Critical Phenomena and the Renormalization Group

Petar Simidzija[1]

[1]*Department of Physics and Astronomy, University of British Columbia, Vancouver, BC, Canada*

We study second-order phase transitions, otherwise known as *critical phenomena*, by computing their critical exponents — six numbers which quantitatively describe the physics of the phase transition. After discussing the classical thermodynamics derivation of the critical exponents and discussing its shortcomings, we introduce a powerful technique known as the *renormalization group*, which allows for a more accurate prediction of critical exponents. We illustrate this technique using an example system — the 2D Ising model.

Critical Phenomena— Phase transitions are one of the most interesting phenomena in thermodynamic systems. A prototypical example of a phase transition is the liquid-vapor transition in water. To visualize this transition, it is useful to consider the phase diagram of water, as shown in Fig. 1. In this diagram, the liquid-vapor transition occurs on the liquid-vapor *coexistence curve*, which begins at the triple-point of water (273 K, 612 Pa), and terminates at the *critical point*: $T_c = 647.29$ K and $P_c = 22.09$ MPa. The transition between liquid and vapor that occurs on the bulk of this coexistence curve (e.g. the transition $A \to B$ in Fig. 1) is called a *first-order* phase transition, and a characteristic feature of such a transition is a finite latent heat. Meanwhile, the transition $C \to D$ in Fig. 1, occurring precisely at the critical point, is called a second-order phase transition. Beyond the critical point there is no distinction between the liquid and vapor phases, and so a second-order transition is more subtle than the first-order kind. In this paper we will study the physics of second-order phase transitions, which, because they occur at the critical point, are also known as a form of *critical phenomena*.
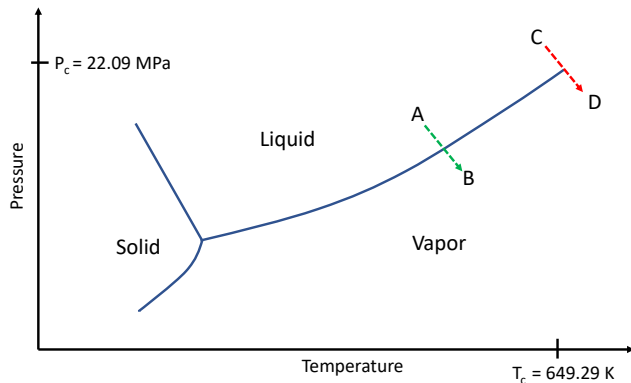

FIG. 1. Phase diagram of water, showing the first-order liquid-vapor phase transition $A \to B$, and the second-order phase transition $C \to D$. The latter passes through the critical point (T_c, P_c), at which the liquid-vapor coexistence curve terminates.

Besides the liquid-vapor transition in water, there are many other examples of critical phenomena, all of which occur at the endpoints of coexistence curves. These include the ferromagnetic-paramagnetic phase transition in Fe, the superconductivity transition in Pb and other superconductors, the order-disorder transition in Cu-Zn, and many others [1].

In each of these critical systems, there is an extensive quantity M, called the *order parameter*, which characterizes the phase that the system is in. For instance, in the liquid-vapor transition we can define M to be the molar volume, such that M is larger in the vapor state and smaller in the liquid state. Another example is the magnetic system Fe, where we take M to be the magnetization: M is zero in the paramagnetic phase, and nonzero in the ferromagnetic phase. Furthermore, we denote the variable conjugate to M by h, which, as usual in thermodynamics, plays the role of a "generalized force". For example, in the liquid-vapor transition h is the pressure difference between the phases, while in the magnetic transition h is the applied magnetic field.

