1D Bosonic SPT Phases and Its Classification

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This paper gives a brief introduction to the classification of 1D bosonic symmetry protected topological (SPT) phases. We first give a big picture of the map of quantum phase of matter and then motivate the idea of SPT phases via concrete examples of 1D Haldane phase and exact solvable AKLT model. We formulate the classification 1D bosonic SPT phases by the classification of the projective representation of its symmetry group G_Ψ i.e. the second group cohomology $H^2(G_\Psi, U(1))$ via illustration upon the concrete example of $\mathbb{Z}_2 \times \mathbb{Z}_2$ upon the AKLT model.

I. MAP OF QUANTUM PHASE OF MATTER

A. Phase and Order

One of the central theme of condensed matter physics is the classification and understanding phase of matter. Different organization of the constituent particles which is formally called **order** leads to different physical properties of the matter and thus can be used to distinguish difference matters. When the change of the order leads to some non-analytical behaviour of the physical property, we say such process a **phase transition**, and states of matter that admits no phase transition in a same **phase**. Different phases are revealed by different physical conditions[1]. We can probe the order using the **order invariant** which stays unchanged for transformation but become distinct under phase transition, thus labelling the different phase of the matter. Each particular phase of the matter can be regarded as an **equivalence class** where every element is **smoothly deformed** to a representative state and associated with an particular value of the order invariant. Therefore, different orders denote different spaces of phases of the matter and different order invariants identify different equivalence classes (i.e. phases) within the space of phases. In essence, the classification of phases hinges at finding the order invariants.

B. Symmetry Breaking Order

Landau realized that many phase transitions accompany symmetry breaking, and subsequently developed a general theory with Ginzburg to describe the phase transition using **symmetry group** of the matter[5–8]. The full symmetry group G_H Hamiltonian H is spontaneously broken into the symmetry group of the ground state $G_\Psi \subset G_H$. Therefore different symmetry breaking characterizes different phases of matter, thus we call it **symmetry breaking order**. We therefore naturally have symmetry breaking order invariant as the pair (G_H, G_Ψ) which leads to full classification. This is a very elegant and powerful paradigm which allows us to classify all of the 230 different kinds of crystals via their associated space group in 3 dimensions[2].

C. Topological Order

Yet, new discoveries of fractional quantum Hall (FQH) effect[9] and high T_c superconductors[10] posed a theoretical challenges to Landau's paradigm which fail to explain both in a satisfying manner[3]. We soon realized that there are many distinct phases lies beyond symmetry breaking order which indicates a new kind of order to classify those phases, thus entering the **topological order**[4]. This leads to a new theme of condensed matter physics to go beyond Landau's symmetry description of phase transition. Analogously, to describe the topological order, we need to identify its order invariants called **topological invariants**[5], where it stays invariant for transformation without phase transition but any change of it indicates a phases transition. This gives us a macroscopic probe for those distinct phases that symmetry group cannot. It is an ongoing question

[1] We are very familiar to the four fundamental phases of matter: solid, liquid, gas and plasma, manifested in different pressure and temperature. Magnetization could reveal some other phases of matter to be either paramagnetic, ferromagnetic or anti-ferromagnetic. In low temperature, we have observed superfluid He_4 characterized by zero viscosity[1, 2] and superconducting material characterized by zero resistance[3] and Meissner effect[4].

[2] For interested readers who want to see all those 230 crystals, I would recommend the following blog as a starting point: `https://crystalsymmetry.wordpress.com/230-2/`

[3] Some early attempts to explain high T_c superconductor leads to the construction of quantum spin liquid[11]. Later chiral spin liquids was constructed to study FQH[12] and was subsequently understood to have different phases under same SSB order[13].

[4] The nomenclature follows from early attempt to explain chiral spin state using the low energy effective field theory - Chern-Simons theory which is a topological quantum field theory (TQFT).

[5] Later works identified some topological invariants as follows[14]: (1) the robust ground state degeneracy (on closed space manifolds), (2) the non-abelian geometric phases (the modular matrices) of the degenerate ground state, (3) the chiral central charge c of the edge states. It was conjectured at least in 2D, the above description based on those topological invariants provides a complete characterization of topological orders[14, 15].

and active research to investigate and understand the nature of **topological phase**s.

D. Gapped Quantum Phases

In many-body quantum system, it is remarkable that we still find different phases even at zero temperature called **quantum phase**, and can study the phase transition without introducing heat to the system. We usually focus on the quantum phase of a **gapped local Hamiltonian**, whose Hilbert space is the product of local site Hilbert spaces, $\mathcal{H} = \bigotimes_i \mathfrak{h}_i$ and exists a gap Δ between the ground state and the first excited state, and remains gapped even in **thermodynamic limit** when system size goes to infinity[6]. Similarly, we call it a **gapped quantum state** if there exists a gapped Hamiltonian which admits it as the grounds state.

