Bethe Ansatz for the one dimensional Heisenberg Model

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In this short article we have a look at the Bethe ansatz and how it was first used to solve the one dimensional Heisenberg model. Today variants of the Bethe ansatz are used to solve other many body problems, for example a modified Bethe ansatz can be used to solve the Kondo model. We will see how dividing the Hilbert space of the Heisenberg model into subspaces helps to first solve the simpler cases and from these the general case can then be derived.

Submission 14. December 2021

1 Introduction

The Heisenberg model named after Werner Heisenberg is a mathematical model for describing magnetic systems in condensed matter, and can be used to study critical points and phase transitions. It was proposed by Heisenberg in 1928 [1]. The model can give qualitative results for insulators but fails at the description of most metals, where the Hubbard model often is better suited. In 1931 Hans Bethe proposed the correct ansatz to solve the Heisenberg model in the one dimensional case [2].

2 Heisenberg Model

Heisenberg and Dirac discovered that the ferromagnetic properties in a solid can be described by interacting, localized electron spins on the lattice [1, 3]:

$$H = -\sum_{i,j} J_{ij} \vec{S}_i \vec{S}_i \tag{1}$$

The exchange integrals J_{ij} usually decrease very fast with distance, which is why one usually focuses on nearest neighbor interaction:

$$H = -J \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_i \tag{2}$$

For positive $J > 0$ the energy is minimized for parallel neighbor spins, so the system prefers ferromagnetic (FM) order. For $J < 0$ it prefers anti ferromagnetic order (AFM). In the one dimensional case, this system can be solved exactly with the Bethe ansatz.

3 Bethe Ansatz

The one dimensional Heisenberg Hamiltonian with nearest neighbor interaction and periodic boundary conditions (PBC) looks as follows:

$$H = -J \sum_{n=1}^{N} \vec{S}_n \vec{S}_{n+1} \tag{3}$$

$$= -J \sum_{n=1}^{N} \left[S_n^z S_{n+1}^z + \frac{1}{2} \left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ \right) \right] \tag{4}$$

where $S_n^{\pm} = S_n^x \pm i S_n^y$ are the spin ladder operators. This Hamiltonian acts on a Hilbert space with dimension 2^N built by the orthogonal basis functions $|\sigma_1, \ldots, \sigma_N\rangle$. The $\sigma_n = \{\uparrow, \downarrow\}$ represent up and down spin at site n along the arbitrarily chosen z-axis. Choosing the z-axis as quantization axis implies that the z-component of the total spin $S^z = \sum_{n=1}^{N} S_n^z$ is conserved, which can be easily checked with the spin commutation relations:

$$\left[S_n^z, S_m^{\pm}\right] = \pm S_n^{\pm} \delta_{nm} \,, \tag{5}$$

$$\left[S_n^+, S_m^-\right] = 2 S_n^z \delta_{nm} \,, \tag{6}$$

$$[H, S^z] = 0 \,. \tag{7}$$

This allows us to separate the Hilbert space into $N+1$ independent subspaces distinguished by the number of down spins r [2, 4].

3.1 The Case $r = 0$

The subspace with $r = 0$ down spins contains only the state with all spins up $|GS\rangle = |\uparrow, \ldots, \uparrow\rangle$. This

state has the eigenenergy:

$$H\left|GS\right\rangle = -\frac{JN}{4}\left|GS\right\rangle \,, \tag{8}$$

and forms the ground state with energy $E_0 = -JN/4$ [3].

3.2 The Case $r = 1$

The basis states in the $r = 1$ subspace are labeled by the position of the down spin:

$$\left|n\right\rangle = S_n^-\left|GS\right\rangle \,. \tag{9}$$

It can be seen that these aren't eigenvectors of the Hamiltonian:

$$\begin{aligned} H\left|n\right\rangle &= -J\sum_{m=1}^{N} S_m^z S_{m+1}^z S_n^-\left|GS\right\rangle \\ &\quad + \frac{1}{2}\left(S_m^+ S_{m+1}^- + S_m^- S_{m+1}^+\right) S_n^-\left|GS\right\rangle \\ &= -J\left(\frac{N}{4} - \frac{2}{2}\right)\left|n\right\rangle \\ &\quad - \frac{J}{2}\left(+\left|n+1\right\rangle + \left|n-1\right\rangle\right) \end{aligned} \tag{10}$$