Although the critical phenomena we described above occur in very different physical systems, they share many features in common. For example, experiments show that in any critical system near the critical temperature T_c, the heat capacity $C \equiv (dQ/dT)_M$, the order parameter M, and the *generalized susceptibility* $\chi \equiv -[(\partial M/\partial h)_T]/M$ behave as

$$C \sim \frac{1}{(T - T_c)^\alpha}, \tag{1}$$

$$M \sim (T_c - T)^\beta, \tag{2}$$

$$\chi \sim \frac{1}{(T - T_c)^\gamma}, \tag{3}$$

as we move up the coexistence curve towards the critical point. Meanwhile, at $T = T_c$, as a function of h the order parameter M behaves as

$$M \sim h^{1/\delta}. \tag{4}$$

The values $\alpha, \beta, \gamma, \delta$ are known as *critical exponents*, and understanding the physics of critical phenomena requires their computation. In particular, we might naively expect that different critical systems possess different critical exponents — after all, a metal becoming superconducting is clearly very different from a liquid evaporating. However, we will discuss how this naive expectation is often incorrect, and that very different physical systems often possess the same critical exponents.

Landau Theory— Landau described the physics of critical phenomena by using classical thermodynamics [1, 2]. The starting point of his calculation is to recall that the equilibrium state of a thermodynamic system with order parameter M occurs at the point where the appropriate *free energy* $F(M)$ is minimized. For example, in a fluid at constant temperature T and pressure P, we take F to be the Gibbs free energy, and recall that M is the molar volume. Then $F(M)$ will have two local minima; one at a smaller value of M corresponding to the liquid phase, and one at larger M corresponding to the vapor phase. The system will then randomly fluctuate between these local minima, but, in the thermodynamic limit, it will spend an overwhelming majority of its time in the phase corresponding to the global minimum of F, i.e. the lower of the two local minima. If we adjust the conditions (such as T and P) so that the two local minima of F are equal, as in Fig. 2-a, then the system is equally likely to be in both phases — this corresponds to being on the coexistence curve in a phase diagram, on which two phases are in equilibrium.
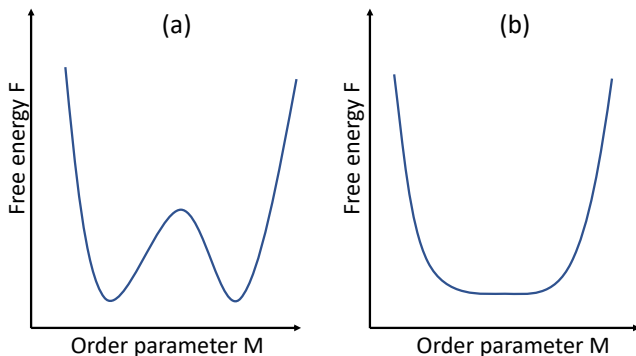

FIG. 2. Plots showing the free energy F as a function of order parameter M for a system located (in phase space) on the coexistence curve separating two phases. Plot (a) is representative of a system on the coexistence curve, but away from the critical point. Plot (b), with a characteristically flat bottom to the free energy surface, is representative of a system precisely at the critical point.

As we move up along the coexistence curve towards the critical point, the barrier separating the two local minima decreases, and precisely at the critical point the barrier vanishes and F develops a flat bottom (Fig. 2-b). Here, the system is able to fluctuate along the flat bottom at no free energy cost. We thus expect that a system at criticality undergoes large-scale fluctuations of its order parameter. This prediction is strongly corroborated by experiments.

With this perspective, Landau was able to compute the critical exponents $\alpha, \beta, \gamma, \delta$. He noted that the qualitative form of the free energy F shown in Fig. 2 leads to a Taylor expansion $F = a + bM^2 + cM^4 + \dots$ with $c > 0$, where we have assumed that there is a symmetry which makes F even (e.g. in the magnetic system, F is invariant under a simultaneous flip of all the spins). The flat bottom which develops as we move towards the critical point means that b goes from a negative value on the coexistence curve to zero at the critical point. Hence at criticality, F is of the form $a + cM^4$, i.e. it has a quartic minimum. As always, it is possible to compute all thermodynamic quantities of interest by taking derivatives of F, and in this way compute the critical exponents. In this way Landau found [1]:

$$\text{Landau:} \quad \alpha = 0, \quad \beta = 1/2, \quad \gamma = 1, \quad \delta = 3. \quad (5)$$

Thus we obtain the prediction that the critical exponents of *all* second-order phase transitions — from superconductors to magnetic systems to binary alloys, etc. — are identical. Unfortunately however, experiments show that this is generally not the case. However, Landau's argument was based on very general thermodynamic reasoning, with very few assumptions. This raises the question: where does this argument fail?

