Criticality and Decimation in the Ising Model for $d = 1$ and $d = 2$.

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The Ising ferromagnet is used as context for a discussion of critical behaviour and the application of renormalization. The notion of criticality is introduced, and its importance for understanding correlations amongst the spin variables of the Ising model is emphasized. For the 1D Ising model, renormalization, in the form of the decimation technique, is used to analyze correlations and obtain an exponential behaviour for the correlation length. In 2D, the expected power law behaviour is observed, and a modified version of decimation is used to compute the critical exponent of correlation length.

Phase Transitions and the Ising Model— The behaviour of materials around second-order phase transitions is among the most fascinating aspects of collective phenomena. Phase transitions are generally characterized by immediate changes in macroscopic system properties that occur as some controlled parameter crosses a particular threshold. First-order transitions are discontinuous in nature, where a finite jump occurs in some macroscopic variable. Conversely, in second-order transitions such variables are continuous across the phase transition boundary, but their derivatives are discontinuous. In the 2D Ising ferromagnet, a second-order phase transition occurs at a finite value of temperature, $T = T_C$, which is denoted as a critical point of its phase diagram; see Fig. 1. The model Hamiltonian describing the Ising ferromagnet system is

$$H = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i \qquad (1)$$

where $S_i = \pm 1$ are spins on a to-be-defined lattice, J is a coupling between spins and H is an external magnetic field. In this paper we will examine this model in 1D and in 2D, with a particular focus on its behaviour at criticality, which is to say near its second-order phase transition.

The macroscopic variable used to distinguish the phases of a material is known as an *order parameter*, so named since it is a measure of the degree of ordering within the system. In the Ising ferromagnet, the order parameter is the average magnetization of the system, M. Consider the phase behaviour of a typical ferromagnetic material sample, as presented in Fig. 1. The two parameters that we tune to move about the phase space are temperature, T, and the strength of an external magnetic field, H. Suppose we were to experimentally traverse curve ‘1’ on the diagram. For a constant value of T we vary H and will observe a sharp, first-order transition as the material goes from a phase in which M is positive (average spin-up) to one in which it is negative (average spin-down). Next consider a traversal of the curve ‘2’. At zero field, coming from high temperature through the vicinity of the circled region about $T = T_C$, we observe that the system will have to spontaneously choose a path along the parabolic curve toward non-zero field.

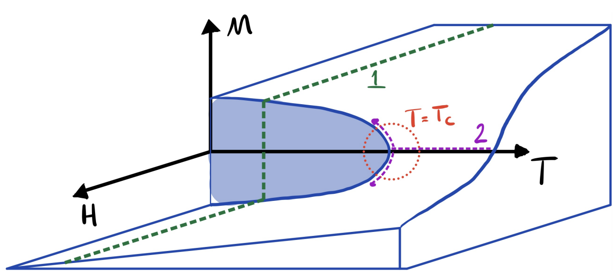

FIG. 1. Phase diagram of the 2D Ising ferromagnet.

Correlations and Critical Exponents— Although the magnetization varies continuously on this path through T_C, its derivative with respect to the external magnetic field - the susceptibility χ - diverges as $T \to T_C$ according to,

$$\chi = \frac{\partial M}{\partial h} \approx (T - T_C)^{-\gamma} \qquad (2)$$

where γ is a characteristic, *critical exponent* that describes the divergence. The susceptibility of the system is also related to the *connected correlation function*

$$\Gamma_c(i-j) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \qquad (3)$$

which is a measure of the mutual statistical dependence of the spins S_i and S_j. The susceptibility may be expressed in terms of $\Gamma_C(i)$ as,

$$\chi = \beta \sum_{i=0}^{\infty} \Gamma_C(i), \qquad (4)$$

revealing that the susceptibility provides us with a measure of how statistically correlated the system is as a whole. As each spin S_i is more statistically dependent on the other spins, χ increases.

When the system is far from the critical point, these correlations $\Gamma_C(i)$ decay exponentially with distance,

$$\Gamma_C(i-j) \approx \exp\left(-\frac{|i-j|}{\xi(T)}\right). \qquad (5)$$

Here $\xi(T)$ is a temperature-dependent length, called the *correlation length*. But, in line with the divergent behaviour of χ, as T approaches T_C, the correlations also diverge, with

$$\xi(T) \approx \frac{1}{|T - T_C|^{\nu}}. \tag{6}$$

Here ν is a different critical exponent, similarly characterizing physical behaviour near the critical point. This divergence is a fundamental characteristic of critical phenomena. The average magnetization of the Ising model system is a macroscopic property that is computed by summing up the microscopic values of all the different spins. The correlation behaviour for the system far from the critical point, described by Eq. (5), suggests that the system can be thought of as consisting as an aggregate of various domains, or groups of correlated spins, that are, at their largest, of the size defined by ξ. At T_C, Eq. (6) says that $\xi \to \infty$; the correlation length extends across the entire system, and this picture of sub-domains ceases to be accurate.