Therefore we consider superpositions of these states:

$$\left|\psi\right\rangle = \sum_{n=1}^{N} a(n)\left|n\right\rangle \,, \tag{11}$$

which satisfy the eigenvalue equation $H\left|\psi\right\rangle = E\left|\psi\right\rangle$ if $a(n)$ satisfy the following N equations:

$$(E - E_0)a(n) = \frac{J}{2}\left(2a(n) - a(n-1) - a(n+1)\right). \tag{12}$$

The solutions are plane waves:

$$a(n) = e^{ikn} \quad \text{with} \quad k = \frac{2\pi}{N}\lambda\,, \tag{13}$$

for $\lambda = 0, \ldots, N-1$. From this follows the normalized eigenvectors:

$$\left|\psi\right\rangle = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{ikn}\left|n\right\rangle \,, \tag{14}$$

with energies:

$$\begin{aligned} E &= E_0 + \frac{J}{2}\left(2 - e^{-ik} - e^{ik}\right) \,, \\ E &= E_0 + J\left(1 - \cos(k)\right) \,. \end{aligned} \tag{15}$$

3.3 The Case $r = 2$

For $r = 2$ we consider the following eigenvectors and assume without loss of generality $n_1 < n_2$:

$$\left|\psi\right\rangle = \sum_{n_1<n_2}^{N} a(n_1, n_2)\left|n_1, n_2\right\rangle \,, \tag{16}$$

with the states

$$\left|n_1, n_2\right\rangle = S_{n_1}^- S_{n_2}^-\left|GS\right\rangle \,. \tag{17}$$

When inserting (16) into the eigenvalue equation $H\left|\psi\right\rangle = E\left|\psi\right\rangle$ we have to consider two cases [2]. Either the down spins are separated from another:

$$\begin{aligned} (E - E_0)\, a(n_1, n_2) = \frac{J}{2}\,&(4a(n_1, n_2) - a(n_1 - 1, n_2) \\ &- a(n_1 + 1, n_2) - a(n_1, n_2 - 1) \\ &- a(n_1, n_2 + 1)) \,, \end{aligned} \tag{18}$$

or they are on neighboring sites:

$$\begin{aligned} (E - E_0)\, a(n_1, n_2) = \frac{J}{2}\,&(2a(n_1, n_2) - a(n_1 - 1, n_2) \\ &- a(n_1, n_2 + 1)\ ) \,. \end{aligned} \tag{19}$$

The Bethe ansatz for $a(n_1, n_2)$ are plane waves with-still unknowen amplitudes [2]:

$$a(n_1, n_2) = C_1 e^{i(k_1 n_1 + k_2 n_2)} + C_2 e^{i(k_2 n_1 + k_1 n_2)} \,, \tag{20}$$

This solves the equation (18) for arbitrary C_1, C_2, k_1, k_2 with the energy:

$$E = E_0 + J\sum_{i=1,2}\left(1 - \cos(k_i)\right) \tag{21}$$

Equation (19) is not automatically satisfied. By subtracting (19) from (18) for $n_2 = n_1 + 1$ one gets N conditions:

$$a(n_1, n_1) + a(n_1 + 1, n_1 + 1) = 2a(n_1, n_1 + 1)\,. \tag{22}$$

The term $a(n_1, n_1)$ doesn't have a concrete physical interpretation but is defined through (20). Inserting the Bethe ansatz (20) into equation (22) one gets a ratio for the amplitudes:

$$e^{i\phi} \equiv \frac{C_1}{C_2} = -\frac{e^{i(k_1+k_2)} + 1 - 2e^{ik_1}}{e^{i(k_1+k_2)} + 1 - 2e^{ik_2}}\,. \tag{23}$$

This is embedded in the ansatz by setting the amplitudes to

$$C_1 = e^{i\frac{\phi}{2}} \,, \quad C_2 = e^{-i\frac{\phi}{2}} \,, \tag{24}$$

where the phase depends on the k_1, k_2 as follows:

$$2\cot\left(\frac{\phi}{2}\right) = \cot\left(\frac{k_1}{2}\right) - \cot\left(\frac{k_2}{2}\right), \tag{25}$$

giving the ansatz:

$$a(n_1, n_2) = e^{i(k_1 n_1 + k_2 n_2 + \frac{1}{2}\phi)} + e^{i(k_2 n_1 + k_1 n_2 - \frac{1}{2}\phi)}\,. \tag{26}$$

