How Majorana fermions can be used for quantum computing

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Abstract

Majorana paricles are a special type of fermion as they are their own antiparticle. In this article, we provide a first introduction to the theory of Majorana fermions before we discuss the emergent phenomenon of their non-abelian exchange statistics. Finally we delineate how Majoranas could in theory be used to execute quantum computations.

1 Introduction

In the wake of his research on neutrino masses, in his final article the Italian particle physicist Ettore Majorana[1] presented an alternative representation of the relativistic Dirac equation in terms of real wave functions. In his representation, the wave function describes particles that are their own antiparticle, and therefore carry no charge [2]. These solutions have later been called *Majorana fermions*, owing the name to their discoverer.

Recently, Majoranas got a lot of scientific attention when several research groups published theories on how Majoranas or particles that behave likewise can be created in solid state systems. Exploiting the feature that certain zero energy excitations called Majorana zero modes (MZMs) are topologically protected from noise and decoherence and obey non-abelian statistics [3], Majoranas are in the spotlight for potential candidates for topological quantum computing because this enables them to be used to encode quantum bits that can assume values *between* 0 and 1.

2 Majorana fermions

Although invented as a theory for the electron, the Dirac equation (DE)

$$(i\gamma^\mu \partial_\mu - M)\psi(\mathbf{x}) = 0, \qquad (1)$$

where $\gamma^{0\ldots3}$ are four 4×4 matrices that obey the Dirac algebra given by the anticommutator[2]

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbb{1}_4, \qquad (2)$$

does per se not 'know' anything about electrodynamics or electric charges [4]. In fact, there exists an infinitely large set of bases for γ^μ for which the Dirac equation is entirely real-valued. The most famous example is the *Majorana representation*[3], in which

$$\gamma^0 = \begin{pmatrix} 0 & -i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & i\sigma^0 \\ i\sigma^0 & 0 \end{pmatrix}, \qquad (3)$$
$$\gamma^2 = \begin{pmatrix} i\sigma^0 & 0 \\ 0 & -i\sigma^0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}.$$

[1] After that, E. Majorana disappeared while on a cruise from Palermo to Naples in 1938. His fate has never been fully resolved and several articles have been written that explore the mystery itself. For more on this, we recommend Reference [1].

[2] The minus sign on the right hand sign is caused by the 'mostly plus'-convention we use in this article.

[3] The Majorana representation can be mapped to any other entirely real-valued representation by orthogonal transformations. Therefore, various different definitions can be found in literature.

Inserting these into Eq. (1) and carrying out the sum yields a linear system of partial differential equations of *purely real* coefficients, telling us that the particle the Dirac equation describes is electrically neutral. This is contrasted to the assumption coming from the Schrödinger equation that the wave function should be complex-valued[4] [4].

In order to gain further insight, we now decompose the field operator for the Dirac fermion into the eigenstate $e^{\mp iEt}\phi_{\pm E}$ of the stationary Dirac equation at energy E,

$$\hat{\psi}(\mathbf{x}) = \sum_{E>0} a_E e^{-iEt}\phi_E(\mathbf{x}) + \sum_{E<0} b^\dagger_{-E} e^{-iEt}\phi_E(\mathbf{x}). \tag{4}$$

Here $a^\dagger_E$ and $b^\dagger_E$ are the creation operators for the particle and antiparticle with energy E, respectively. They obey the canonical fermionic anticommutation rules

$$\{c^\dagger_\alpha, c^\dagger_\beta\} = \{c_\alpha, c_\beta\} = 0, \quad \{c^\dagger_\alpha, c_\beta\} = \delta^\beta_\alpha. \tag{5}$$

Using the fact that for every solution of the DE $\phi(\mathbf{x})$ with energy E there exists another solution of opposite charge

$$\hat{\psi}^c(\mathbf{x}) = C\hat{\psi}^*(\mathbf{x}) \tag{6}$$

with energy $-E$, where C is the charge conjugation matrix[5] and * denotes complex conjugation, we can recast Eq. (4) into a sum over only positive energies

$$\hat{\psi}(\mathbf{x}) = \sum_{E>0} a_E e^{-iEt}\phi_E(x) + b^\dagger_E e^{iEt}\phi^c(\mathbf{x}). \tag{7}$$

