1D model of an intrinsically gapless SPT phase

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We provide a very brief overview of a topological phase of matter that has a gapless bulk and edge modes that are protected solely by an on-site symmetry action and the gaplessness of the bulk. To this extent, we study the Ising-Hubbard model as a 1+1D IGSPT and derive its phase diagram under tuning the external field and chemical potential. We also provide an effective field theoreric construction of the 1+1D IGSPT that allows us to explicitly obtain the protected edge modes and also characterize its topological properties.

I. INTRODUCTION

Topological Insulators (TIs) are a novel phase of matter that have an insulating bulk but admit conducting states on the edge [1]. Thus, their properties are intrinsically linked to the topology of the manifold the system is placed on [2]. Such systems are generally described by free electron theories. A generalization that includes interactions is given by Symmetry Protected Topological Phases (SPTs) [3].

SPTs typically have a bulk gap protected by a global on-site symmetry of the system, but admit non-trivial surface states. The trivial and topological phases of an SPT are connected by a phase transition that involves closing the bulk gap. It can also be continuously connected to the trivial phase if one of the symmetries is spontaneously broken (Hence "Symmetry Protected").

The topic of this paper however is *intrinsically gapless* SPT phases (IGSPTs). Such phases, initially proposed by Thorngren et al. [4], are fundamentally different from standard SPTs in that the bulk is gapless but the system hosts topological edge modes. In fact, opening up the bulk gap tunes the system out of the topological phase. Furthermore, the phase can be studied by looking at charges at the end of long-range string operators. The topological properties of the system can be linked to the edge and the bulk transforming differently under the on-site symmetry. A simple 1+1D IGSPT is realized by the Ising-Hubbard model.

Another mechanism for generating an IGSPT is by examining the 1D edge of a 2+1D SPT. The bulk of the SPT can be trivialized by adding fermionic couplings, thereby changing the action of the on-site symmetry. We provide a simple bosonized description of such a model using K-matrix theory to complement the Ising-Hubbard construction, and to explicitly obtain the edge modes.

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II. ISING-HUBBARD IGSPT

The 1D Ising-Hubbard model describes a 1D lattice that can host spinful fermions (electrons) with an on-site repulsion and a nearest neighbour spin-spin coupling. The Hamiltonian is given by:

$$\mathcal{H} = \mathcal{H}_{Hub} + \mathcal{H}_{Ising} \tag{1}$$

$$\mathcal{H}_{Hub} = -t\sum_{j,\sigma}(\hat{c}^{\dagger}_{j\sigma}\hat{c}_{j+1\sigma} + h.c) - \mu\sum_{j,\sigma}\hat{n}_{j\sigma} + U\sum_{j}\hat{n}_{j\uparrow}\hat{n}_{j\downarrow} \tag{2}$$

$$\mathcal{H}_{Ising} = J\sum_{j}\hat{S}^z_j\hat{S}^z_{j+1} + h\sum_{j}\hat{S}^x_j \tag{3}$$

where $\hat{c}_{j\sigma}$ are the usual electron annihilation operators and $\hat{n}_{j\sigma} = \hat{c}^{\dagger}_{j\sigma}\hat{c}_{j\sigma}$ is the on-site number operator. To obtain the spin operators, we can assemble the electron operators into vectors and take a matrix product with the Pauli sigma matrices. Let $\hat{\boldsymbol{c}}_j = (\hat{c}_{j\uparrow},\ \hat{c}_{j\downarrow})$. Then $\hat{S}^{\alpha}_j = \hat{\boldsymbol{c}}^{\dagger}_j\sigma^{\alpha}\hat{\boldsymbol{c}}_j$. We can tune the filling-fraction of the system using the chemical potential μ. We also assume an anti-ferromagnetic coupling $J > 0$ and the external field strength parameter h allows us to tune the Ising phase of the system.

We begin by studying the ground state of the system at half-filling. The on-site Hubbard repulsion favours a single electron per site, giving us an effective Ising system. At $h = 0$, we have a standard Ising anti-ferromagnet, eventually undergoing a phase transition to a paramagnet at some critical value $h = h_c$.

