occur in a first-year graduate condensed matter course may make the content here more digestible. Condensed matter theory will provide a better preparation for this than a standard quantum mechanics course. In addition, there will, hopefully, be concepts that are a bit more familiar to the reader in between the talk of qubits and entropy.

II. MEASURES AND GROUPS

Before the full discussion can begin, it is pertinent that new terminology be introduced to the reader. Specifically, it is important to talk about the topics needed to set up our circuits and qubits. Let us begin by recalling what a qubit is. A qubit is a quantum bit, which is a bit redundant to say. To be more rigorous about it, a qubit is a two-state system, such as the spin of a spin-$\frac{1}{2}$ particle[4]. The quantum behavior of this system, which makes it fundamentally different from a classical bit, is that while the qubit could be in one state or the other, it can also exist in a coherent superposition of those two states as well. This is what makes quantum computing so unique and desirable.

Granted, this only tells us how the units of data is set up. In order to make proper use of quantum com-

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puting, we need to understand how to manipulate these qubits, and, thus, how to set up circuits. In classical computers, we define logic gates to mediate our manipulation of bits. However, in quantum computers, we use unitary operators, or "quantum logic gates", to mediate our manipulation of qubits. If we were looking at making a quantum algorithm, we would carefully pick and choose our unitary operators, such as the Pauli spin gates or the Hadamard gate, in order to complete the desired algorithm. However, we are not looking at an algorithm, but, instead, looking at a behavior that we suppose would hold for any circuit of a particular size. As such, we consider a random circuit constructed from randomly determined unitaries. While we can consider picking this operators from a list of all possible unitaries, we can instead consider only certain unitaries that have certain properties assigned to them. We will refer to this as a measure or a measure of unitaries. Once we understand how the system behaves with respect to a specific measure of unitaries, we can attempt to generalize it to a more stringent group.

In the following sections, we will look at two measures in particular. The more general of the two is the Haar measure[5, 6]. A rigorous definition of the Haar measure will not be fully discussed here, as there is a lot of mathematics and formalism that needs to be discussed beforehand. For now, we will simply say that the Haar measure is one of the most used measures of unitary groups that are used in quantum information due to its properties as well as its generality.

At the same time, the reasons why the Haar measure may be useful are the same reasons why it may be a hindrance. When we are looking at a large amount of entanglements, which will make use of the Rényi entropy discussed in section IV, or working with a large number of qubits, using the Haar measure can result in challenges with numerical simulation, even with simply modeling the unitaries. As such, we introduce the more stringent Clifford measure[7]. The n-qubit Clifford measure of unitaries, or rather the Clifford unitaries that interact with n-qubits, is the group of unitaries such that the following is true:

$$\hat{U}e^{i\theta\pi/2}\sigma_{j_1}\otimes\cdots\otimes\sigma_{j_n}\hat{U}^{\dagger}=e^{i\theta'\pi/2}\sigma_{j'}\otimes\cdots\otimes\sigma_{j'_n} \quad (1)$$

where $\theta,\theta'\in\{0,1,2,3\}$; $j_n,j'_n\in\{0,1,2,3\}$; and σ_j is the jth Pauli matrix. The less specific Clifford measure should make it easier for people to perform numerical circuit simulations while still maintaining some interesting behavior. After all, as we have learned, it is important to consider the simplest interesting model.

In addition to manipulating our qubits, we will also need a way to read the qubits, which happens through a process known as measuring. Just as we restricted our unitaries above, it is useful for us to restrict the measurements that we perform that have a similar construction to the unitaries that we are making use of. As the Clifford measure is closely connected to the Pauli matrices, it seems natural that we would consider only measurements from the single-qubit Pauli group. Rather, we only make use of spin-$\frac{1}{2}$, single-qubit measurements that can be derived from the Pauli matrices.

III. CIRCUIT

Now that we have established what operators and measures we are working with, we can begin to understand the model that is being investigated. Consider a one-dimensional chain of L qubits, which those at the University of California, Santa Barbara called a prototypical quantum circuit model[1]. The chain has a lattice constant a to mark the distance between neighboring qubits. As the chain evolves with time, we will consider two major factors, namely the evolutions caused by applying unitaries and the local, single-qubit measurements that are applied. Let us look at these individually.