This relation predicts the most general particle-antiparticle pair of Dirac fermions, in which the particle is distinguishable from the antiparticle as it is oppositely charged [5]. If we now require the field operator to be real-valued as in Eq. (6), we straightforwardly find $a^\dagger_E = b^\dagger_E$, indicating we obtain a pair of chargeless particles that are *their own antiparticle*, called *Majorana fermions.*

[4]From a physical point of view a complex wave function becomes necessary when coupling the electron to an electromagnetic field, though.

[5]For the Majorana representation, $C = \mathbb{1}_4$.

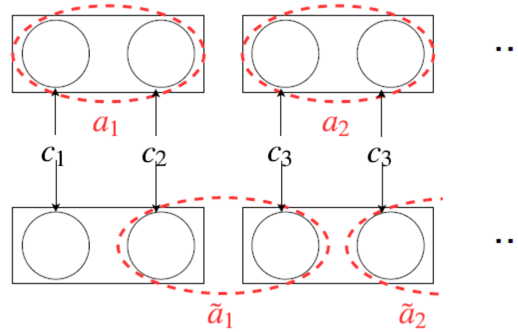

Figure 1. Kitaev's model of a p-wave superconducting tight-binding chain. Each rectangle represents an electron and the dashed circles contain two Majoranas representing a fermionic state [3].

In operator language, we call every particle a Majorana fermion if its corresponding creation and annihilation operators γ are fermionic and square to 1 (i.e. they are self-adjoint).[6] If they also commute with the Hamiltonian[7]

$$[H, \gamma] = 0, \tag{8}$$

we call the particle a *Majorana zero mode* (MZM). Eq. (8) indicates that MZMs lead to ground state degeneracies, because clearly the ground state $|\mathrm{GS}\rangle$ and the excited states $\gamma_i|\mathrm{GS}\rangle$ have the same energy [6, 7].

After they have been predicted by Kitaev [8], the first potential experimental evidence for MZMs has been achieved in semiconductor quantum wires [9] whose ends are connected to a superconductor. For a 1D-system with vanishing chem-

[6]We trust the reader to distinguish between Dirac matrices and Majorana operators albeit both are called γ. From here onward, γ refers to the operator.

[7]This is an idealization. For real physical systems, $[H, \gamma] \sim e^{-x/\xi}$, where x can be construed to be the separation between two MZMs and ξ is a correlation length associated with H. For further limitations, we refer to Reference [6].

ical potential $\mu = 0$, starting from the non-interacting tight-binding Hamiltonian with real superconducting gap $\Delta = t > 0$

$$H = \sum_n \left[-t\, a_n^\dagger a_{n+1} + \Delta\, a_n a_{n+1} + \text{h.c.} \right], \quad (9)$$

where $a_n^\dagger$ and a_n denote the usual electron creation and annihilation operators, Kitaev was able to show that this Hamiltonian can be expressed in terms of the Majorana operators as

$$H = it \sum_{n=1}^{N-1} \gamma_{j,2}\gamma_{j+1,1}. \quad (10)$$

Here $\gamma_{i,j}$ represents the j-th Majorana operator ($j = 1$ or 2) associated with the i-th electron. This transition to Majorana operators can intuitively understood by appealing to Fig. 1.

Now, in order to find the ground state, we recast this Hamiltonian in terms of the new set of fermionic operators

$$\tilde{a}_n = \frac{1}{2}(\gamma_{2j,2} + i\,\gamma_{2j+1,1}), \quad (11a)$$

$$\tilde{a}_n^\dagger = \frac{1}{2}(\gamma_{2j,2} - i\,\gamma_{2j+1,1}) \quad (11b)$$

yielding

$$H = 2t \sum_{j=1}^{N-1} \left(\tilde{a}_n^\dagger \tilde{a}_n - \frac{1}{2} \right). \quad (12)$$

The remarkable thing is that here $\gamma_{1,1}$ and $\gamma_{N,2}$ do not appear, representing *zero-energy* MZMs localized at the end of the chain. The pair encodes one Dirac fermion that is highly delocalized between the ends of the wire [3, 10, 11].