The place where Landau's argument fails is in the assumption that the equilibrium state is precisely the state which minimizes the free energy F. In actuality, there are always random fluctuations that allow a system to briefly occupy states with a higher free energy, before decaying back to the stable minimum, and the true equilibrium state is a time average over all occupied states. At a temperature β^{-1} these fluctuations are suppressed by a factor of $\exp(-\beta \Delta F)$, with ΔF the increase in free energy during the fluctuation. Thus in the case of a very deep and narrow potential well — as is almost always the case in the thermodynamic limit — these fluctuations are highly suppressed, and the equilibrium state is, to a very good approximation, the state which minimizes F. However, we saw that at criticality F develops a very wide minimum. In this case, the fluctuation-averaged equilibrium state could deviate significantly from the state with minimum F. Thus, in order to improve upon Landau's calculation of the critical exponents, we must take fluctuations into account.

Fluctuations and Scaling— We can quantify the fluctuations present in a system through a *correlation function*. As an example, we consider the Ising model of classical spins $\sigma_i \in \{\pm 1\}$ on a lattice, with Hamiltonian

$$H(k, h; \{\sigma\}) = -h \sum_i \sigma_i - k \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \quad (6)$$

where h is an external field and k is a spin-spin coupling. This system undergoes a paramagnetic-ferromagnetic second-order phase transition at a critical temperature T_c. The connected two-point correlation function is

$$\Gamma_c(i, j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle, \quad (7)$$

where the angle brackets correspond to a thermal average, and it quantifies the correlated fluctuations between spins i and j. Away from T_c, it can be shown that the correlation function decays exponentially with the distance

$|i-j|$ between sites i and j, namely [3],

$$\Gamma_c(i,j) \sim \exp\left(-\frac{|i-j|}{\xi(T)}\right) \quad \text{for } T \neq T_c, \qquad (8)$$

where $\xi(T)$ is a temperature dependent *correlation length*. On the other hand, we expect that at T_c the fluctuations are large, i.e. sub-exponential in $|i-j|$,

$$\Gamma_c(i,j) \sim \frac{1}{|i-j|^{d-2+\eta}} \quad \text{for } T = T_c. \qquad (9)$$

Here we have written the exponent in a conventional form, where d is the dimension of the system and η is an unknown parameter. In order to have Eq. (8) approach Eq. (9) as $T \to T_c$ we must have [3]

$$\xi(T) \sim \frac{1}{|T-T_c|^\nu} \quad \text{for } T \text{ near } T_c. \qquad (10)$$

Thus, in order to quantify the fluctuations we introduced two additional unknown exponents, ν and η. To see why this is beneficial, we first define the free energy per unit volume, $f \equiv F/V$. For the Ising model (6) we can write f in terms of h and k, or equivalently in terms of $\delta k \equiv k - k_c$ and $\delta h \equiv h - h_c$, where k_c and h_c are the values of k and h at the critical point. Then, it can be shown that in order for f to give the required forms of the correlators in Eqs. (8) and (9), near $T = T_c$ it must satisfy the *scaling relation* [3]

$$f(\lambda^{ad}\delta k, \lambda^{bd}\delta h) \overset{!}{=} \lambda^d f(\delta k, \delta h), \quad \text{for any } \lambda > 0, \quad (11)$$

where a and b are some unknown constants. Then, by differentiating this expression, we can compute the heat capacity, the generalized susceptibility, etc., and thus obtain all the critical exponents in terms of a and b. The result is [3]:

$$\begin{array}{lll} \alpha = 2 - \frac{1}{a}, & \beta = \frac{1-b}{a}, & \gamma = \frac{2b-1}{a}, \\ \delta = \frac{b}{1-b}, & \nu = \frac{1}{ad}, & \eta = 2 + d(1-2b), \end{array} \qquad (12)$$

Hence all six critical exponents can be written in terms of only a and b. All that is left is to find a and b.