The order parameter M, other thermodynamic functions such as χ, the heat capacity C, as well as the correlation length all exhibit power law divergence in the neighbourhood of the critical point, and the divergence is defined by a series of the aforementioned critical exponents. It turns out to be the case that these critical exponents have far more generality than might appear from this Ising model context. There are many other examples of critical phenomena, such as the liquid-vapor transition in water and the transition undergone in Pb to become superconducting [1]. In fact, very different physical systems often have very similar critical exponents, despite their apparent microscopic differences. This phenomenon is known as *universality*, and references the way in which the behaviour of certain parameters near criticality is universal across certain classes - consequently named *universality classes* - of systems. In some sense, the macroscopic similarities across a universality class are the result of large distance correlations coming to dominate the behaviour at criticality, such that the microscopic details become increasingly ‘washed out’.

The Kadanoff Construction— Kadanoff first concretized this notion of criticality leading to a washing out of microscopic details [2]. Consider what happens to the correlation length $\xi(T)$ when we ‘zoom out’ from the system. Here we’re assuming the system to be infinitely large, and that we are at any time viewing some finite subset of it. At T far from T_C, the exponential decay of Eq. (5) governs the behaviour of $\xi(T)$. Thus we should expect that zooming out by a factor of two will lead to a shrinking of $\xi(T)$ by a corresponding factor of two. As we view the system at our new perspective, the sub-domains we observe will look half as large as they did before. However, at $T = T_C$ the correlation behaviour is governed by Eq. (6), and is itself infinite. No amount of zooming out will change this, i.e. $\xi(T)$ is invariant to any such zooming out procedure. The assumption of Kadanoff was to take this one step further, and suggest that near criticality one could zoom out from a system and all of its properties would as a whole remain invariant. The assumption of Kadanoff was to take this one step further, and suggest that if new, composite variables are appropriately defined to capture the behaviour of groups of spins, then one should be able to zoom out from a system and see that all its macroscopic properties remain invariant.

All thermodynamic properties of the system are computable from the partition function, so let us now define,

$$Z = \sum_{\{S_i = \pm 1\}} \exp\left(-\beta H\right) \tag{7}$$

$$= \sum_{\{S_i = \pm 1\}} \exp\left(K \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i\right) \tag{8}$$

where we’ve redefined the coupling constants as $K = J\beta$ and $h = H\beta$. The preceding sum is over all configurations of the spins S_i We anticipate a second order phase transition to occur at $K = K_C$, and $h = 0$. Our zooming out procedure will have the effect of modifying the couplings between the spins, and will redefine Z in terms of the modified couplings. More specifically, it will leave us with a new Hamiltonian,

$$-\beta\tilde{H} = K(b) \sum_{\langle ij \rangle} \tilde{S}_i \tilde{S}_j + h(b) \sum_i \tilde{S}_i \tag{9}$$

with the couplings $K(b)$ and $h(b)$ being functions of the factor, b, by which we zoom out. This procedure of zooming out and calculating the induced modifications to the coupling constants for a system is known as *renormalization.* The expressions $K(b)$ and $h(b)$ are known as the *renormalization group flow equations* since they can be thought of as describing how the coupling constants ‘flow’ through their own parameter space in response to the action of a renormalization procedure. Perhaps the most well-known example of this type of sensitivity to scale is that of the renormalization of the electron charge. In early work on QED, the notion of an ’effective charge’ that varied with distance scales was fundamental to understanding the validity of the theory; the closer to the electron one gets, the more one penetrates its positive cloak, and the more charge is experienced [3]. With this picture in mind, the critical point, where the system is invariant to zooming out, must correspond to a point from which the coupling constants do not flow under renormalization. We identify the critical point with non-trivial *fixed points* of these equations. Formally, the transformations generated by renormalization do not form a symmetry group in the rigorous mathematical sense, but the nomenclature is conventional.