3 Non-abelian braiding

The primary significance of MZMs is that they, arising from their ground-state degeneracy, obey non-commutative exchange statistics, in this context often referred to as non-abelian braiding statistics. The existence of such non-trivial behavior has been theoretically proposed for the first time in References [12] and [13]. The origin of the term braiding becomes clear when considering the trajectories through time of two Majoranas that are being exchanged. The following derivation is oriented towards Reference [11].

For our purposes, imagine a 2D system in which a Majorana zero mode γ_2 encircles a vortex supporting another Majorana γ_1 as shown in Fig. 2. Fu and Kane envisioned this system to be realizable on a thin layer of superconducting film on the surface of a 3D topological insulator [14].[8]

Now it can be shown that there exists a normalizable zero mode with the associated Majorana wave function

$$\chi(\mathbf{r}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i(\alpha/2-\pi/4)} \\ e^{i(\alpha/2-\pi/4)} \end{pmatrix} f_0(\mathbf{r}), \quad (13)$$

where $f_0(\mathbf{r})$ is a real-valued function irrelevant for our purposes and will henceforth be ignored.

How does this wave function change when γ_2 circles around γ_1? Assume the correlation length ξ of the Hamiltonian H is negligible compared to the separation of the vortices such that $[H, \gamma_i] = 0$ is fulfilled and the orbiting process is adiabatic[9]. Then the only effect will be the change of the superconducting phase near the origin due to the phase field produced by the distant vortex. Upon inspection of Fig. 2, we express the phase change as

$$\alpha(\mathbf{R}) = \alpha_0 + \Omega(\mathbf{R}) + \pi. \quad (14)$$

α_0 denotes a constant phase offset we may tune at will to be $\alpha_0 = -\pi/2$ without changing the physics. For this choice the wave function of the Majorana mode at the origin is

$$\chi(\mathbf{r}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\,\Omega(\mathbf{r})/2} \\ e^{i\,\Omega(\mathbf{r})/2} \end{pmatrix} f_0(\mathbf{r}). \quad (15)$$

Note that since $\chi(\mathbf{R})$ is dependent on the time parameter it acquires a Berry phase which can be

[8] In their model MZMs are supported by Abrikosov vortices which can be moved about the plane through an array of Josephson junctions by tuning fluxes [6, 14].

[9] The braiding operation has to be sufficiently slow ('adiabatic') compared with the topological gap energy but at the same time fast enough (so that one is in the topologically protected regime) compared with the Majorana splitting energy [6].

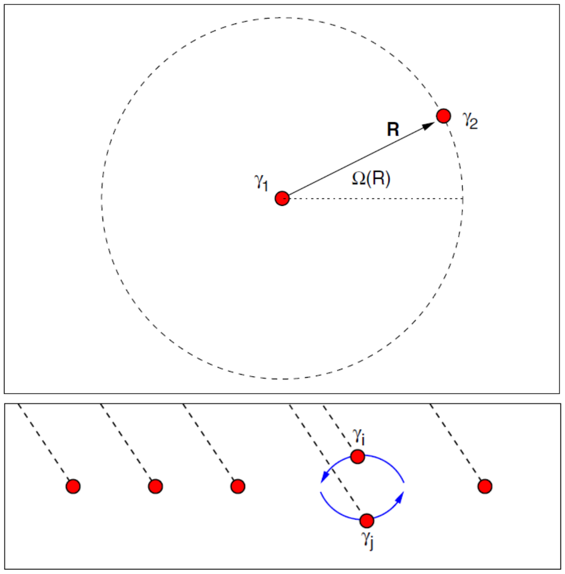

Figure 2. Top: A vortex with Majorana zero mode γ_2 encircles a vortex placed at the origin with Majorana γ_1 [11]. Bottom: The exchange of two Majoranas γ_1 and γ_2 [11].