Things start to get interesting when we dope the system with electrons by tuning the chemical potential [7]. The doped electrons pair with up with the singly occupied sites (termed a "doublon") and and form a gapless state. This state is gapless since the paired electrons do not contribute to the Ising energy and we essentially have just the hopping term dictating the dynamics of the system. This phase is described by a gapless Luttinger liquid, which is a model used to describe interacting electrons in 1D.

The unpaired spins still retain a trace of their anti-ferromagnetic order due to the dynamics of the paired "doublons". To see this, consider a doublon sandwiched between two unpaired spins. The energy cost for the doublon hopping to the next site is determined by the energetics of the final state. If the unpaired spins are aligned, the energy is raised by J, while if they are anti-aligned, the energy is lowered by J. Thus, in the ground state, we obtain an anti-ferromagnetic spin-1/2 chain, albeit with doublons interspersed between the spins. This phase is identified as the topological Luttinger liquid phase of the system. We will address what makes it topological shortly.

The discussion proceeds in a similar manner when the system is an Ising paramagnet and is then doped with electrons to form doublons. This bulk state is described by a gapless Luttinger liquid, as before. We however do not have the long range anti-ferromagnetic order that was present in the toplogical phase. Thus, this phase is identified as the trivial Luttinger liquid. The four phases of the system as functions of the parameters h and μ are shown pictorially in FIG. 1.

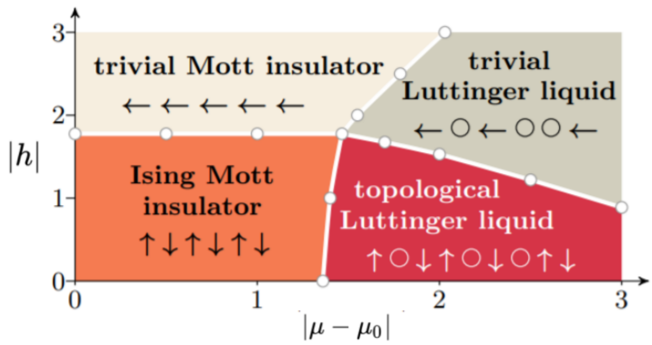

FIG. 1: Phase diagram for the Ising-Hubbard model obtained using DMRG calculations by Thorngren et al. [4] The circles represent the paired doublons away from half filling at $\mu = \mu_0$. The anti-ferromagnetic and paramagnetic Ising states are labelled as Ising and trivial Mott insulators in this graphic. h is the external magnetic field applied to the system and μ is the chemical potential.

Returning to the topological phase, consider the fermion parity operator $\hat{P}_j = (-1)^{\hat{n}_{j\uparrow}+\hat{n}_{j\downarrow}}$. This operator checks the number of fermions at a site mod 2 and assigns a phase of ± 1. We can thus create a string of parity operators $\hat{P}_i ... \hat{P}_j$ that has the effect of "squeezing" the doublons out of the chain, since the $\hat{P}_j$ acts trivially on them. This allows us to map spin correlations between the doped chain and the undoped chain at half filling.

$$\langle \hat{S}_i^z \hat{P}_{i+1} ... \hat{P}_{j-1} \hat{S}_j^z \rangle_{doped} \sim \langle \hat{S}_i^z \hat{S}_j^z \rangle_{undoped} \tag{4}$$

Thus, the expectation value of the string operator is non-zero in the topological phase, while it decays exponentially with distance in the trivial phase (FIG. 2). Observe that in order to diagnose the topological phase, we are required to make non-local measurements of the fermion parity (through the parity string operator). This is characteristic of the topological phase and is not captured by the local order parameter of Landau-Ginzburg theory. The non-trivial expectation value also points towards the existence of edge-modes in the topological phase. To see this, consider a finite length chain (open boundary conditions). Then, the string order points towards a non-zero expectation value between the ends of the chain. However, the sandwiched parity string now simply measures the total fermion parity of the chain.

$$0 \neq \langle \hat{S}_1^z \hat{P}_2 ... \hat{P}_{N-1} \hat{S}_N^z \rangle = (-1)^N \langle \hat{S}_1^z \hat{P}_1 \hat{P}_N \hat{S}_N^z \rangle \tag{5}$$

Thus, we have non-trivial $\hat{S}^z$ charge on either end of the string with a two-fold degeneracy (corresponding to states of fermion parity).