To begin with, we can describe how our unitaries are set up. For this model, we will solely be using two-qubit Clifford gates. These gates take two qubits as an input, apply their unitaries, and output two qubits after the transformation has been completed. Furthermore, the Clifford gates will only be used such that a qubit interacts only with its neighbor qubits. In other words, the two-qubit Clifford gate applied to a qubit at location x is applied in conjunction with the qubit located at either $x-1$ or $x+1$. These Clifford gates are then arranged in what is commonly referred to as a brick-layer pattern. At some time $t=2n\tau$ for $n\in\mathbb{Z}$ and for some time constant τ, every qubit at a position $x=2ma$ for $m\in\mathbb{Z}$ experiences a random Clifford gate with the qubit located at $x=(2m+1)a$. Then, immediately after at time $t=(2n+1)\tau$, the same qubits at positions $x=2ma$ experience a random Clifford gate with the qubit located at $x=(2m-1)a$, in the opposite direction as the previous time step. The result is a circuit of unitaries that are periodic in both space and time, as shown in Fig. 1, and that mimics brickwork, hence the name.

Before we look into the single-qubit measurements, let us define some useful terms when working with these cir-

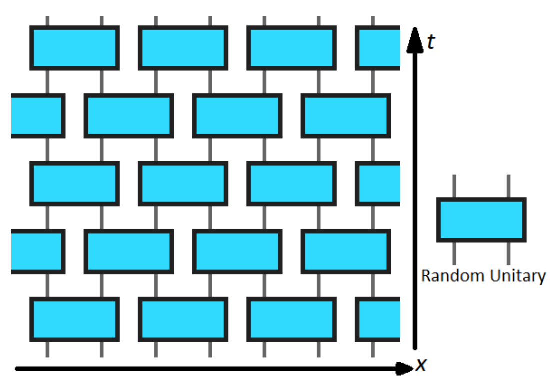

FIG. 1. The random circuit model. This circuit displays the commonly used brick-layer pattern for random unitaries. Notice that the above circuit shows both spatial symmetry, with a lattice constant a, and temporal symmetry, with a time constant 2τ.

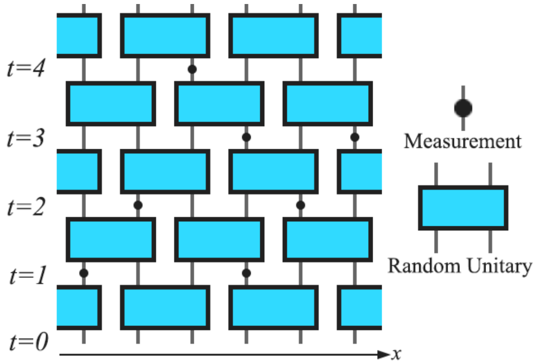

FIG. 2. The random circuit model. This circuit displays the commonly used brick-layer pattern for random unitaries with single-qubit measurements randomly interspersed between the unitaries. The measurements are randomly placed with a Poissonian distribution, where each position and time combination has a probability p. The total number of possible combinations is simply the number of positions multiplied by the depth.

cuits. Particularly, we want to adjust how we discuss the nature of the time evolution. We, thus, introduce the term *depth* which is defined to be the number of unitary layers that have been applied. We will denote this as D. It follows directly from this that a circuit with a depth D has a number of time cycles such that $T = D/2$.

Now, let us move onto the single-qubit measurements. We place these single-qubit measurements in between brick-layer pattern of the random unitaries. As such, for a system of L qubits and of depth D, there are a total of $L \times D$ sites where measurements can be placed. By extension, since the depth D is defined to be the number of unitary layers and each unitary operates on two unique qubits, the total number of unitaries applied to the same system is $D \times L/2$. We do want to consider different ratios of measurements to unitaries. As such, we define the parameter p as the fraction of total possible sites that these single-qubit measurements have been made. These sites can be chosen either deterministically or randomly, depending on what is to be investigated. However, for the purposes here, we will be randomly determining the sites via a Poisson distribution as depicted in Fig. 2. We, then, identify this fraction p as the rate of measurements. We can also see that the ratio of measurements to unitaries is simply $2p$, which is bounded below at 0 and above at 2. By extension, we can investigate only the unitary patterns by setting this rate such that $p = 0$.

