Majorana fermions as emergent quasiparticles

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Majorana fermions are a special type of fermion predicted by the Dirac's equation that are their own antiparticles. Currently they have rapidly gained interest in condensed matter physics as emergent quasiparticles in certain systems like topological superconductors. In this article, we review the theory of Majorana fermions starting from the Dirac equation. Then we discuss using Bogoliubov-deGennes formalism how Superconductors form ideal hunting grounds for Majorana particles and introduce the notion of Majorana zero modes. Finally we discuss the Kitaev model- a paradigmatic model to look for unpaired Majorana zero modes.

I. INTRODUCTION

In 1928 developed the wave equation that describes relativistic spin 1/2 particles. The solutions of this equation are complex valued four component *spinors* which can be interpreted as a spin 1/2 particle-antiparticle pair. It was Etorre Majorana's insight to look for a completely real set of solutions for the Dirac equation in order to create a symmetric theory of particles and anti-particles. As a result in the year 1937 he introduced the notion of fermions which are their own anti-particles; known today as Majorana fermions [1]. As Majorana fermions are their anti-particles, they must be chargeless.

While many elementary particles are well-described as Dirac fermions, but so far there seem to be no examples of those that could be thought of as Majorana fermions. Although there are strong theoretical reasons to believe that neutrinos could be Majorana fermions[2] (they seem to have a small mass and are chargeless), convincing experimental evidence is yet to be found.

On the other hand, there is considerable interest in the field of condensed matter physics about *emergent* quasi-particles that behave as Majorana fermions[3]. It so happens that excitations in superconducting systems carry signatures of Majoranas. What is more interesting is that certain zero energy excitations called Majorana zero modes(MZM) often have the additional feature of being topologically protected; which means that any continuous deformation of the Hamiltonian does not destroy the state. Because of the topological properties and interesting exchange statistics that MZMs follow, they have also been thought of as candidates for storing quantum information[4]. Thus observation of Majorana quasi-particles in solid-state systems is of great interest.

The aim of this article is to review the emergence of Majorana particles in superconducting systems. We begin by briefly reviewing solutions of Dirac equations and how Majorana fermions can be obtained from them. Subsequently we shall find physical motivation for systems that might show emergent Majoranas and then provide theoretical justification for the same. Finally, we discuss the Kitaev model which shows how unpaired MZMs can arise in a one dimensional system.

II. WHAT ARE MAJORANA FERMIONS?

In this section, we briefly review how Majorana fermions come about from Dirac's equations. The Dirac equation for a free particle [1]is

$$(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m)\Psi(x) = 0 \tag{1}$$

Where $\Psi(x) = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ is a four-component spinor field and γ^{μ} are 4×4 matrices satisfying the following algebra:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}, \quad \gamma_0\gamma_{\mu}\gamma_0 = \gamma_{\mu}^{\dagger}. \tag{2}$$

the choice of the γ matrices is not unique. Any set of matrices satisfying (2) can be chosen to solve the Dirac equation and the solutions $\Psi(x)$ for a particular choice of γ_{μ} is related to solutions for other choices of γ through a unitary transformation Majorana himself used a basis known today as the *Majorana basis* (see [2]), where all the γ matrices are purely imaginary and hence in this basis, the complex conjugate of a solution $\Psi(x)$ is a also a solution. Another convenient set that is often used is the *Weyl basis*:

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \hat{\sigma}^{\mu} & 0 \end{pmatrix} \tag{3}$$

where we have denoted $\sigma^{\mu} = (I, -\sigma^i)$ and $\hat{\sigma}^{\mu} = (I, \sigma^i)$. σ^i are the usual Pauli matrices. In the following discussions we shall use the Weyl basis everywhere.

Let us now consider the stationary solutions for the Dirac equation with energy E(which can be both positive and negative), which are nothing but $\Psi(x) = e^{-\mathrm{i}Et}\Phi_E(\boldsymbol{x})$. Here, $\Phi_E(x)$ satisfies the Dirac equation (1) with $i\partial_0$ replaced with E everywhere. The stationary states provide a complete basis, and any general solution $\Psi(x)$ can be expanded in terms of it. Moreover, they help us see an important internal symmetry of the Dirac equation called the *Charge conjugation symmetry*. It happens that if $\Phi(x)$ is a state associated with energy E, we can find a corresponding *charge-conjugated* state defined as

$$\Phi^c(x) = C\Phi^*(x) \tag{4}$$

[1] Through out the article we use Einstein summation convention; the same raised and lowered index in an expression such as μ here, are considered to be summed over.

with energy $-E$. In the above equation, C is a 4×4 matrix called the charge conjugation matrix, and in the Weyl basis $C = \mathrm{i}\gamma^2$.