Finally, we want to discuss the symmetries of the

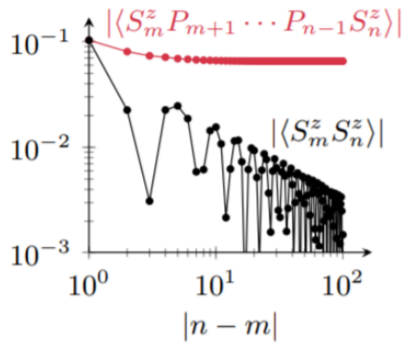

FIG. 2: The string order, with the sandwiched parity string, has non-trivial expectation value with increasing distance, whereas the standard two-point correlator decays exponentially, displaying the topological nature of the phase [4].

Hamiltonian and the phases above. The full Hamiltonian has a $\mathbb{Z}_4$ symmetry which corresponds to rotating the system by π around the x-axis (recall that spin-1/2 fermions get a $-$ sign on a 2π rotation). At half filling, the anti-ferromagnet spontaneously breaks the $\mathbb{Z}_4$ symmetry to its $\mathbb{Z}_2$ subgroup of 2π rotations, while the paramagnet preserves the full symmetry. In the topological phase, however, the gapless bulk has a "hidden" symmetry-breaking, since we explicitly see the $\mathbb{Z}_2$ symmetry only when we squeeze out the doublons. The edge in fact is anomalous in that it respects the full $\mathbb{Z}_4$ on-site symmetry, which is characteristic of the topological phase. Explicitly breaking the symmetry by tuning to the anti-ferromagnetic phase drives us to a topologically trivial phase. Thus, the edge modes are protected by the $\mathbb{Z}_4$ symmetry.

III. BOSONIZED EFFECTIVE FIELD THEORY

Thorngren et al. remark that a possible way to construct IGSPTs is to take the gapless edge of an SPT and trivialize the rest by gapping out the relevant degrees of freedom [4]. This means we add interaction terms to the Lagrangian that creates a gap in the excitation spectrum of certain degrees of freedom. Thus, at low energies, this degree of freedom is not excited since it requires a finite energy cost.

We consider the edge of a 2+1D bosonic SPT and trivialize the bulk by adding fermions. Then we establish a boundary on the 1D edge by gapping out only the fermions in the topological phase and gapping out all the degrees of freedom in the trivial phase. The following model has solely been constructed by the author.

We begin by employing the K-matrix formulation of 2+1D SPTs to describe a simple bosonic SPT edge [5][6]. Consider 2π periodic fields $\boldsymbol{\phi}(x,t) = (\phi_1, \phi_2)$ whose dynamics is determined by the Lagrangian density:

$$\mathcal{L}_\phi = \partial_x \boldsymbol{\phi}^T \sigma^x \partial_t \boldsymbol{\phi} - \partial_x \boldsymbol{\phi}^T V_\phi \partial_x \boldsymbol{\phi} \tag{6}$$

where σ^x is the usual Pauli matrix and V_ϕ is an undetermined 2×2 matrix of parameters. It might seem a little strange to use fields that only take values on $[0, 2\pi)$, but they resolve several complications that arise if we use the base boson degrees of freedom. In particular, the edge description in terms of a Lagrangian density becomes exceedingly complicated due to interactions. To recover the original bosonic creation/annhilation operators, we simply exponentiate linear combinations of the $\boldsymbol{\phi}$ fields.

The $\boldsymbol{\phi}$ fields are endowed with the following commutation relations:

$$[\partial_x \phi_i(x), \partial_y \phi_j(y)] = 2\pi i (\sigma^x)_{ij} \partial_x \delta(x-y) \tag{7}$$

This algebraic structure is referred to as the "Kac-Moody algebra" and is chosen to reproduce the commutation relations for the original bosonic operators.