Now we have to consider the PBC:

$$a(n_1, n_2) = a(n_2, n_1 + N)\,, \tag{27}$$

where the notation $a(n_2, n_1+N)$ comes from the fact that we always list the n_i in the order of size. Inserting (26) into (27) gives the following equation:

$$\begin{aligned} &e^{i(k_1 n_1 + k_2 n_2 + \frac{1}{2}\phi)} + e^{i(k_2 n_1 + k_1 n_2 - \frac{1}{2}\phi)} \\ &= e^{i(k_1 n_2 + k_2 (n_1+N) + \frac{1}{2}\phi)} + e^{i(k_2 n_2 + k_1 (n_1+N) - \frac{1}{2}\phi)} \end{aligned} \tag{28}$$

This must be true for all n_1 and n_2 and therefore leads to the conditions:

$$Nk_1 = 2\pi\lambda_1 + \phi\,, \tag{29}$$

$$Nk_2 = 2\pi\lambda_2 - \phi\,, \tag{30}$$

$$\lambda_1, \lambda_2 = 0, \ldots, N-1\,. \tag{31}$$

The k_1 and k_2 individually do not have the usual form $2\pi\lambda/N$, but their sum does

$$k = k_1 + k_2 = \frac{2\pi}{N}(\lambda_1 + \lambda_2) \tag{32}$$

While k_1 and k_2 specify the Bethe ansatz wave function, the wave number k is the quantum number connected to translation of both down spins. Now we want to discuss the phase ϕ as a function of k_1 and k_2. For $k_1 = k_2$ we can distinguish into two cases:

$$\phi = +\pi\,, \qquad \lambda_1 = \lambda_2 - 1 = \frac{Nk_1}{2N} - \frac{1}{2}\,, \tag{33}$$

$$\phi = -\pi\,, \qquad \lambda_1 = \lambda_2 + 1 = \frac{Nk_1}{2N} + \frac{1}{2}\,. \tag{34}$$

In both cases the amplitudes (26) are zero for all n_1, n_2:

$$a(n_1, n_2) = e^{ik_1(n_1+n_2)} \cos\Big(\frac{\pi}{2}\Big) = 0\,. \tag{35}$$

Therefore $k_1 = k_2$ does not lead to a sensible solution and we find for a given λ_2:

$$\lambda_1 \neq \lambda_2, \lambda_2 + 1, \lambda_2 - 1\,. \tag{36}$$

Since we have the permutation symmetry $(k_1, k_2) = (k_2, k_1)$ we can assume without loss of generality that $k_1 < k_2$. From this follows, that for a given λ_2 we get $\lambda_2 - 1$ solutions $\lambda_1 = 0, 1, \ldots, \lambda_2 - 2$. With λ_2 going from 2 to $N-1$ we get the total amount of solutions:

$$\sum_{k_2=2}^{N-1} \lambda_2 - 1 = \binom{N-1}{2}\,. \tag{37}$$

It's no surprise that there should be as many solutions as spin distributions $\binom{N}{2}$, so we are missing $N-1$ solutions. We can find these solutions if we consider k_1, k_2 as a complex conjugate pair:

$$k_1 = u + iv\,, \qquad k_2 = u - iv\,. \tag{38}$$

This is further elaborated in the original paper [2]. We see that all eigenstates can be specified by a set of quantum number with which they can be distinguished according to their physical properties.

3.4 The Case for arbitrary r

We now proceed to the general case of r down spins. Generalizing the eigenstates (16):

$$|\psi\rangle = \sum_{n_1 < \ldots < n_r}^{N} a(n_1, \ldots, n_r)\, |n_1, \ldots, n_r\rangle\,. \tag{39}$$

The conditional equations of the eigenvalue equations again decompose into different types. For no neighboring down spins we get:

$$\begin{aligned} (E - E_0)\, a(n_1, \ldots, n_r) = \frac{J}{2} \sum_{i=1}^{r} (&a(n_1, \ldots, n_r) \\ &- a(n_1, \ldots, n_i + 1, \ldots, n_r) \\ &- a(n_1, \ldots, n_i - 1, \ldots, n_r))\,. \end{aligned} \tag{40}$$

and for neighboring down spins we get:

$$\begin{aligned} &(E - E_0)\, a(n_1, \ldots, n_r) = \\ &\frac{J}{2} \sum_{i \neq j_\nu, j_\nu + 1}^{r} \sum_{\sigma} (a(n_1, \ldots, n_r) - a(n_1, \ldots, n_i + \sigma, n_r)) \\ &+ \frac{J}{2} \sum_{\alpha} (2a(n_1, \ldots, n_r) \qquad\qquad (41) \\ &\qquad - a(n_1, \ldots, n_{j_\nu} - 1, n_{j_\nu+1}, \ldots, n_r) \\ &\qquad - a(n_1, \ldots, n_{j_\nu}, n_{j_\nu+1} + 1, \ldots, n_r))\,, \qquad (42) \end{aligned}$$