Renormalization Group— Before computing a and b, let us first, somewhat arbitrarily, consider what happens to the correlation length $\xi(T)$ as we "zoom out" of the system, i.e. as we look at the system from further away. For $T \neq T_c$, $\xi(T)$ is finite, and if we zoom out by a factor of two, $\xi(T)$ will effectively decrease by a factor of two. However at $T = T_c$, Eq. (10) says that $\xi(T)$ is infinite, and zooming out by a finite amount does not change that fact. In other words, at criticality the correlation length is invariant under a "zooming out" procedure. (For a visual representation of a "zooming out" procedure, see Fig. 3 below.)

Let us now make the bold conjecture that, at criticality, all properties of the system are invariant under a "zooming out" procedure. More precisely, suppose our system is described by a Hamiltonian $H(t_1, t_2, \dots)$,

which is a function of some coupling constants t_i. Then, under a general "zooming out" procedure, the Hamiltonian will change to $H(t'_1, t'_2, \dots)$, where t'_i are generally different from the t_i. (In order to write the "zoomed out" Hamiltonian in the same functional form as the original Hamiltonian, we must include couplings t_i for all terms permitted by symmetries — see Sec. 12.3 in [4] for more discussion.) For reasons that come from quantum field theory, we call such a "zooming out" procedure a *renormalization group (RG) flow*. (For more on how our discussion here fits into the context of quantum field theory, see the early works by Wilson [5, 6].) Our conjecture is thus that, *at criticality, the coupling constants are invariant under RG flow.* If we view a continuous RG flow (i.e. continuous "zooming out") as continuously moving us around the space of couplings t_i, then our conjecture is that critical points are *fixed points* of the RG flow, i.e. points which are invariant under RG flow.

Before we show how this can be used to compute the values a and b, and thus all the critical exponents, let us mention that this conjecture is strongly supported by experimental evidence: real critical systems are indeed found to be scale invariant, i.e. invariant under RG flow. Furthermore, we note that the RG process of "zooming out", i.e. of ignoring the small-scale structure of the system, is equivalent to integrating out the high-energy degrees of freedom of the system (since high energy $\sim$ high momenta $\sim$ small wavelength). Hence the Hamiltonian which we obtain after RG flow, $H(t'_i)$, is analogous to the effective Hamiltonian $H_{\text{eff}} = H_0 + V + P_0 V \frac{1-P_0}{E_0 - H_0} V P_0 + \dots$, which we computed in class when we were interested in the dynamics of a degenerate low energy subspace of the Hilbert space. The only difference between what we do here and what we did in class, is that there we integrated all degrees of freedom above some degenerate ground state subspace, while here we integrate everything above some energy cutoff scale E_{cutoff}.

Example: 2D Ising Model— Let us now illustrate, using the 2D Ising model in Eq. (6), how the RG procedure can be used to compute the exponents a and b in Eq. (11). For concreteness we consider the spins σ_i to be on a triangular lattice [3], as shown in Fig. 3-a. Now, let us group the spins into blocks of three, as shown in Fig. 3-b, and define the *block spin* Σ_I at block site I by

$$\Sigma_I = \text{sgn}(\sigma_1^I + \sigma_2^I + \sigma_3^I), \qquad (13)$$

where σ_1^I, σ_2^I and σ_3^I are the original spins making up the block I. In other words, Σ_I takes on the majority value of its constituent spins, and so in a sense it represents the effective spin that we would see if we "zoomed out" a bit, i.e. if we performed an RG flow. More precisely, we are zooming out by a factor of $\lambda = \sqrt{3}$, since a unit cell increases in size by a factor of 3. Now, the original three spins σ_i^I making up block I lead to $2^3 = 8$ different states, but the block spin Σ_I by itself can only be in two states, ± 1. To represent the additional degrees of freedom, which we are "integrating out" when we perform the RG flow, we introduce the internal variable ξ_I, which