This allows us to formulate quantum phase as a property of a class of Hamiltonians called H**-class**, and refine our definition of **phase transition** at zero temperature related to the inevitability of gap closing during the transformation. Consider a H-class $H(g)$ as a collection of Hamiltonian that depends on some parameter g. Then for every $H(g)$, it admits corresponding ground state $|\Psi(g)\rangle$, finite gap $\Delta(g)$ and expectation value of any physical observable with respect to the ground state $\langle\mathcal{O}\rangle(g) \equiv \langle\Psi(g)|\mathcal{O}|\Psi(g)\rangle$. Then $H(g)$ defines a path when we smoothly change parameter g with corresponding $|\Psi(g)\rangle$ and $\langle\mathcal{O}\rangle(g)$ change accordingly. We can use perturbation theory to calculate the change of $\langle\mathcal{O}\rangle(g)$ for a small g only if $\Delta(g) > 0$ along the path. If the gap closes at some particular g_0, $\Delta(g_0) = 0$, we can have a singularity in any physical quantity. Therefore, two system $H(g = 0)$ and $H(g = 1)$ are said to be in the same **phase** if-and-only-if (iff) there exists a smooth path $H(g)$, $g \in [0, 1]$ connecting two and the gap remain finite $\Delta(g) > 0$, $\forall g$.

This formalism can be very generally defined using **local unitary** (LU) transformations[16, 17] which corresponds to product of local unitary operators acting locally on the wavefunction, i.e. LU transformation generates the path $H(g)$. Therefore, we can say that two gapped ground states are in the same phase if there is exist a LU transformation for one into another. The quantum phase of matter can be identified to the equivalence classes of LU transformation. We can have a "**trivial**" **product state** of form $|\Psi\rangle = \bigotimes_i |\Psi_i\rangle$, and define all states that can be transformed into this "trivial" product state as **short-ranged entanglement** (SRE) states, i.e. they differed from the "trivial" product state only by local fluctuations. All SRE states are in same phase under LU transformations. For the states cannot be transformed into a "trivial" product state, we call them **topologically ordered** state as they exhibits **long-ranged entanglement** (LRE) features like non-local correlations. Therefore, under this paradigm, the product states and the topologically ordered states are not in the same phases. LU transformation can also include symmetry breaking ordered states differed from the aforementioned two phases. Therefore we arrived at a map of quantum phases in terms of **LU classes**, shown in Figure. 1. We included the gapless states for completeness, but they are in generally very hard to analyze thus out the scope of this brief survey.

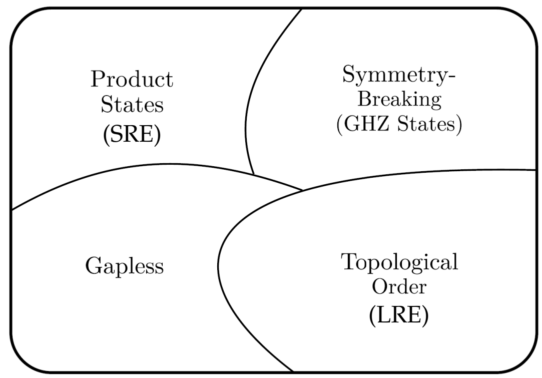

FIG. 1. LU classes of quantum phases maps the space of phases for different orders. We include gapless phase into our picture indicating all other three phases in consideration are gapped. (Credit to [18]).

The product states fall outside of both symmetry breaking order and topological order. Both symmetry breaking space of phases and topological order space of phases admit very rich phases. By parallel, We want to investigate whether inside the spaces of product states admits different phases or not as well. This leads us to a very interesting kind of quantum phase and order within the space of the product state when there is a symmetry constraint. We call them **symmetry protected topological** (SPT) phases, which is the main subject of this brief survey.

II. SPT PHASE

It turns out that those gapped product states with no symmetry breaking order and topological order, shown in Figure. 1, are not as trivial as to simply sit inside a single phase when there is a symmetry around. There are different phases between them and go across them requires a gap-closing phase transition if we retain its symmetry G_Ψ on the ground state. We therefore say

[6] The introduction of this limit is to exclude the trivial gap due to finite size of the Hamiltonian. Therefore, technically, when we talk about gapped Hamiltonian, we are thinking about a series of Hamiltonian of increasing size towards thermodynamic limit. All of them admits a finite gap between the ground state and the first excited state.

that the symmetry "protects" the states from smoothly deformed into a trivial product state, hence the nomenclature[7]. *what SPT phases exist? What properties do they have? Is there a way to classify them?* This are the three main questions to be addressed in the rest of this brief survey. We will see that there is a non-trivial **SPT order** by looking at the edge states on the boundary as long as the symmetry is imposed. We will mainly use the Haldane phase[19, 20] and AKLT model[8][21, 22] as a primary example of the 1D bosonic SPT phases to illustrate how we approach to a mathematical classification of SPT phases.