For any system where we have specified the initial L qubits and we have defined the positions of both the unitaries and the measurements, the state of the qubits at any depth d is determined if we know the outcomes of the measurements. This time evolution is known as *quantum trajectory.*

IV. PHASES AND CRITICALITY

As we move to understand possible phases and critical behavior, we need to discuss how to understand what to look for. We will be looking specifically for how the entanglement varies as we vary the rate of measurements p. It has been shown in research that these unitary gates increase the entanglement of a system while the use of local measurements reduces the entanglement[8]. More specifically, as the researchers at the University of California, Santa Barbara have described, the increase to the entanglement that the unitary gates supply is merely local, while the decreases given by the local measurements is thought to be non-local (or at least not entirely local). The competition to increase or decrease the entanglement provided by the unitary gates and the measurements produces interesting results, such as the phases a critical behavior we will talk about here.

However, in order to observe such a thing, we need a way to measure entanglement. Thus, we will employ the commonly used Rényi entropy for this exact case[9, 10]. A full description of the Rényi entropy is unfortunately unable to be recreated here without a good deal of effort and time devoted to it. However, we can at least give a brief overview. The Rényi entropy can be found for a specified contiguous subregion A and it is dependent not only on the size L of the system, but also the size $|A|$ of the subregion and the pure state wavefunction that is given through the quantum trajectory. Typically, Rényi entropies are defined with some order n, where n is a non-negative real number with $n \neq 1$, such that:

$$S_A^n = \frac{1}{1-n} \log_2 \left(\mathrm{Tr}_A \left[\left(\mathrm{Tr}_{\bar{A}} \left[|\psi\rangle\langle\psi| \right] \right)^n \right] \right) \tag{2}$$

where $\left(A, \bar{A}\right)$ is some bipartition of our L-qubit system and $|\psi\rangle$ is the resulting quantum trajectory wavefunction. However, Clifford gates have a flat entanglement spectrum[11, 12], meaning that the Rényi entropies are all equal to each other regardless of the order n. Thus, we do not need to worry about this order n and will consider the single Rényi entropy of a subregion A as S_A.

The Rényi entropy S_A serves an additional purpose as well. In addition to being a measure of entanglement, the Rényi entropies can be used to define our measure of mutual information. The mutual information between two subregions A and B can then be written as:

$$I_{A,B} = S_A + S_B - S_{A \cup B} \tag{3}$$

It is this mutual information that we will investigate as it will preserve information through the unique entanglement between two different regions.

Now we can start to get into the more interesting properties of this circuit. To begin with, we are mostly concerned with the long term behavior, or steady state behavior, of the circuits as $T \to \infty$. We expect that the steady state behavior will be dependent on the measurement rate p but will not be dependent on the unitary dynamics at any finite time. To characterize this, we will

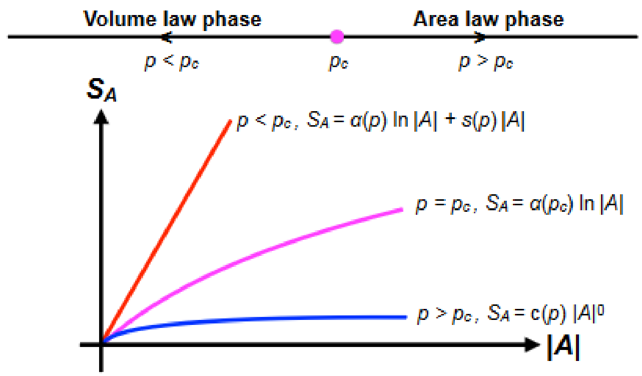

FIG. 3. A graphic description of the resulting phase and criticality behavior created by Li et al.[1]. The above shows the differences in the scaling of the entanglement entropy in both phases and at criticality.

look at the Rényi entropy in the large time limit. For a given circuit of size L and a given subsection of size $|A|$ it has been shown that the Rényi entropy saturates to some value that is dependent on the value of the measurement rate p and independent of the choice of initial state of the system[1]. Because of this behavior, this state is generally referred to as a *bulk property* of the system.

This saturated entropy, which will be referred to as the entanglement entropy, has some unique properties to it. The entanglement entropy has some small fluctuations not only in time, but in different random circuits with the same L, $|A|$, and p. The distribution of these entropies are not only sharply peaked for the specified measurement rate p; they also appear to be Gaussian. As such, we will refer only to the average of these distributions. Furthermore, for an arbitrary system, this entanglement entropy is dependent solely on the measurement rate p, the size of the circuit L, and the size of the subsystem $|A|$. The location of the subsystem A does not affect the entanglement entropy due to the translational symmetry of the circuit.