A standard approach to obtain particle interpretation from Dirac equation is to quantise (1) by elevating $\Psi(x)$ to the status of an operator field and imposing the following anti-commutation relations:

$$\{\hat{\Psi}_a(x), \hat{\Psi}_b^\dagger(x')\} = \delta_{ab}\delta^4(x - x'), \quad \{\hat{\Psi}_a(x), \hat{\Psi}_b(x')\} = 0 \tag{5}$$

Then we can expand a general field operator $\hat{\Psi}(x)$ in terms of the stationary state solutions as

$$\hat{\Psi}(x) = \sum_{E>0} a_E e^{-iEt}\Phi_E(\boldsymbol{x}) + \sum_{E<0} b_{-E}^\dagger e^{-iEt}\Phi_E(\boldsymbol{x}) \tag{6}$$

Note that we have separated the positive and negative energy part of the expansions. Here a_E and b_E are operators that play the role of arbitrary coefficients in the expansion. Using the anti-commutation relations (5) we can show that a_E and b_E also follow the canonical anti-commutation rules. Thus $a_E^\dagger$ and $b_E^\dagger$ are interpreted as the creation operators for a particle and an antiparticle with energy E, as they correspond to the positive energy and negative energy parts of the expansion respectively. By reversing the sign of the dummy summation variable in the second term and using the charge conjugation property (4), we may write (6) as a sum over positive energy states only:

$$\hat{\Psi}(x) = \sum_{E>0} \left[a_E e^{-iEt}\Phi_E(\boldsymbol{x}) + b_E^\dagger e^{iEt} C\Phi_E^*(\boldsymbol{x}) \right]. \tag{7}$$

Equation (7) predicts the most general particle-antiparticle pair of Dirac fermions. The particle is distinguishable from the anti-particle as it has an opposite charge[2]. We find an interesting special case when we impose the so-called *Majorana condition* on $\hat{\Psi}(x)$:

$$\hat{\Psi}^c(x) = C\hat{\Psi}^*(x) = \hat{\Psi}(x). \tag{8}$$

The significance of this constraint can be more clearly seen in the Majorana basis in which C turns out to be just the identity matrix. Hence in this basis the Majorana condition simply says that $\hat{\Psi}(x)$ is real. Moreover, it can be shown that a solution satisfying the Majorana constraint always exists. Consider any $\hat{\Psi}(x)$ that solves the Dirac equation. It can be shown that $\hat{\Psi}^c(x)$ is also a solution and since the Dirac equation is linear the superposition: $\hat{\Psi}_{Maj}(x) = 1\sqrt{2}(\hat{\Psi}(x) + \hat{\Psi}^c(x))$ also is a solution. It is obvious that $\hat{\Psi}_{Maj}(x)$ satisfies the Majorana condition.

Using (7) and the Majorana condition, we straightforwardly get $a_E^\dagger = b_E^\dagger$. Moreover, the Noether current that gives meaning to charge identically vanishes in this case. This indicates that under the constraint (8) we get a pair of particles which are charge-less and are the same as their anti-particle. We call them *Majorana fermions*. For more information about properties of Majorana fermions the reader may refer to [2][3].

[2] In field theoretic language, the conserved Noether current corresponding to global phase symmetry has an opposite sign[5]

III. EMERGENCE OF MAJORANA FERMIONS IN SOLIDS

A. Ideal hunting grounds

As we saw in our course, condensed matter physics has plenty of examples where the system as a whole behaves very differently compared to the constituent particles. The idea that 'more is different' has been beautifully expressed in the article [6] Under certain approximations, the collective excitations of a given condensed matter system is best described in terms of *quasiparticles* - particles that do not exist at the microscopic level but seem to *emerge* from the microscopic description of the system and explain the physical observations obtained in experiments. Examples of such particles include phonons, polarons, magnons and plasmons, which we are familiar with. In solid-state physics the most important fermionic particles-electrons, are Dirac fermions. However it turns out Majorana fermions can occur in certain solids as emergent quasiparticles.

To see how that comes about we recall that in the second quantisation formalism, electrons are represented by a set of creation and annihilation operators where $c_j^\dagger$ creates an electron with quantum numbers denoted by index j while c_j annihilates it. The index j includes the quantum degrees of freedom appropriate for the set up we are describing, typically spin, position or crystal momentum etc. These operators satisfy the canonical commutation relations. Without any loss of generality, we can perform a canonical transformation of any other operator of interest to the *Majorana Basis* defined as:

$$c_j = \frac{1}{2}(\gamma_{j1} + \mathrm{i}\gamma_{j2}) \quad c_j^\dagger = \frac{1}{2}(\gamma_{j1} - \mathrm{i}\gamma_{j2}) \tag{9}$$

$$\{\gamma_{i\alpha}, \gamma_{j\beta}\} = 2\delta_{ij}\delta_{\alpha\beta}, \quad \delta_{i\alpha}^\dagger = \delta_{i\alpha} \tag{10}$$