A. Prototypical Two Spins System

Let us first consider the ground states of two spin-S system. For a classical spin $\mathbf{S} = (S^x, S^y, S^z) \in \mathbb{R}^3$ as a vector and $|\mathbf{S}| = S$, we have two possible way to construct a classical two spins system, either via ferromagnetic interaction $E_{\mathrm{FM}} = -\mathbf{S}_1 \cdot \mathbf{S}_1$ or antiferromagnetic interaction $E_{\mathrm{AFM}} = \mathbf{S}_1 \cdot \mathbf{S}_1$. Both classical two spins systems admit infinitely many ground states. Now, we promote $\mathbf{S}$ to quantum spin operator $\hat{\mathbf{S}} = (\hat{S}^x, \hat{S}^y, \hat{S}^z)$ and respect $so(3)$ algebra $[\hat{S}^i, \hat{S}^j] = i\epsilon^{ijk}S^k$, and spin is quantized to take values $\frac{1}{2}, 1, \frac{3}{2}, \cdots$. Then similar to the classical cases, we can construct two Hamiltonian $\hat{H}_{\mathrm{FM}} = -\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ for ferromagnetic interaction and $\hat{H}_{\mathrm{AFM}} = \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ for antiferromagnetic interaction, and we denote their ground states as $|\Psi_{\mathrm{FM}}\rangle$ and $|\Psi_{\mathrm{AFM}}\rangle$ respectively. It is quite easy to check that for spin-$\frac{1}{2}$, $|\Psi_{\mathrm{FM}}\rangle$ is the linear combination of the spin triplets $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, thus admits infinitely many ground states. However, $|\Psi_{\mathrm{AFM}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is the spin singlet, which is unique and rotational invariant ground state and it has an energy gap to any first excited states. Therefore, we see that the quantum antiferromagnetic Hamiltonian $\hat{H}_{\mathrm{AFM}}$ yield a strong quantum effect on the ground state, and is very different than other models.

B. Haldane's Conjecture

This observation motivates us to consider 1D quantum Heisenberg antiferromagnetic (HAF) chain with Hamiltonian $H = \sum_{j=1}^{L} \mathbf{S}_j \cdot \mathbf{S}_{j+1}$ with system size $L >> 1$. In 1931, Bethe obtained the first exact solution for fermions and is now celebrated as the **Bethe ansatz**[23]. The result can be briefly summarized as: (1) the ground state $|\Psi_{\mathrm{HAF}}\rangle$ is unique both for finite[9] and infinite L; (2) there is no energy gap above ground state energy in $L \uparrow \infty$ limit[10]; and (3) the ground state correlation function decays as a power law[11]. These results was believed to be hold for both fermionic and bosonic system.

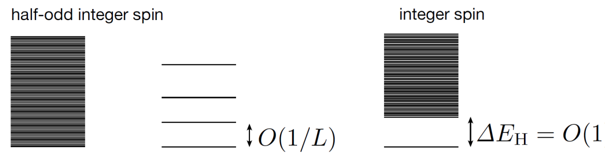

FIG. 2. Haldane's conjecture on energy spectra of the spin S Heisenberg antiferromagnetic chain. For half-odd integer spin, in the thermodynamic limit, we have continuous energy spectra while for integer spin, we have a unique gapped ground state with Haldane gap ΔE_H scales linearly with L. (Credit to [34]).

However, in 1983, Haldane discovered that 1D Heisenberg antiferromagnetic chain with integer spin has different low energy properties to those of half-odd-integer spin as shown in Figure. 2. He conjectured[19, 20] the integer spin chain to possess a unique gapped ground state with a exponential decay in the ground state correlation function[12], in sharp contrast to Bethe's results (2) and (3). Such gap is called **Haldane gap** which is subsequently checked via numerical calculations[24–27] and experimental data[28–33], and also hinted by the exact solution of a slightly modified model called AKLT model[21, 22] which exhibits the exponential decay in correlation. We will then focus on the AKLT model below as a primary example of SPT phases.

C. AKLT Model and VBS State

The **AKLT model** will be our prototypical example for SPT phases and the Hamiltonian is given by,

$$H_{\mathrm{AKLT}} = \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{1}{3}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2, \qquad (1)$$

where $\mathbf{S}_i$ is the spin-1 operator, i.e. we are now looking at a spin-1 HAF chain with an additional quadratic interaction. It is important to notice H_{AKLT} has a $SO(3)$ spin rotation symmetry which will play a key role to "protect" the non-trivial product states. The ground state wavefunction can be explicitly constructed and is called **valence-bond solid** (VBS) state[21, 22] and denoted as

[7] It was originally named **symmetry protected trivial** state as it is connected to trivial product state under LU transformation.

[8] Named after I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki.

[9] For finite L, it is known as the **Marshall-Lieb-Mattis theorem**

[10] For finite L, there is a energy gap due to the finite size, but the gap scales with L like $\mathcal{O}(L^{-1})$.

[11] $\langle\Psi_{\mathrm{HAF}}|\mathbf{S}_x\mathbf{S}_y|\Psi_{\mathrm{HAF}}\rangle \sim (-1)^{x-y}\frac{\sqrt{\log|x-y|}}{|x-y|}$, $1 \ll |x-y| \ll L$.

[12] $\langle\Psi_{\mathrm{HAF}}|\mathbf{S}_x\mathbf{S}_y|\Psi_{\mathrm{HAF}}\rangle \sim \frac{(-1)^{x-y}}{\sqrt{|x-y|}}\exp\left(-\frac{|x-y|}{\xi}\right)$, $\xi \ll |x-y| \ll L$, where ξ is the correlation length.

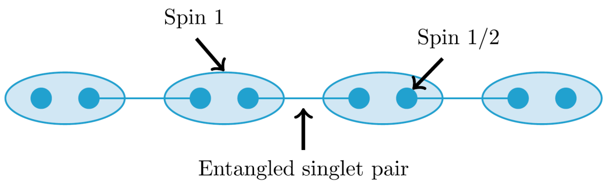

FIG. 3. The VBS state in projected entangled pair picture where the entangled singlet pair (blue line) $|\Psi_{AFM}\rangle$ which is called **valence bond** and each lattice site (blue oval) contains two spin-$\frac{1}{2}$s (blue dot) forming the triplets $|\Psi_{FM}\rangle$. (Credit to [18]).