The spins $\tilde{S}_i$ must be defined by some procedure which takes in several spins in the original system, and computes some aggregate spin quantity for use in the zoomed out system. In Kadanoff’s original proposal, the spins would be grouped into ‘blocks’ of linear size ba, with a being the lattice spacing. Thus each block contained $N_b = b^d$ many spins, where d is the dimension of the system. The new spin variables could be a simple averaging of the spins in each block, or some other such procedure. Note that a major assumption has been made here that the Hamiltonian can be written in the same form upon rescaling. We will later observe that this assumption can fail for quite simple models.

It can be demonstrated that the zooming out by a factor of b induces a change in the correlation length of the following form,

$$\xi(K,h) = b\xi(K(b),h(b)) \tag{10}$$

as long as we are in the vicinity of the critical point and can take advantage of its invariance. Furthermore, introducing $t = K - K_c$ as the deviation from the critical point, it can be shown that the correlation length also satisfies,

$$\xi(t,h) = b\xi(b^{-\frac{1}{\nu}}t, b^{\lambda_h}h) \tag{11}$$

where ν is precisely the critical exponent which is our aim, and λ_h is related to another critical exponent which we will not discuss. The computation of ν can be achieved by expanding the function $K(b)$ about the critical point, which we will ultimately demonstrate in an example.

Decimation in the 1D Ising Model— Instead of instituting an averaging procedure over block spins, we will apply a cruder technique of removing every other spin from the system. This is known as *decimation.* Beginning with a 1D Ising model described by the partition function of Eq. (8). The procedure of decimation achieves the same end as the zooming out procedure of Kadanoff's block spin construction. We will carry out the sum over spin configurations, but only for the S_i with even-valued i in the spin chain. The result will be a new partition function, in terms of a new Hamiltonian, where the remaining sum is only over the configuration possibilities for the remaining, odd-valued i spins S_i. This new, effective result, will represent a renormalized system of half as many spins, or equivalently, of twice the lattice constant: $a \to a' = ba$ where $b = 2$. The procedure is illustrated schematically in Fig. 2. This procedure returns the following modifications to the coupling constants,

$$K(b=2) = K' = \frac{1}{2}\ln\left(\cosh(2K)\right), \tag{12}$$

$$h(b=2) = h' = h\left(1+\tanh(2K)\right) + \mathcal{O}(h^2). \tag{13}$$

Next we can analyze the thermodynamic properties implied by these equations. Note that if we were to iteratively apply this zooming out procedure, decimating more and more spins in the chain, these equations tell us how the coupling constants K and h would be altered upon each successive iteration. As discussed, the critical point must be a point where the system is invariant to such iterations, and so we wish to identify fixed points in these equations. Specifically, we are looking for values of K^*, in the equation,

$$K^* = \frac{1}{2}\ln\left(\cosh(2K^*)\right). \tag{14}$$

This equation has no non-trivial fixed points, for finite K^*. This is observed by noting that the function $\frac{1}{2}\ln\left(\cosh(2x)\right) < x$ for all $0 < x < \infty$. If we begin our zooming out procedure from any finite value of K, successive applications of this equation will eventually bring $K \to 0$. We interpret this as the $K = 0$ ($T = \infty$) point corresponding to the infinite-temperature, paramagnetic phase of the system. Here correlations are short range and $\xi \approx a$, i.e. on the order of the separation between the spins. Next consider the point $K = \infty$. This is an unstable, non-trivial, fixed point and thus it represents a critical point of the model where correlations will extend across the entire system ($\xi \to \infty$). We can analyze the behaviour of K near this point to understand how ξ behaves. For $K \gg 1$,

$$K' = \frac{1}{2}\ln\left(\frac{e^{2K}+e^{-2K}}{2}\right) \tag{15}$$

$$= \frac{1}{2}\ln\left(e^{2K}\left(1+e^{-4K}\right)\right) - \frac{1}{2}\ln(2) \tag{16}$$

$$= K + \ln\left(1+e^{-4K}\right) - \frac{1}{2}\ln(2) \tag{17}$$

$$\approx K - \frac{1}{2}\ln(2) + \mathcal{O}(e^{-4K}). \tag{18}$$

If $K = \infty$ then $K' = \infty$ and the system remains at the fixed point. However if the system begins at some finite $K \gg 1$, then the above expansion tells us that K' will move slowly in the negative direction. Retrieving the parameter t quantifying the deviation from the critical point, we wish to understand how t changes as we step infinitesimally away from the critical point. We define just such an object, the β-*function*:

$$\beta_K(K) = \frac{dK(b)}{d\ln b} = -\frac{1}{2}. \tag{19}$$

With this definition, we identify the critical point to be $\beta_K(K_C) = 0$. The result for this problem of $-\frac{1}{2}$ comes from identifying dK with the result in Eq. (18), and