is a vector that can take on four possible values

$$\xi_I : (+,+,+),\ (+,+,-),\ (+,-,+),\ (-,+,+). \quad (14)$$

Then, the 8 possible states of the σ_i^I are obtained by multiplying the 2 possible states of Σ_I by the 4 possible states of ξ_I, namely $\sigma_i^I = \Sigma_I \xi_I^i$, where ξ_I^i is the ith component of ξ_I. For example the state $\sigma_1^I = +1, \sigma_2^I = -1, \sigma_3^I = -1$ is obtained by taking $\Sigma_I = -1$ and $\xi_I = (-,+,+)$.
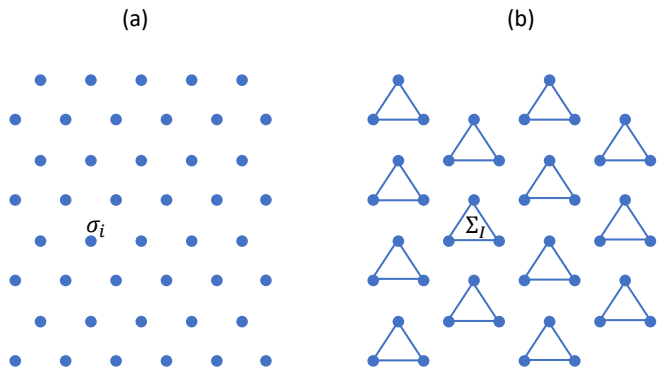

FIG. 3. (a) Triangular lattice with spin $\sigma_i \in \{\pm 1\}$ at site i. (b) Triangular lattice with block spin $\Sigma_I \in \{\pm 1\}$ at site I. If we are only interested in what happens on scales larger than the block spin spacing, then we can "zoom out", and study the effective dynamics of the Σ_I, without worrying about what happens to individual σ_i. This form of renormalization group argument was pioneered by Kadanoff [7].

To perform the RG flow, we would like to write the Hamiltonian in the same form as before, Eq. (6), only now in terms of the block spins Σ_I, i.e.

$$H(k',h';\{\Sigma\}) = -h'\sum_I \Sigma_I - k'\sum_{\langle I,J\rangle}\Sigma_I\Sigma_J, \quad (15)$$

where h' and k' are the RG flowed couplings. On scales larger than the spacing between blocks, this *block Hamiltonian* should describe the same physics as the original Hamiltonian; from the effective Hamiltonian picture we are simply integrating out the degrees of freedom on length scales smaller than the block size. To enforce this requirement, we recall that all thermodynamic information is encoded in the partition function, $Z = \sum \exp(-\beta H)$, where the sum is over all possible states of the system. Hence we require that the partition functions, written in the block picture and the original picture, are equal:

$$\sum_{\{\Sigma\}} e^{-H(k',h';\{\Sigma\})} = \sum_{\{\Sigma\},\{\xi\}} e^{-H(k,h;\{\Sigma\},\{\xi\})}, \quad (16)$$

where we have written the states $\{\sigma\}$ in terms of the block states $\{\Sigma\}$ and the internal degrees of freedom $\{\xi\}$. (We are also absorbing the inverse temperature β into a redefinition of the couplings h and k.) This equation is satisfied if, for all block configurations $\{\Sigma\}$,

$$e^{-H(k',h';\{\Sigma\})} = \sum_{\{\xi\}} e^{-H(k,h;\{\Sigma\},\{\xi\})}$$

$$\Longrightarrow H(k',h';\{\Sigma\}) = -\ln\left(\sum_{\{\xi\}} e^{-H(k,h;\{\Sigma\},\{\xi\})}\right). \quad (17)$$

Thus, to compute the renormalized couplings h' and k', we simply need to compute the right hand side of Eq. (17) and compare the result to Eq. (15). We perform this calculation in Appendix A, and find

$$k' = 2k\left(\frac{e^{3k}+e^{-k}}{e^{3k}+3e^{-k}}\right)^2, \quad h' = 3h\left(\frac{e^{3k}+e^{-k}}{e^{3k}+3e^{-k}}\right). \quad (18)$$

Recall that we would like to identify critical points with RG fixed points, i.e. values of (h,k) which remain unchanged by the transformation (18). The only non-trivial fixed point is $(h_c, k_c) = (0, \ln(1+2\sqrt{2})/4)$. Letting $\delta h = h - h_c$ and $\delta k = k - k_c$, the RG transformation near the fixed point has the linearization

$$\begin{pmatrix}\delta k' \\ \delta h'\end{pmatrix} = \begin{pmatrix}1.62 & 0 \\ 0 & 2.12\end{pmatrix}\begin{pmatrix}\delta k \\ \delta h\end{pmatrix}. \quad (19)$$