$|\Phi_{VBS}\rangle$ (whose precise **matrix product state** (MPS) representation is included in Appendix. A 1) and we can represent it as a 1D chain of entangled singlet pairs with two spin-$\frac{1}{2}$s on each lattice site forming triplets, as shown in Figure. 3. Therefore we see a very interesting correlation between $|\Psi_{FM}\rangle$ at lattice site and $|\Psi_{AFM}\rangle$ at the valence bond.

This pictorial representation of VBS state can help us intuitively understand the properties of the state. Notice that so far, we have not assumed any boundary conditions and we have two possibilities. If periodic boundary condition (PBC) is imposed, our projected entangled pair chain forms a ring and the ground states preserves the $SO(3)$ spin rotation symmetry and is unique and gapped. However, on a open chain with boundary, we have two edge spin-$\frac{1}{2}$s sit freely at the edge as shown in Figure. 4 which leads to the fourfold degeneracy[13]of the ground states. We can write them as $\left|\Phi_{VBS}^{\sigma_L,\sigma_R}\right\rangle$, with $\sigma_L, \sigma_R = \{\uparrow, \downarrow\}$ denote the edge spins at left and right boundaries respectively.

FIG. 4. This is one of the four degenerate ground states, $\left|\Phi_{VBS}^{\uparrow,\downarrow}\right\rangle$. For open chain, we have two free edge spin-$\frac{1}{2}$ at the boundary which leads to fourfold degeneracy of the ground states (Credit to [34]).

We provide a heuristic explanation that the VBS states as SPT phases as follows: For spin-1 chain, we could have a "trivial" product state $|\Phi\rangle_{trivial} = |\cdots 000 \cdots\rangle$, i.e. a chain (or ring if PBC is imposed) of spin with S^z eigenvalue 0 at each site. admitted by a trivial Hamiltonian $H_{trivial} = \sum_i (S_i^z)^2$. We would like to know whether our VBS states sit in the same phase with the "trivial" product state. If we allow symmetry of the VBS states to break, we can freely rotate our spin at each site to arrive at a "trivial" product $S^z = 0$ chain, therefore they sit in the same phase. However, if we retain the symmetry of the VBS states at any point, it does not sit in the same phase as the "trivial" product state. One quick way to see this is to consider their transformation properties under $SO(3)$ rotation. the VBS picks up a factor -1 under 2π rotation because of the non-trivial presence of edge spin$\frac{1}{2}$s[14], but the trivial $S^z = 0$ state picks up nothing under 2π rotation. Therefore, they can not be smoothly deformed into each other as there will be a singular point that the symmetry transformation in any path. This gives us a very simple example of bosonic SPT phases in 1D[15]. The key observation from this heuristic analysis is that the symmetry acts independently on the two edges. The symmetry transformation leads to a relative phase at the edge which may only admit a discrete set of possible values, which is our hint for a non-trivial topological index as the order invariant. But if the symmetry is removed, this distinction of between SPT phases and "trivial" product state vanishes.

D. Topological Phase Transition

We have argued how SPT phases arise in context of the ground states associated to it. We know the exact ground state of AKLT model on a ring has unique gapped ground state. We have also considered a trivial model of spin-1 ring $H_{trivial} = \sum_i (S_i^z)^2$, whose ground state easily to be found to be uniquely gapped $|\Phi_{trivial}\rangle = |\cdots 000 \cdots\rangle$ a trivial product state. Therefore, both model admits a unique gapped ground state and also breaks no symmetry. We can carry out a more formal analysis via their Hamiltonians which admits those ground state.

To understand that, we can write down an interpolating model $H(g) = gH_{AKLT} + (1-g)H_{trivial}$ with $g \in [0, 1]$, i.e. constructed a path $H(g)$ to connect the trivial model at $g = 0$ to the AKLT model at $g = 1$ and see whether smooth change of g leads any gap-closing behavior or not. The numerical results suggest there is a gap-closing point at some g_0, as shown in Figure. 5. We call g_0 a **gapless/critical point** which signals a topological phase transition which is not characterized by symmetry breaking order. Similar behaviour was seen for the interpolating model of HAF model and trivial model numerically, where the parallel analysis on the ground states can be difficult.

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[13] More precisely, a nearly-fourfold degeneracy. The reason comes from the correlation between the two edge spin$\frac{1}{2}$s, for the finite chain, the correlation is not zero. However, if one take the infinite size limit, we will have a unique gapped ground state as the correlation decays exponentially to zero as suggested by Haldane's conjecture.

[14] As a spin-$\frac{1}{2}$ particle return to itself after a 4π rotation around any axis under $SO(3)$ rotation.

[15] One can in fact generalize this construction using the projected entangled singlet pairs to higher spin chain or to higher dimensions on different lattices, such as 1D spin-2 chain, 2D square spin-2 lattice and 2D honeycomb spin-$\frac{3}{2}$ lattice, etc..