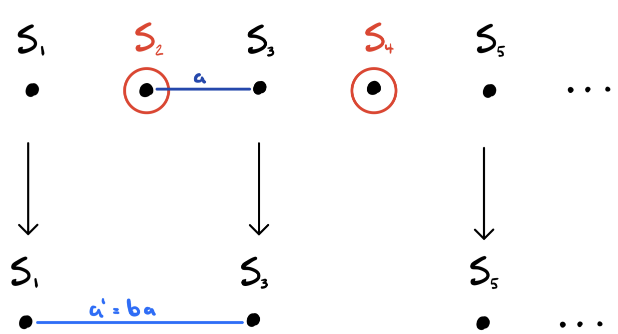

FIG. 2. The procedure of decimation in the 1D Ising model is presented schematically. The sum over configurations of every second spin is carried out, yielding a remaining partition function for the odd-value indexed spins.

that $d\ln(b) = \ln(2)$ for our $b = 2$ example. Integrating the above differential equation yields the explicit b-dependence of the coupling constant,

$$K(b) = K_0 - \frac{1}{2}\ln(b). \tag{20}$$

Now suppose we iteratively apply our zooming out procedure many times, such that $K(b) \to 0$. This corresponds to returning the system to the stable fixed point; the paramagnetic phase where $T \to \infty$ and $\xi \approx a$. To reconcile this with Eq. (20), we have $b \approx \exp(2K_0) = \exp(2J/T)$, and recalling Eq. (11) we have

$$\xi(T) \approx b \cdot a = \exp(2J/T) \cdot a \tag{21}$$

in this regime. The fact that we achieve exponential, rather than power law behaviour for $\xi(T)$ is expected for the 1D Ising model, and is quite general for systems of sufficiently low dimension. We will see that in the 2D model this will cease to be the case.

The Migdal-Kadanoff Procedure— The fact that the 1D Ising model Hamiltonian could be expressed in the same form following the renormalization procedure was critical to the method's success. In higher dimensions, this simplicity disappears, and renormalization introduces new interactions as we zoom out. To observe this concretely, consider the partition function for the 2D Ising model,

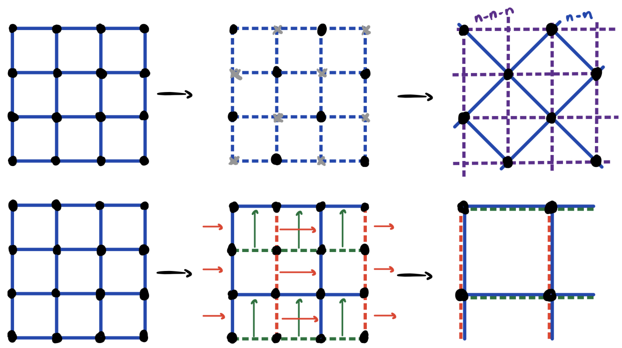

FIG. 3. Illustration of the Migdal-Kadanoff, or 'bond-moving' procedure for a 2D square lattice. The objective is to retain the same interaction structure in the zoomed out lattice and its model Hamiltonian. The straightforward decimation approach outlined in the upper row, would yield next-to-nearest neighbour interactions, thus making the problem invariably more complicated. The bond-moving method of Migdal, in the lower row, proceeds by shifting the bonds from the summed-over lattice sites on to the remaining nearest-neighbour bonds [4, 5]. Thus the Hamiltonian remains of the same form by design.

$$Z = \sum_{\{S=\pm 1\}} \exp\left\{ \sum_{\langle (ij),(kl) \rangle} K S_{ij} S_{kl} \right\} \tag{22}$$

for simplicity we'll focus on the case of $h = 0$ here. We'll try to follow the same decimation procedure for this model, and observe the complications that arise. Suppose, that following the upper set of lattice diagrams in Fig. 3, we identify half of the spins to be summed over. Denote those to-be-summed spins by B_{ij} and those to remain as A_{ij}. We have,

$$\begin{aligned}
Z &= \sum_{\{A\}} \sum_{\{B\}} \exp\left\{ \sum_{\langle (ij),(kl) \rangle} K A_{ij} B_{kl} \right\} && (23)\\
&= \frac{1}{2} \sum_{\{A\}} \sum_{\{B\}} \prod_{kl} \exp\left\{ \sum_{\eta} K A_{(k,l)+\eta} B_{kl} \right\} && (24)\\
&= \frac{1}{2} \sum_{\{A\}} \prod_{kl} \Bigg(\exp\left\{ \sum_{\eta} K A_{(k,l)+\eta} (+1) \right\} && (25)\\
&\qquad + \exp\left\{ \sum_{\eta} K A_{(k,l)+\eta} (-1) \right\} \Bigg) \\
&= \sum_{\{A\}} \prod_{kl} \cosh\left(\sum_{\eta} K A_{(k,l)+\eta} \right) && (26)
\end{aligned}$$