In a similar vein, we can describe the edge of a 2+1D fermionic SPT using the bosonized fields $\boldsymbol{\theta}(x,t) = (\theta_1, \theta_2)$ with the corresponding Lagrangian and commutation relations:

$$\mathcal{L}_\theta = \partial_x \boldsymbol{\theta}^T \sigma^z \partial_t \boldsymbol{\theta} - \partial_x \boldsymbol{\theta}^T V_\theta \partial_x \boldsymbol{\theta} \tag{8}$$
$$[\partial_x \theta_i(x), \partial_y \theta_j(y)] = 2\pi i (\sigma^z)_{ij} \partial_x \delta(x-y) \tag{9}$$

Our full theory is described by the sum of the two Lagrangians $\mathcal{L}_0 = \mathcal{L}_\phi + \mathcal{L}_\theta$. We endow the system with a $\mathbb{Z}_4$ symmetry by requiring invariance under the transformation:

$$\boldsymbol{\phi} \to \boldsymbol{\phi} + \pi, \quad \boldsymbol{\theta} \to \boldsymbol{\theta} + \pi/2 \tag{10}$$

Note that the $\boldsymbol{\phi}$ fields transform trivially under a $\mathbb{Z}_2$ subgroup of the full symmetry - this will be important later. $\mathcal{L}_0$ trivially satisfies the symmetry requirements since only derivative terms appear in it. However, our system right now isn't particularly interesting - it's two SPT edges stacked on top of each other. We proceed to couple them together as follows - (i) First, we obtain just the boson SPT edge for $x \geq 0$ by gapping out the $\boldsymbol{\theta}$ fields (ii) Then, we trivialize the entirety of the system for $x \leq 0$ by gapping out all fields (iii) We examine the interface at $x = 0$ to determine the edge mode and diagnose the topological character of the phase.

In order to gap out degrees of freedom, we need to add additional terms to the Lagrangian that are symmetry allowed. Consider the term $M_1 cos(\theta_1 - \theta_2)$ (referred to in the literature as a Higgs term). It is invariant under the $\mathbb{Z}_4$ symmetry and corresponds to back-scattering between the θ_1 and θ_2 fields. Adding this term pins the $\boldsymbol{\theta}$ fields, thereby gapping them out and leaving just the gapless boson SPT edge. We can see this by taking the coupling constant M_1 to be very large. In order to minimize the action, the cosine value is pinned to $-M_1$. This pins the ground state expectation value of the $\boldsymbol{\theta}$ fields to 0, effectively killing those degrees of freedom at low energy.

Now consider the Higgs terms $L_1 cos(\phi_1 + \theta_1 + \theta_2)$

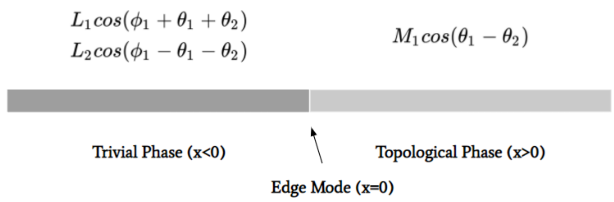

FIG. 3: A pictorial representation of the system in consideration. The Higgs terms are chosen for $x < 0$ and $x > 0$ such that we have a trivial and topological phase respectively. The edge mode is found at the interface of the two.

and $L_2 cos(\phi_1 - \theta_1 - \theta_2)$. Both of them are symmetry allowed and it can be verified from the commutation relations that they are mutually commuting. Thus, we can simultaneously pin both the combination of fields that appear in the arguments of the cosines. This gaps out all the fields, trivializing the system.