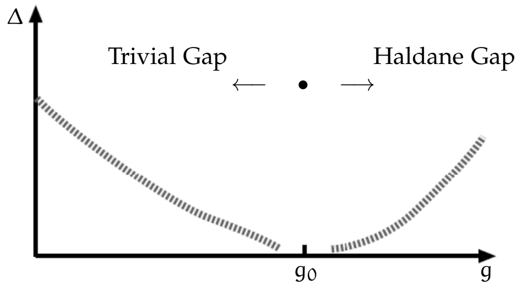

FIG. 5. The energy gap of the interpolating model where we see the gap closing at some g_0, indicating phase transition. (Modified upon [34])

It turns out that H_{AKLT} and $H_{trivial}$ can be smoothly connected if any short-ranged Hamiltonian is allowed to interpolate[16], which reflects the idea of existence of a LU transformation to connect SPT phases to "trivial" product state. However, if certain symmetry are allowed, then these two Hamiltonian cannot be smoothly deformed into each other, which is the hallmark of a SPT phase[37]. Therefore, in the space of Hamiltonian that admits a uniquely gapped state, we can understand the SPT phases have the behaviour shown in Figure. 6.

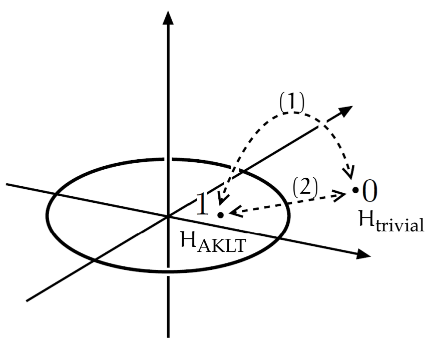

FIG. 6. A schematic diagram of SPT phases: Two models represented by dots 0 and 1 are along path-(1) continuously connected in the larger parameter space. But when certain symmetry restrict the larger parameter space to smaller one represented here by the plane, path-(2) separated by a definite phase boundary (g_0) (Credit to [34])

As the above example illustrate, AKLT model and its ground state $|\Phi_{VBS}\rangle$ in Haldane phase serves as a simple example of 1D SPT phases. It has a gapped bulk SRE states with non-trivial SPT order manifested in the existence of non-trivial edge states on the boundary and cannot be smoothly deformed into a trivial product states when symmetry group G_Φ is present.

[16] Rigorously proved by [35, 36].

III. SYMMETRIES OF 1D BOSONIC SPT PHASES AND ITS CLASSIFICATION

So, *what are those symmetries that can protect the SPT phases?* For Haldane phase of a spin-1 chain, the non-trivial SPT phase is protected either by: (1) π spin rotation about three axes, i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry; (2) time-reversal symmetry; (3) bond-centered reflection symmetry. We focus on $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry to illustrate how it leads to the classification of SPT phases. We first pay a quick revisit of the unitary representation and projective representation of a generic symmetry group G.

A. Unitary and projective representation

A unitary matrix $u(g)$ is called **unitary representation** if it forms a representation of symmetry group G, i.e. satisfies $u(g_1)u(g_2) = u(g_1, g_2)$, $\forall g_1, g_2 \in G$. If such group multiplication is preserved only up to some phases $\omega \in U(1) \equiv \{z \in \mathbb{C} : |z| = 1\}$, we call $u(g)$ **projective representation** of G, i.e. $u(g_1)u(g_2) = \omega(g_1, g_2)u(g_1, g_2)$, $\forall g_1, g_2 \in G$. We have two additional condition of the phases:

(1) From the associativity of representation, we have $u(g_1)[u(g_2)u(g_3)] = [u(g_1)u(g_2)]u(g_3)$,

$$\implies \frac{\omega(g_2, g_3)\omega(g_1, g_2 g_3)}{\omega(g_1, g_2)\omega(g_1 g_2, g_3)} = 1,\ \forall g_1, g_2, g_3 \in G. \quad (2)$$

We call ω that satisfies this condition a **2-cocycle**, and this relation **cocycle condition**[17]. And we denote the set of all 2-cocylce of group G to be $Z^2(G, U(1))$, where $U(1)$ denotes that $\omega \in U(1)$ is a phase.

(2) The projective representation is equivalent to each other up to a phase $\beta \in U(1)$, i.e. $u'(g) = \beta(g)u(g)$. Then, we have for all $g_1, g_2 \in G$, $u'(g_1)u'(g_2) = \beta(g_1, g_2)u'(g_1 g_2) \implies \omega'(g_1, g_2) = \frac{\beta(g_1)\beta(g_2)}{\beta(g_1 g_2)}\omega(g_1, g_2)$. Therefore we have an new equivalence relation condition called **coboundary condition**: for some β, $\forall g_1, g_2 \in G$

$$\omega \sim \omega' \text{ iff } \omega'(g_1, g_2) = \frac{\beta(g_1)\beta(g_2)}{\beta(g_1 g_2)}\omega(g_1, g_2). \quad (3)$$

And we call the set of the **index** of the equivalence classes of 2-cocycle ω **second group cohomology**, denoted as $H^2(G, U(1)) \equiv Z^2(G, U(1))/\sim$. Therefore, we can label those equivalence classes of the projective representation of G by an index ind $\in H^2(G, U(1))$. In essence, the projective representation of symmetry group G is classified by the second group cohomology $H^2(G, U(1))$.

[17] seen by multiply both numerator and denominator by $u(g_1 g_2 g_3)$.