Recall that in this example we have zoomed out by a factor $\lambda = \sqrt{3}$, and so the volume per unit cell increased by a factor of $\lambda^2 = 3$. Hence the free energy per unit volume also increases by a factor of λ^2, i.e.

$$f(\delta k', \delta h') = \lambda^2 f(\delta k, \delta h). \quad (20)$$

Comparing Eqs. (19) and (20) to Eq. (11) we find $a = 0.44$ and $b = 0.68$. Finally, from Eq. (12) we obtain the critical exponents of the Ising model, as calculated by RG techniques:

$$\text{RG:}\quad \alpha = -0.27, \quad \beta = 0.72, \quad \gamma = 0.84,$$
$$\delta = 2.17, \quad \nu = 1.13, \quad \eta = 1.26. \quad (21)$$

Conclusions— To summarize, in this paper we have discussed second-order phase transitions, otherwise known as critical phenomena, which occur at the endpoints of coexistence curves in thermodynamic phase diagrams. Critical phenomena are characterized by a set of critical exponents, which dictate how physical quantities behave near the critical point. After discussing Landau's calculation of the critical exponents, which predicts the same exponents for all systems at criticality, we went on to compute the critical exponents for the 2D Ising model using a different method — the *renormalization group*. This latter approach is based on the assumption that critical points are fixed points in coupling space under the process of renormalization group flow, which we can think of as the integrating out of high energy degrees of freedom, or as "zooming out" of our system.

It turns out that the 2D Ising model has been solved exactly [8], so we can compare the Landau and RG predictions to the exact values, which are

$$\text{Exact:} \quad \alpha = 0, \quad \beta = 1/8, \quad \gamma = 7/4,$$
$$\delta = 15, \quad \nu = 1, \quad \eta = 1/4. \tag{22}$$

We see that both the Landau and RG predictions deviate significantly from the exact values. We already explained that Landau's predictions are inaccurate because of his failure to account for fluctuations, which are particularly important at criticality. In fact, our RG calculation are inaccurate for the same reason: as detailed in Appendix A, for simplicity we only kept leading order terms in our perturbative RG calculation, which amounts to neglecting fluctuations within blocks. Keeping higher order terms in this expansion would result in a more accurate RG prediction.

While we have focused on the 2D Ising model, for which the exact solution is known, RG calculations are extremely powerful because they allow us to compute critical exponents for systems which we cannot solve exactly. For example, more sophisticated RG calculations, formulated in the framework of *conformal field theory* [3], have resulted in the most precise calculation to date of the critical exponents for the 3D Ising model [9], finding, e.g., $\eta = 0.0362978(20)$. Notice that these more sophisticated methods are also able to give an error bound (in the sixth decimal digit!) to the precision RG calculation.

As a final remark, we note that unlike the Landau prediction, RG calculations show that not all critical systems have the same critical exponents. However, Landau was closer to the truth than we might naively expect: critical systems fall into broad families, known as *universality classes*, such that members of the same universality class share the same critical exponents [10]. The universality class to which a system belongs depends only on the dimension of the system, and the symmetry group of its order parameter. Hence, for example, the liquid-vapor critical phase transition in a fluid is in the same universality class as a ferromagnetic-paramagnetic phase transition in a uniaxial spin system [10]. Hence, the renormalization group allows us to simultaneously study the quantitative behaviour of these systems at criticality, even though naively we might expect the different systems to have little in common with one another. This exemplifies one reason for why the renormalization group is a very valuable tool to keep in one's theoretical physics toolbox.