We consider the ansatz:

$$a(n_1, \ldots, n_r) = \sum_{\mathcal{P}}^{r!} \exp\left(i \sum_{j=1}^{r} k_{\mathcal{P}_j} n_j + \frac{i}{2} \sum_{j<l} \phi_{\mathcal{P}_j \mathcal{P}_l} \right), \tag{43}$$

where $\mathcal{P}$ is any permutation of the r numbers and $\mathcal{P}_i$ is the number that is placed at position i in this permutation. This gives the energy:

$$E = E_0 + J \sum_{i=1}^{r} (1 - \cos(k_i))\,. \tag{44}$$

The ansatz satisfies the first equation type (40). The other equations are satisfied by requiring:

$$\begin{aligned} &2a(n_1, \ldots, n_{j_\nu}, n_{j_\nu} + 1, \ldots, n_r) = \\ &\quad a(n_1, \ldots, n_{j_\nu}, n_{j_\nu}, \ldots, n_r) + \\ &\quad a(n_1, \ldots, n_{j_\nu} + 1, n_{j_\nu} + 1, \ldots, n_r)\,, \end{aligned} \tag{45}$$

analogously to section 3.3, for $\nu = 1, \ldots, r$. These equations again leed to a phase factor:

$$e^{i\phi_{ij}} \equiv -\frac{e^{i(k_i + k_j)} + 1 - 2e^{ik_i}}{e^{i(k_i + k_j)} + 1 - 2e^{ik_j}}\,, \tag{46}$$

that relates the phase ϕ_{ij} to the k_i. We use again the periodic boundary conditions:

$$a(n_1, \ldots, n_r) = a(n_2, \ldots, n_r, n_1 + N) \tag{47}$$

giving the following N equations:

$$\sum_{j=1}^{r} k_{\mathcal{P}_j} n_j + \frac{1}{2}\sum_{j<l}\phi_{\mathcal{P}_j\mathcal{P}_l} = \sum_{j=2}^{r} k_{\mathcal{P}'_{j-1}} n_j + k_{\mathcal{P}'_r}(n_1+N) + \frac{1}{2}\sum_{j<l}\phi_{\mathcal{P}'_j\mathcal{P}'_l} - 2\pi\lambda_{\mathcal{P}'_r}\,, \tag{48}$$

where the permutations have the relation $\mathcal{P}'_{j+1} = \mathcal{P}_j$ and $\mathcal{P}'_r = \mathcal{P}_1$. Considering that all terms that do not have the index $\mathcal{P}'_r = \mathcal{P}_1$ cancel out, we get r relations between k and ϕ:

$$Nk_i = 2\pi\lambda_i + \sum_{j\neq i}\phi_{ij}\,, \tag{49}$$

with $\lambda_i = 0, \ldots, N-1$, which is similar to the case $r = 2$. The sum of the k_i give again the wave number:

$$k = \frac{2\pi}{N}\sum_{i=1}^{r}\lambda_i\,. \tag{50}$$

Analogously to section 3.3, one can show that no two k_i can be the same because the coefficients $a(n_1, \ldots, n_r)$ would vanish. And for real valued k_i two successive λ_i have to have a difference of two. Therefore the amount of real valued solutions is:

$$\binom{N-r+1}{r}\,, \tag{51}$$

which is much smaller than $\binom{N}{r}$. So one can consider again the complex k_i which give the missing solutions. This is further elaborated in the original paper [2].

4 Summary

The Bethe ansatz is shown to be the correct ansatz to solve the one dimensional Heisenberg model. We saw that the Hilbert space of the Heisenberg model can be separated into subspaces that can be distinguished by the number of down spins relative to the quantization axis. The ground state $|GS\rangle = |\uparrow, \ldots, \uparrow\rangle$ is a state with all spins up respective to a chosen axis. Looking at subspaces of one and two down spins helped to derive the general case for arbitrary down spins.

References

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