As it turns out, the projective representation of the symmetry emerges naturally in the MPS representation of a general quantum spin chain[18] which includes the Haldane phase and AKLT VBS states. We now move on to the discussion of generic on-site symmetry, and use $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the VBS states as a concrete example to illustrate how this index could be used to distinguish different SPT phases.

B. General Formalism for Infinite Spin Chain

We can now set up a formal discussion of the general quantum spin chain. For local gapped Hamiltonian, with the finite-dimensional local site Hilbert space, we can define the notion of local operator if it only act on the tensor product of the local Hilbert space of the finite portion of the chain. They, after taking the completion wit respect to the operator norm, forms a C^*-algebra to be denoted as $\mathfrak{U} := \overline{\{\text{all local operators}\}}$. Then for a generic symmetry group G, its unitary projective representation $u_g^{(j)} := u^{(j)}(g)$ acts on local site Hilbert space $\mathfrak{h}_j$, and has the index $\mathrm{ind}_j \in H^2(G, U(1))$. Then the group action $g \in G$ can transform a local operator $A \in \mathfrak{U}$ as $\Xi_g(A) := \left(\bigotimes_{j=1}^{L} u_g^{(j)}\right) A \left(\bigotimes_{j=1}^{L} u_g^{(j)}\right)^\dagger$, which is a *-automorphism on $\mathfrak{U}$, and satisfies $\Xi_{g_1}\Xi_{g_2} = \Xi_{g_1 g_2}$, i.e. a representation of a group G. In essence, we set up the on-site symmetry for local operator in terms of *-automorphism from the projective representation of G.

we have previous observed that SPT phases need the symmetry to protect it from smoothly connected to trivial phase. Therefore, we consider G-invariant short ranged Hamiltonian of form $H = \sum_{j\in\mathbb{Z}} h_j$, where $h_j = h_j^\dagger$ is a self-adjoint operator and acts only on $\bigotimes_{k;|k-j|\leqslant r_0} \mathfrak{h}_k$, i.e. a finite size of chain localized near site j with size r_0, and h_j is also G-invariant, i.e. $\Xi_g(h_j) = h_j$, $\forall j \in \mathbb{Z}$, $g \in G$. We can define a ground state denoted as $|\Phi\rangle$ of H to be unique and with a nonzero energy gap, via a linear function denoted as $\Phi : \mathfrak{U} \to \mathbb{C}$ the expectation value of any local operator $A \in \mathfrak{U}$ that satisfies (1) $\Phi(\mathbf{1}) = 1$ (i.e. normalization) and $\Phi(A^\dagger A) \geqslant 0$, $\forall A \in \mathfrak{U}$; (2) $\Phi(A^\dagger[H, A]) =\geqslant 0$, $\forall A \in \mathfrak{U}$ which make sure that it is indeed taking expectation value respect to the ground state[19]; (3) $\exists \gamma > 0$, $\Phi(A^\dagger[H, A]) \geqslant \gamma A^\dagger A$, $\forall A \in \mathfrak{U}$ such that $\Phi(A) = 0$, which makes sure that $|\Phi\rangle$ has a nonzero gap[20].

We therefore have define a index ind for local site spin via the group cohomology. *How can we define an index Ind to denote a G-invariant unique gapped ground state Φ on infinite chain to distinguish SPT phases?* We defined Φ to be G-invariant, but if we make a fictitious cut at site j and perform symmetry transformation only on the half portion of the infinite chain then there might be some non-trivial index Ind_j[21] account for the transformation property of the half infinite chain. We will see that this indeed leads use to construct a topological index as order invariant for different SPT phases. We now proceed to illustrate a way to construct an index for the infinite chain using our $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant chain for both AKLT model and trivial model as an concrete example, which will holds true for generic symmetry group, and with some modifications to higher dimensional cases as well.

C. $\mathbb{Z}_2 \times \mathbb{Z}_2$ Symmetry

The symmetry group of π spin-rotation around three axes, $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ has four elements, $G = e, x, y, z$ satisfying relation $x^2 = y^2 = z^2 = e, xy = yx = z$, $yz = zy = x$, $zx = xz = y$. And the computation of the second group cohomology $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2 = \{0, 1\}$. The projective representation $u(g)$, $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by the four elements of G, i.e. $u(g) = \exp(-i\pi S^g)$, $g \in \{x, y, z\}$, together with identity element $u(e) = \mathbf{1}$. They satisfies $u(x)u(y) = u(z)$, $u(y)u(z) = u(x)$, $u(z)u(x) = u(y)$. But for $g_1, g_2 \in \{x, y, z\}$, $g_1 \neq g_2$, we see there is a difference: (1) the integer spin, where $(u(g))^2 = \mathbf{1}$, $[u(g_2), u(g_2)] = 0$, respecting the bosonic nature of the spin; (2) the half-odd-integer spin, where $(u(g))^2 = -\mathbf{1}$, $\{u(g_2), u(g_2)\} = 0$, respecting the fermionic nature of the spin. Therefore, we can use $\mathrm{ind} \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ to label the two different equivalence classes as,

$$\mathrm{ind} = \begin{cases} 0, & S = 1, 2, \cdots \quad \text{(trivial)} \\ 1, & S = \frac{1}{2}, \frac{3}{2}, \cdots \quad \text{(non-trivial)} \end{cases} \tag{4}$$

The fact that there are two distinct indices illustrate Haldane's claim that integer spin chain is transforms qualitatively different under same symmetry group G to half-odd-integer spin chain.