Appendix A: RG Calculation for 2D Ising Model

For the interested reader, in this Appendix we will explicitly compute the Hamiltonian $H(k', h'; \{\Sigma\})$ in Eq. (17). The calculation is based on Section 3.3.2 of Ref. [3], with some alterations made to clarify the presentation. We start by writing the original Hamiltonian $H(k, h, \{\sigma\})$, given by Eq. (6), as the sum of a free part

H_0 (containing only interactions within blocks I), and an interaction part V (containing interactions between different blocks I and J, and also interactions with the external field):

$$H_0 = -k \sum_I \sum_{\langle ij \rangle} \sigma_i^I \sigma_j^I, \tag{A1}$$

$$V = -k \sum_{\langle IJ \rangle} \sum_{\langle ij \rangle} \sigma_i^I \sigma_j^J - h \sum_I \sum_i \sigma_i^I. \tag{A2}$$

We can then write Eq. (17) as

$$H(k', h', \{\Sigma\}) = -\ln\left(Z_f \langle e^{-V} \rangle\right), \tag{A3}$$

where we introduce the notation

$$\langle (\dots) \rangle = \frac{1}{Z_f} \sum_{\{\xi\}} (\dots) e^{-H_0(\{\Sigma\}, \{\xi\})}, \tag{A4}$$

for the statistical average of any quantity $(\dots)$ with respect to the free Hamiltonian H_0, and where Z_f is the free Hamiltonian partition function,

$$Z_f = \sum_{\{\xi\}} e^{-H_0(\{\Sigma\}, \{\xi\})}. \tag{A5}$$

Let us first compute Z_f. If we let N denote the number of sites of the initial triangular lattice, then the lattice of block spins has $N/3$ sites, and hence the first sum in the expression (A1) for H_0 contains $N/3$ terms. Substituting this expression for H_0 into the definition (A5) for Z_f, we see that Z_f factorizes into a product of $N/3$ terms:

$$Z_f(\{\Sigma\}) = \prod_I Z_0(\Sigma_I), \tag{A6}$$

where $Z_0(\Sigma_I)$ is the partition function of block I, without any interactions with other blocks, i.e.

$$Z_0(\Sigma_I) = \sum_{\xi_I} e^{k(\sigma_1^I \sigma_2^I + \sigma_1^I \sigma_3^I + \sigma_2^I \sigma_3^I)}. \tag{A7}$$

We recall that we can write $\sigma_i^I = \Sigma_I \xi_I^i$, where ξ_I^i is the ith component of the three component vector ξ_I describing the internal degrees of freedom of block I, which are being integrated out. Hence $Z_0(\Sigma_I)$ reads

$$Z_0(\Sigma_I) = \sum_{\xi_I} e^{k(\xi_I^1 \xi_I^2 + \xi_I^1 \xi_I^3 + \xi_I^2 \xi_I^3)}. \tag{A8}$$

Recalling from (14) that ξ_I can only take on 4 possible values, the above sum explicitly evaluates to

$$Z_0(\Sigma_I) = 3e^{-k} + e^{3k}, \tag{A9}$$

which, in particular, is independent of Σ_I. Therefore by Eq. (A6), Z_f is given by

$$Z_f = (e^{3k} + 3e^{-k})^{N/3}. \tag{A10}$$

Thus to obtain $H(k', h', \{\Sigma\})$ from Eq. (A3) all that remains is to compute $\langle e^{-V} \rangle$. This can be expressed as a *cumulant expansion* [3]:

$$\langle e^{-V} \rangle = \exp\left(-\langle V \rangle + \frac{1}{2}\left(\langle V^2 \rangle - \langle V \rangle^2\right) + \dots\right). \quad (A11)$$

We will only keep the lowest order term in this expansion. This should work well if we expect that the fluctuations within each block are small, such that $\langle V^2 \rangle \approx \langle V \rangle^2$, and similar for higher order terms. Of course, near the critical point we expect large fluctuations at all scales, including within each block, so this approximation is not ideal. It is because of this rather crude approximation that our results for the critical exponents will vary so significantly from the exact values (see the Conclusions section). By keeping more terms in the cumulant expansion, we can get much more accurate results, but for simplicity here we will just keep the lowest order term. To that end,

$$\langle e^{-V} \rangle \approx e^{-\langle V \rangle}, \quad (A12)$$

and all that is left is to compute $\langle V \rangle$.