where $\eta = (1,0), (0,1), (-1,0), (0,-1)$ to cover the four A sites that are nearest neighbours of the B_{kl} site. The factor of $\frac{1}{2}$ is introduced to compensate for double counting. In Eq. (25), the sum has been carried out over the $\{B\}$ configurations, and this necessarily introduces terms that couple sites which were next-to-nearest neighbours in the original lattice. For example, terms of the form $A_{i+1,j}A_{i,j+1}$ will appear, corresponding to the diagonal nearest neighbour lines in the upper row of Fig. 3. We cannot, without further approximation, return this expression to the form of the original model as we did in the 1D case.

The proposal from Migdal is to instead implement a procedure which deliberately retains the form of the original Hamiltonian. Bonds are selectively translated on to new sites of the retained lattice, as illustrated in the lower row of Fig. 3. The underlying motivation for this choice is to retain the longer-distance behaviour of the interactions, with each spin still being connected to only two adjacent neighbours per direction, in the same manner as the $d = 1$ model. The movement of the bonds achieves this goal precisely, with the upshot of renormalization group flow equations that have identical form to what we observed in 1D, except with twice the coupling constant value, i.e.

$$K' = \frac{1}{2}\ln\left(\cosh(4K)\right). \tag{27}$$

As before, we're looking for fixed points, K^* of this equation. The situation is qualitatively different in this case, note that

$$\frac{1}{2}\ln\left(\cosh(4K)\right) = \begin{cases} 4K^2 \ll K & \text{if } K \ll 1 \\ 2K > K & \text{if } K \gg 1 \end{cases} \tag{28}$$

Again we find that the $K = J\beta = 0$ point is a stable phase of the system. However, in contrast to the 1D case we observe that the $K = \infty$ fixed point is also stable, as successive applications of $K' = 2K$ will of course not decrease K. This indicates that there is some finite value of K for which the function $K - \frac{1}{2}\ln(\cosh(4K))$ must change sign, i.e. a value where the modification of the coupling constants will be so as to increase them, or to decrease them, depending on which side of the point the system finds itself. Numerical solutions provide that $K^* = 0.30469$ [6].

Now consider replacing $K = K^* + t$, with $t = K - K_C$. Recalling Eq. (11) and expanding in small t,

$$K' \approx \frac{1}{2}\ln(\cosh(4K^*)) + 2\tanh(4K^*)t + \dots \quad (29)$$

$$\implies t' = 1.6786t = b^{\frac{1}{\nu}}t \quad (30)$$

in our present setup with $b = 2$, we find $\nu = 1.3392$.

Conclusions— In summary, we have demonstrated the use of renormalization as a formal technique of zooming out in our perspective of a physical system. We've found that the method of decimation can be used as an example of this technique, to characterize the behaviour of the correlation length, $\xi(T)$, near critical points in particular. In the 1D Ising model, we found that $\xi(T)$ exhibits exponential behaviour, as is typical of systems of low dimension. In 2D we observed that decimation is not so straightforward, and without the adjustments provided by the Migdal-Kadanoff procedure, leads to proliferation of the complexity of interactions in the zoomed out system. Following adjustment, we were able to compute the critical exponent for $\xi(T)$, finding $\nu = 1.3392$.

It should be noted that an exact solution for this critical exponent has been computed [7]. The result, $\nu_{\text{exact}} = 1$, deviates significantly from the crude result of our renormalization through decimation. Granted, we truncated the expansion of Eq. (29) for simplicity, and this leads inevitably to our inaccuracy. Furthermore, had we chosen a different scheme for moving the bonds, for example by carrying this out on a different lattice, we would have computed different critical exponents. Unfortunately this is the price paid for the simplicity of the Migdal-Kadanoff approach. A scheme-invariant understanding would require a more controlled renormalization procedure. Nevertheless, this exploration demonstrates the power of renormalization, and its utility for analysis of models which do not yet admit exact solutions. Second-order phase transitions are ubiquitous in the study of physical systems, and renormalization offers a tool that can be applied to disparate physical systems that may differ broadly in their microscopic character.

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