D. Towards MPS Index

Now imagine we have $|\Phi_{\text{trivial}}\rangle$ and $|\Phi_{\text{VBS}}\rangle$ and we make a fictitious cut at left-hand side of site j. When

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[18] We refer the reader to [18] for explicit presentation. The essence is that after the simplification of the MPS, the symmetry acts via a projective representation of G at the fixed point of renormalization flow, where the technique is to long to include in this brief essay.

[19] This can be clearly seen by in finite chain case that $H|\Phi\rangle = E|\Psi\rangle$ and $\langle\Phi|A^\dagger[H, A]|\Phi\rangle \geqslant 0 \implies \frac{\langle\Phi|A^\dagger HA|\Phi\rangle}{\langle\Phi|A^\dagger A|\Phi\rangle} \geqslant 0$, thus Φ is indeed a ground state.

[20] This can be clearly seen analogously in the finite chain case.

[21] which might dependent on the location of the cut, thus the subscript j

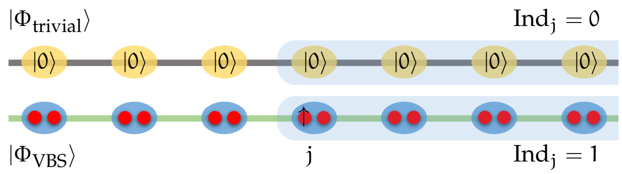

FIG. 7. Index for both $|\Phi_{\text{trivial}}\rangle$ and $|\Phi_{\text{VBS}}\rangle$ in presence of a fictitious cut. The difference edge spin state leads to distinguishable Ind_j.

we act G on the left portion of the infinite spin chain via the projective representation $u_g^{(j)}$, we have to use different indice due to the difference of the effective edge spin at the cut: (1) for $|\Phi_{\text{trivial}}\rangle$, the edge spin-0 leads to $\text{Ind}_j = 0 = \text{ind}_j(S = 0)$; (2) for $|\Phi_{\text{VBS}}\rangle$, the edge spin$\frac{1}{2}$ leads to $\text{Ind}_j = 1 = \text{ind}_j(S = \frac{1}{2})$. So, we see that this would indeed distinguish $|\Phi_{\text{VBS}}\rangle$ and $|\Phi_{\text{trivial}}\rangle$, as shown in Figure. 7.

But this is dependent on local site j thus is not a property of the ground state. *How can we define a index for the ground state?* One might try to define the index as the sum of local site index for a large but finite chain as $\text{Ind}_j = \sum_L^{k=j} \text{ind}_k$, with ind_k tells us the transformation property of the projective representation $u_g^{(k)}$ at k site. However, we observe that such definition leads to same index for both $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant chain. But MPS representation come to rescue, as we can define such an index analogously on the MPS which will distinguish different SPT phases. We Start with a G-invariant injective MPS given by (A3), where the symmetry acts on MPS via the transformation of matrices as $M^\sigma \to u_g M^\sigma u_g^\dagger$. Therefore, if we introduce a fictitious cut at j-th site on the MPS, at the edge, we will have free u_g not be able to contract with $u_g^\dagger$, and therefore $\text{Ind}_j = \sum_L^{k=j} \text{ind}_k$ will detect SPT order, as shown in Figure. 8. The detail construction can be found in [38]. The upshot is that this allows us to define a valid index $\text{Ind}_j \in H^2(G, U(1))$ that can detect SPT phases of injective MPS which include AKLT model as $\text{Ind} = 1$ and trivial model as $\text{Ind} = 0$ for infinite chain. In fact, such construction of the index is generic and apply to a general class of SPT phases.

Therefore, we arrive at the classification of the 1D bosonic SPT phases using the group cohomology $H^2(G_\Psi, U(1))$, whose elements are the topological indices labelling those different SPT phases. If we combine with Landau's symmetry breaking order, we can in fact classify all 1D bosonic SPT state using the order invariant $(G_H, G_\Psi, H^2(G_\Psi, U(1)))$!

IV. CONCLUSION

This conclude our brief introduction to the classification of 1D bosonic SPT phases using group cohomology $H^2(G, U(1))$, where each element of $H^2(G, U(1))$ gives index for a distinct SPT phase. When G is imposed, different SPT phases associated with different MPS index cannot be smoothly deformed into one another without closing the gap. Therefore, we can refine our map of the quantum phase shown as Figure. 9, where now we have more information about the structure the space of the product state. There are some recent work on constructing a more general index called Ogata index[39] which leads to a full classification of SPT order. For higher dimensional bosonic SPT phases, it was shown and partially verified that $(d+1)$-D SPT states with on-site symmetry G are labeled by the elements in group cohomology class $H^{d+1}(G, U(1))$. And as we only discussed bosonic case, one can easily obtain classification of fermionic SPT phases via the Jordan-Wigner transformation to account for the intrinsic $\mathbb{Z}_2$ parity of fermions. But we leave the reader to explore those interesting results. We therefore conclude a brief glimpse of the rich world of the quantum phase of the matter.