From Eq. (A2) we find that

$$\langle V \rangle = -k \sum_{\langle IJ \rangle} \langle V_{IJ} \rangle - h \sum_I \sum_i \langle \sigma_i^I \rangle$$
$$= -k \sum_{\langle IJ \rangle} \langle V_{IJ} \rangle - 3h \sum_I \langle \sigma_3^I \rangle \quad (A13)$$

where in the second line we have used the fact that, since all spins are equivalent, $\langle \sigma_1^I \rangle = \langle \sigma_2^I \rangle = \langle \sigma_3^I \rangle$, and so $\sum_i \langle \sigma_i^I \rangle = 3 \langle \sigma_3^I \rangle$. Also, for nearest neighbour block sites I and J we have defined V_{IJ} as

$$V_{IJ} = \sum_{\langle ij \rangle} \sigma_i^I \sigma_j^J, \quad (A14)$$

where the sum over i and j is over nearest neighbour elementary sites i and j. As shown in Fig. 4, each pair of nearest neighbour sites I and J has exactly two elementary nearest neighbour links. Thus we can write V_{IJ} as

$$V_{IJ} = \sigma_3^I(\sigma_1^J + \sigma_2^J), \quad (A15)$$

and because the expectation value $\langle (\dots) \rangle$ factorizes over blocks, we obtain

$$V_{IJ} = 2\langle \sigma_3^I \rangle \langle \sigma_3^J \rangle, \quad (A16)$$

and hence $\langle V \rangle$ is given by

$$\langle V \rangle = -2k \sum_{\langle IJ \rangle} \langle \sigma_3^I \rangle \langle \sigma_3^J \rangle - 3h \sum_I \langle \sigma_3^I \rangle. \quad (A17)$$
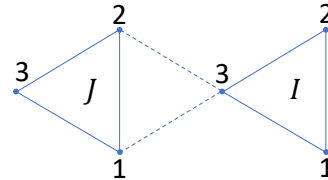

FIG. 4. Nearest neighbour block spins at sites I and J have two elementary nearest neighbour links, shown in dashed lines.

By again using $\sigma_i^I = \Sigma_I \xi_I^i$, we can easily compute $\langle \sigma_3^I \rangle$:

$$\langle \sigma_3^I \rangle = \frac{1}{Z_0} \sum_{\xi_I} \Sigma_I \xi_I^3 e^{k(\xi_I^1 \xi_I^2 + \xi_I^1 \xi_I^3 + \xi_I^2 \xi_I^3)}$$
$$= \frac{1}{Z_0}\left(e^{3k} - e^{-k} + e^{-k} + e^{-k}\right)\Sigma_I$$
$$= \left(\frac{e^{3k} + e^{-k}}{e^{3k} + 3e^{-k}}\right)\Sigma_I. \quad (A18)$$

Finally, combining Eqs. (A3), (A10), (A17) and (A18), we find the block Hamiltonian

$$H(k', h', \{\Sigma\}) = -3h\left(\frac{e^{3k} + e^{-k}}{e^{3k} + 3e^{-k}}\right)\sum_I \Sigma_I$$
$$- 2k\left(\frac{e^{3k} + e^{-k}}{e^{3k} + 3e^{-k}}\right)^2 \sum_{\langle I,J \rangle} \Sigma_I \Sigma_J. \quad (A19)$$

Comparing this to the expected form (15), we find the values in Eq. (18) for the renormalized couplings h' and k'.

[1] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics* (American Association of Physics Teachers, 1998).
[2] L. D. Landau, E. M. Lifšic, E. M. Lifshitz, and L. Pitaevskii, *Statistical physics: theory of the condensed state*, Vol. 9 (Butterworth-Heinemann, 1980).
[3] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory* (Springer Science & Business Media, 2012).
[4] S. Weinberg, *The quantum theory of fields*, Vol. 2 (Cambridge university press, 1995).
[5] K. G. Wilson, Physical review B **4**, 3174 (1971).
[6] K. G. Wilson, Physical Review B **4**, 3184 (1971).
[7] L. P. Kadanoff, Physics Physique Fizika **2**, 263 (1966).
[8] L. Onsager, Physical Review **65**, 117 (1944).
[9] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, Journal of High Energy Physics **2016**, 36 (2016).
[10] M. Peskin, *An introduction to quantum field theory* (CRC press, 2018).