The Lagragian for our model is now given by:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \tag{11}$$
$$\begin{aligned} \mathcal{L}_{int} = & M_1 \Theta(x) cos(\theta_1 - \theta_2) \\ & + \Theta(-x) L_1 cos(\phi_1 + \theta_1 + \theta_2) \\ & + \Theta(-x) L_2 cos(\phi_1 - \theta_1 - \theta_2) \end{aligned} \tag{12}$$

IV. EDGE MODES AND PARITY STRINGS

We can construct the edge modes by looking at the fields pinned by the Higgs term. For $x \leq 0$, note that we can add together the two pinned fields to obtain:

$$2\langle\phi_1\rangle = 0 \implies \langle\phi_1\rangle = n\pi \tag{13}$$

Thus we can construct an operator $e^{i\phi_1(0)}$ that is localized to the edge, commutes with the topological phase of the wire ($x \geq 0$) and has charge ± 1. This operator measures the $\mathbb{Z}_2$ symmetry charge of the edge mode (corresponds to the σ^z operator of a spin-1/2 system). To further investigate the edge, we need to obtain the action of the on-site symmetry on the edge.

We do this by finding an operator that is conjugate to $e^{i\phi_1}$. Naively, we might take this to be $e^{i\phi_2}$. However, this does not commute with the Higgs terms for $x \leq 0$. Thus, the conjugate must include $\boldsymbol{\theta}$ fields to compensate and is given by $e^{\frac{i}{2}[\phi_2+(\theta_1-\theta_2)/2]}$. This operator at $x = 0$ corresponds to the σ^x operators and flips the edge between the two ± 1 charges. Observe that, unlike the topological phase ($x \geq 0$), the edge appears to transform non-trivially under the full $\mathbb{Z}_4$ symmetry. This demonstrates the intrinsically gapless topological character, as seen in the Ising-Hubbard model as well.

Finally, we construct the parity string operator to probe the long range order of the system. Parity acts trivially on bosonic operators and with a minus sign on fermionic operators. Knowing that these operators are obtained by exponetiating the $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ fields, we can construct parity operators by translating them by 2π and π. As an analogy, consider the canonical commutation relations for position and momentum operators $[\hat{x}, \hat{p}] = i$. The momentum operator is said to generate translations in position, which reads $|x + a\rangle = e^{i\hat{p}a}|x\rangle$ Thus, we have for the parity string:

$$\prod_{[x,y]} \hat{P} = e^{i\int_x^y dx'[\partial_{x'}(\phi_1+\phi_2)+(\theta_1-\theta_2)/2]} \tag{14}$$

$$= e^{i[(\phi_1+\phi_2)+(\theta_1-\theta_2)/2]_x^y} \tag{15}$$

Within the topological phase ($x \geq 0$), we can see that the combination of $\boldsymbol{\theta}$ fields is pinned to 0, collapsing the parity string to $e^{i[(\phi_1+\phi_2)]_x^y}$. By attaching the $\mathbb{Z}_2$ charged operator to either end of the string, we recover the string order of the Ising-Hubbard model.

V. CONCLUSION

In this paper, we introduced the concept of an intrinsically gapless topological phase and studied the Ising-Hubbard model as a candidate. We showed that the model exhibits a topological phase under doping away from half-filling, characterized by a gapless bulk with long-range string order. A feature of this phase was that the edge transforms under the full $\mathbb{Z}_4$ symmetry of the system, whereas the bulk transforms only under a $\mathbb{Z}_2$ subgroup. Thus, the edge modes are protected by the $\mathbb{Z}_4$ on-site rotational symmetry of the system.

We then provided an effective field theory description of a 1+1D IGSPT by looking at the surface of a "failed" 2+1D SPT. We used the standard K-marix formulation to describe the edge theory and tuned the phases by coupling to Higgs terms. This allowed us to construct the edge modes and explicitly obtain their symmetry properties.

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010)
[2] Xiao-Gang Wen, Advances in Physics, 44, 405 (1995)
[3] Chong Wang, Andrew C. Potter, T. Senthil, Science 343, 6171 (2014)
[4] Ryan Thorngren, Ashvin Vishwanath, Ruben Verresen, Phys. Rev. B 104, 075132 (2021)
[5] Yuan-Ming Lu, Ashvin Vishwanath, Phys. Rev. B 86, 125119 (2012)
[6] Chong Wang, T. Senthil, Phys. Rev. B 86, 125119 (2012)
[7] The model is actually symmetric to doping with electrons or holes. We consider electrons here for brevity, but the arguments hold for hole-doping as well