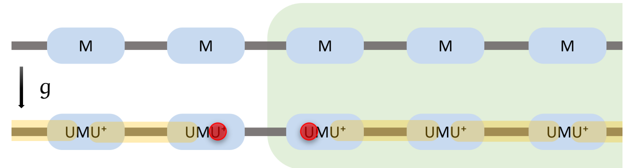

FIG. 8. A schematic diagram of the fictitious cut of the MPS which leads to edge u_g carries corresponding index (represented in red dot). The orange bond shows the bulk site to be G-invariant due to the $u_g^* u_g$ cancellation, but the edge have non-trivial transformation properties.

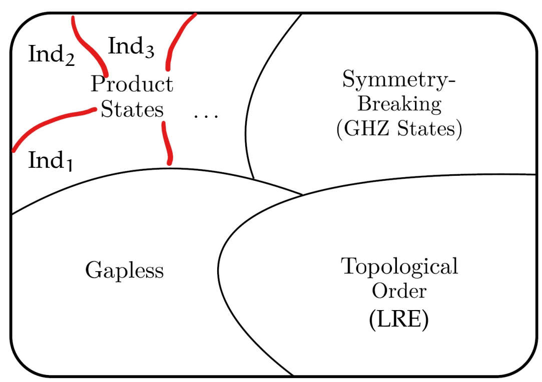

FIG. 9. Map of quantum phases with SPT phases classificed by $H^2(G, U(1))$. (Modified upon [18]).

Appendix A: Matrix Product State (MPS)Representation

1. VBS States

In practice, we can write $|\Phi_{\text{VBS}}\rangle$ using matrix product representation [40–42] in terms of a product of matrices:

$$|\Phi_{\text{VBS}}\rangle = \sum_{\sigma_1,\cdots,\sigma_L=0,\pm1} \text{Tr}[M^{\sigma_1}\cdots M^{\sigma_L}]\,|\sigma_1,\cdots,\sigma_L\rangle\,, \tag{A1}$$

where $\sigma_i = 0, \pm 1$ (spin-1) denotes the eigenvalue of local $\mathbf{S}_i^z$ operator on site i and

$$M^+ = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad M^0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad M^- = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}.$$

The trace reflects the periodic boundary condition where we identify the bond between two ends as taking the trace. The advantage of MPS representation is that they admit a nice diagrammatic representation as shown in Figure. 10.

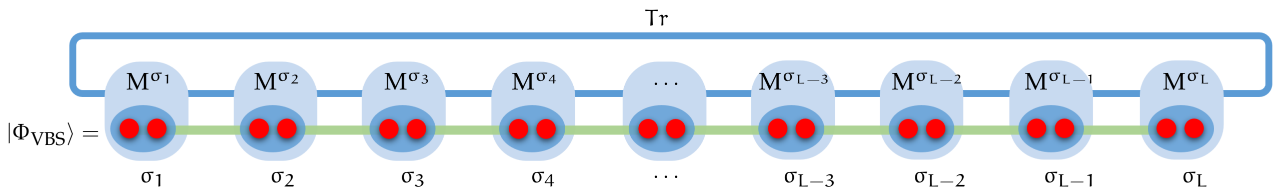

FIG. 10. This is diagrammatic representation of the MPS representation of VBS state (we overlay it on the projected entangled pair picture for clarity).

For an on open chain, we can write $|\Phi_{\text{VBS}}\rangle$ as,

$$|\Phi_{\text{VBS}}\rangle = \sum_{\sigma} l^{\sigma_1} M^{\sigma_2}\cdots M^{\sigma_{L-1}} r^{\sigma_L}\,|\sigma_1,\cdots,\sigma_L\rangle\,, \tag{A2}$$

where now we insert the row vector l^{σ_1} at site 1 and the column vector r^{σ_L} at site L reflecting boundary effect of the two free spin-$\frac{1}{2}$s. And as before the orientation of the edge spins lead to a fourfold degeneracy of the ground states which we can denote as $|\Phi_{\text{VBS}}^{\sigma_L,\sigma_R}\rangle$, with $\sigma_L, \sigma_R = \uparrow, \downarrow$ denote the edge spins.

Using this, the correlation function can be verified to satisfies Haldane's claim of exponential decay behaviour. The correlation function of $|\Phi_{\text{VBS}}\rangle$ indeed decays exponentially as

$$\frac{\langle\Phi_{\text{VBS}}|\mathbf{S}_i\cdot\mathbf{S}_j|\Phi_{\text{VBS}}\rangle}{\langle\Phi_{\text{VBS}}|\Phi_{\text{VBS}}\rangle} = 4(-3)^{-|i-j|}, \quad (i \neq j)$$

.

2. MPS and Symmetry Transformation

VBS states is just a special case of a general class called **matrix product state** (MPS)[40–42] where σ can take eigenvalues of arbitrary spin.

$$|\Phi\rangle = \sum_{\sigma_1,\cdots,\sigma_L=-S}^{S} \text{Tr}[M^{\sigma_1}\cdots M^{\sigma_L}]\,|\sigma_1,\cdots,\sigma_L\rangle\,, \tag{A3}$$

It turns out that any unique gapped ground state of a spin chain can be approximated accurately by MPS. This allows us to calculate easily for some physical quantity.

The symmetry operation on the MPS (in canonical form) is given by the projective representation of the group G, $u(g)$, where the local site matrix transform as $M^\sigma \to u(g)M^\sigma u^\dagger(g)$. And the

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