shown to be equal to π for one full orbit, i.e. the sign of the MZM wave functions changes. Likewise the sign of the wave function of the orbiting Majorana changes because its local superconducting phase advances by 2π, so we can write the effect of one encircling run as

$$\gamma_1 \to -\gamma_1, \tag{16a}$$
$$\gamma_2 \to -\gamma_2. \tag{16b}$$

Likewise, in a system of $2N$ MZMs the adiabatic exchange of two Majoranas is governed by the following rule

$$\gamma_i \to \gamma_{i+1} \tag{17a}$$
$$\gamma_{i+1} \to -\gamma_i \tag{17b}$$
$$\gamma_k \to \gamma_k \text{ for } k \notin \{i,\, i+1\}, \tag{17c}$$

because an exchange is equivalent to one-half of the encircling operation, as can be seen in Fig. 2 [11, 15].

Following References [10], [11], and [16], we now turn to an explicit example illustrating the action of exchange transformations for a system of four MZMs.

Imagine we exchange two Majoranas γ_i and γ_j. The laws of quantum mechanics dictate that the initial and final state have to be connected by a unitary operator U, such that

$$|\psi\rangle \to U\,|\psi\rangle\,. \tag{18}$$

If this process happens sufficiently slow, i.e. if we are in the adiabatic limit, our state never leaves the ground state manifold of $2^N = 4$ states. Therefore we may write U as a 4×4 unitary matrix.

What does U look like? Since the parity of the number of electrons and the fermion parity is conserved under adiabatic exchanges, we infer U can only depend on the product of the hermitian operator $-i\,\gamma_1\gamma_2$. Moreover, the exponential of i times a hermitian operator is unitary, and we therefore conclude that, up to a global phase

$$U = e^{\beta\gamma_i\gamma_j} = \cos\beta + \gamma_i\gamma_j\,\sin\beta \tag{19}$$

where β is a yet unknown real coefficient and we used $(\gamma_i\gamma_j)^2 = -1 = i^2$. In order to determine β, we investigate the evolution of the Majorana operators

$$\gamma_k \to U\gamma_k U^\dagger \quad \forall k \in \{1,2,3,4\} \tag{20}$$

and into this equation plug in our guess for U from Eq. (19), yielding

$$\begin{pmatrix}\gamma_i\\ \gamma_j\end{pmatrix} \to \begin{pmatrix}\cos 2\beta & -\sin 2\beta\\ \sin 2\beta & \cos 2\beta\end{pmatrix}\begin{pmatrix}\gamma_i\\ \gamma_j\end{pmatrix}. \tag{21}$$

By comparison of coefficients with Eq. (17a) and (17b) we conclude $\beta = \pm\frac{\pi}{4}$ and arrive at[10]

$$U = e^{\pm\frac{\beta}{4}\gamma_i\gamma_j} = \frac{1}{\sqrt{2}}\,(1 \pm \gamma_i\gamma_j)\,. \tag{22}$$

Considering the four Majoranas come from two electrons, the ground state is 4-fold degenerate with basis states

$$|00\rangle\,, |10\rangle\,, |01\rangle\,, |11\rangle \tag{23}$$

[10] Finding two solutions for β can be explained by the possibility of exchanging the particles clock- or counter-clockwise. Henceforth, we pick to positive solution.

as a complete basis set for the 4D Hilbert space where the first and second quantum number is the occupation number of the fermionic mode $c_1^\dagger = \frac{1}{2}(\gamma_1 + i\gamma_2)$ and $c_2^\dagger = \frac{1}{2}(\gamma_3 + i\gamma_4)$, respectively. With this we can compute the action of a product of the γ_i's on the basis states of Eq. (23) to find explicit representations[11] of $U_{i,i+1} = e^{\frac{\pi}{4}\gamma_i\gamma_{i+1}}$ that exchange the Majorana i with $i+1$ which we omit for brevity and instead just present some interesting results. When exchanging particle 2 and 3 for example, we obtain a superposition of states

$$|00\rangle \to U_{23}\,|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle - i\ |11\rangle). \tag{24}$$

Recall that for ordinary bosons or fermions this operation would leave the wave function invariant up to a trivial phase ± 1, respectively. Furthermore, since the U's are not diagonal in general, they do not commute, i.e. the order of the product matters.