As we look at this entanglement entropy, we want to investigate how it behaves at specified values of the measurement rate p with respect to the subregion size $|A|$. This behavior has been analyzed many times, especially in the extreme cases[2, 13, 14]. In the limit $p \to 1$, the steady state wavefunction appears to be the trivial wavefunction with all ground state qubits and the entanglement energy obeys an area law, as follows:

$$S_A = c\,(p)\,|A|^0 \tag{4}$$

where $c\,(p)$ is some constant dependent on the measurement rate p. Conversely, in the limit $p \to 0$, the steady state wavefunction appears to be maximally entangled as we just have a random unitary circuit. In this phase, the entanglement entropy obeys a volume law that behaves as:

$$S_A = a\,(p)\ln\left(|A|\right) + b\,(p)\,|A| \tag{5}$$

where $a\,(p)$ and $b\,(p)$ are also constants that depend on the measurement rate p. There also exists a critical phase between these two phases. Measurement rates below this critical rate p_c exhibit the volume-law dependence in entanglement entropy while measurement rates above this critical rate p_c exhibit the area-law dependence. In other words, for measurement rates $p < p_c$, the entanglement entropies increases linearly for large $|A|$. For measurement rates $p > p_c$, the entanglement entropies saturate at some value dependent on p. However, at the critical rate p_c, the entanglement entropies increase logarithmically with respect to $|A|$:

$$S_A = a\,(p)\ln\left(|A|\right) \tag{6}$$

The behavior of both the phases as well as at the critical rate p_c is qualitatively shown in Fig. 3.

While there is more information to be gained from investigating these phases, we will leave this discussion here and move directly into an investigation of this critical measurement rate, as it will provide some insight into mutual information that we are interested in here. In doing so, let us consider, specifically, the result directly associated with the mutual information. When we observe the mutual information, we need to pick particular regions to look at. As such, we choose the regions A and B to be the same size of $|A| = |B| = L/8$, but a distance $r_{A,B} = L/2$ apart. These regions are antipodal in a system with periodic boundary conditions. Regardless of the phase, if the measurement rate p is away from criticality p_c, then this mutual information between the antipodal regions decays exponentially with respect to the size of the system L.

However, this behavior does not occur at the critical rate p_c. Instead, due to the long-range correlations[2], the mutual information should be enhanced as the system grows with L. These long-range correlations are a result of the circuit being at a point where it cannot be disentangled into smaller pieces. By looking at the mutual information at various rates p with a circuit of finite size L, we always seem to have a peak in the mutual information at the critical rate p_c. In addition to this, this peak seems to narrow with increasing system size L. However, while the height of the peak also increases, it saturates to some constant independent of L. As such, there appears to be a critical measurement rate at which there is some entanglement that ensures the mutual encoding of a region of information in two different points. This provides the error detecting feature that we were hoping to find.

V. FURTHER INVESTIGATION

This merely gives a brief insight into the topics presented here today. The information provided here is useful but can be further expanded on by looking at the specifics of the phases or the other properties of criticality. In addition, we could even continue by looking at how our understanding of mutual information relates to our understanding of operators or of correlation functions. However, any further investigation into this topic would require more background to be presented and is not within the scope of this project.

Even though we end here, there are a number of interesting routes that can be pursued from here, some of which we will discuss here. For more experimentally-minded readers, some may want to tackle the problem of figuring out how to incorporate this in a physical system. As the investigation of how to implement quantum computers is still on-going, figuring out how to intersperse these measurements with these unitaries may present and even more unique challenge. In the meantime, there is a variety of other theoretical frameworks that must be developed. As some theorists have pushed to research in the recent years, we can extend this process to other circuits. While the Clifford gates makes the numerical calculations easier, they does so at the risk of losing generality. As such, we can expand our understanding by looking at circuits that make use of the Haar unitaries instead or that have some sort of symmetry, such as the spatial symmetry of measurements, ingrained in it.

Regardless of the path that one takes with this topic, there is some interesting information to be gained that could make headway in the battle to combat quantum decoherence in quantum computers, or at the very least a more unique understanding of the relationship between the application of unitaries and the use of measurements.

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