We conclude this section by briefly addressing the permutation of Majoranas [15], which is of great significance for the next section, where we delineate how this non-trivial behavior of Majoranas can be utilized for quantum computing. For ease of notation, let us introduce the exchange operator $T_i \equiv U_{i,i+1}$. The set of all neighboring particle-interchanges form a braid group $\mathcal{B}_n$ generated by the T_i's, where the index n denotes the number of Majoranas. $\mathcal{B}_n$ is defined by

$$T_iT_j = T_jT_i, \quad |i-j| > 1 \tag{25a}$$
$$T_iT_jT_i = T_jT_iT_j, \quad |i-j| = 1. \tag{25b}$$

The second relation is illustrated in Fig. 3.

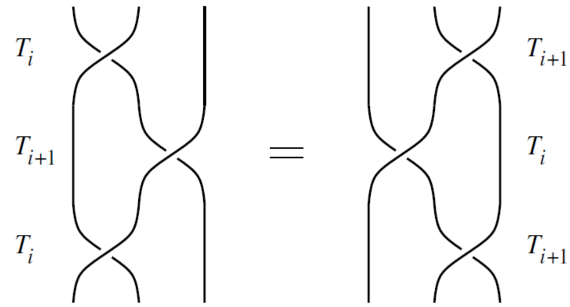

Figure 3. Defining relation for the braid group $\mathcal{B}_n$: $T_iT_jT_i = T_jT_iT_j$ [15].

4 Majoranas in quantum computing

In this section we delineate how Majorana modes can be used to execute quantum computations, which differ from classical computations by the virtue of the register that can be a superposition of different states.

A network of nanowires with $2N$ MZMs can be though of as a small processing unit. Because of the 2^N degeneracy of the ground state we can encode a string of N bits, just like a register but *non-locally.* Now we can let this quantum register execute an algorithm simply by exchanging MZMs, yielding different outcomes depending on the exchange sequence, as shown in Section 3. The stand alone feature herein is the *topological protection* of both the state of the register and the algorithm [16].

What does this mean? The fermion parity degrees of freedom encode the state of the register and are shared non-locally by the Majoranas, implying that no local perturbation can change the state. In order to understand this non-locality, recall the semiconductor nanowire connected to a p-wave superconductor as reviewed in Section 2: The unpaired Majoranas at the ends of the wire are spread out over the whole wire and thus delocalized. Topologically protected simply means that the environment can not access the information as long as the Majoranas are separated by a great enough distance, enabling immediate applications for long-lived *topological quantum memory* [10, 16].

For a more technical description and a discourse about current problems and limitations, we refer the interested reader to Reference [2] and [17].

5 Conclusion

The Dirac equation allows for allows for two fundamentally different types of solutions describing

[11] For these, see Reference [16].

a massive spin-1/2 particle. Dirac fermions such as electrons on the one hand are charged and have a distinct antiparticle, which is related to it via the charge conjugation symmetry $\mathcal{C}$. These are obtained from the "unconstrained" solution of the DE. The Majorana solution on the other hand can be obtained by demanding the time-dependent wave function to be real. Majorana fermions are thus truly neutral spin-1/2 particles that can not be distinguished from their antiparticles, named after the Italian physicist Ettore Majorana, who was the first to predict them in 1937.

From an experimental point of view, however, there is still a dispute whether Majoranas have been detected at all [18]. On both sides there is a plethora of proposals on where to find, how to detect, and how to use Majoranas. The blueprints for a Majorana-based quantum computers are therefore already in place; we simply need to begin assembling the hardware [10].

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