The hermiticity condition of $\gamma_{i\alpha}$ implies that the particle created the gamma operator is the same as its anti-particle, and hence corresponds to a Majorana fermion. While mathematically equivalent, the above description in terms of Majorana fermion operators does not give any benefit in understanding the physics of the system. The primary reason for this is that in most cases the two Majoranas corresponding to a single electron-positron pair, occur paired and intertwined in space (recall that the Dirac equation predicts a pair of Majorana fermions) and thus it makes little sense to describe the Majoranas as individual particles. However in some special systems such as topological superconductors it is possible to spatially separate the

[3] If you found the prediction of Majorana fermions from Dirac theory weird or magical in some way I encourage you to take a look at this article, describing your state of mind using an 'Alice in the wonderland' example

Majoranas corresponding to a single electron. A hint regarding why this might be so is seen by inverting (9) to obtain:

$$\gamma_{j1} = c_j^\dagger + c_j, \quad \gamma_{j2} = \mathrm{i}(c_j^\dagger - c_j) \tag{11}$$

At this point we recall that the superpositions of electron and hole degrees of freedom such as above naturally arise in s-wave superconductors described by BCS theory as the condensation of Cooper-pairs violates conservation of number of particles. Though one might try to find Majorana excitations here, typical s-wave superconductors have a shortcoming too. Recall that the quasiparticle modes in such superconductors combine particles and holes of opposite spin, i.e. the quasiparticle operators take a typical form like $\gamma = uc_\uparrow^\dagger + vc_\downarrow$. This cannot describe a Majorana particle as it is physically different from the its conjugate, $\gamma^\dagger = u^* c_\uparrow + v^* c_\downarrow^\dagger$. The impediment in this situation is the existence of two different fermionic species (a spin up and a spin down). Thus to look for unpaired Majorana fermions it would be beneficial to design systems where a) superconducting order exists, allowing mixing of particle and hole states, and b) a single kind of fermionic species exists at the microscopic level, such as a 'spinless' electron. While the spinless assumption might seem unphysical, models exist that allow approximating electrons as spinless, as we shall see later.

B. Emergence in superconductors

Before we dive into particular models and physical realisations, let us see the mathematical basis behind why Majorana states are supported in a superconductor.

To do so we use the Bogoliubov-deGennes(BdG) formalism which is a self-consistent formalism like we used in class but generalised to the case of spatially non-uniform situations. We start out with a minimal Hamiltonian for superconducting systems[3]:

$$\mathcal{H} = \int \mathrm{d}^d r \left[h_0^{\sigma\sigma'}(\boldsymbol{r}) c_{\sigma\boldsymbol{r}}^\dagger c_{\sigma'\boldsymbol{r}} - V n_{\uparrow\boldsymbol{r}} n_{\downarrow\boldsymbol{r}} \right]. \tag{12}$$

$c_{\sigma\boldsymbol{r}}^\dagger$ creates a particle at the position $\boldsymbol{r}$ in d dimensional space with spin σ. $h_0^{\sigma\sigma'}(\boldsymbol{r})$ includes the kinetic energy of the electrons and any spatially varying single-electron potential and $V > 0$ represents the attractive interaction between electrons[4]. Next, we consider a mean field factorisation[5] of the quartic interaction term:

$$\begin{aligned} &- n_{\uparrow\boldsymbol{r}} n_{\downarrow\boldsymbol{r}} = c_{\uparrow\boldsymbol{r}}^\dagger c_{\downarrow\boldsymbol{r}}^\dagger c_{\uparrow\boldsymbol{r}} c_{\downarrow\boldsymbol{r}} \\ &\simeq \left\langle c_{\uparrow\boldsymbol{r}}^\dagger c_{\downarrow\boldsymbol{r}}^\dagger \right\rangle c_{\uparrow\boldsymbol{r}} c_{\downarrow\boldsymbol{r}} + c_{\uparrow\boldsymbol{r}}^\dagger c_{\downarrow\boldsymbol{r}}^\dagger \left\langle c_{\uparrow\boldsymbol{r}} c_{\downarrow\boldsymbol{r}} \right\rangle - \left\langle c_{\uparrow\boldsymbol{r}}^\dagger c_{\downarrow\boldsymbol{r}}^\dagger \right\rangle \left\langle c_{\uparrow\boldsymbol{r}} c_{\downarrow\boldsymbol{r}} \right\rangle \end{aligned} \tag{13}$$

[4] While V has been considered a constant here, we could more generally consider V as a function of the positions of 2 electrons $\boldsymbol{r}$ and $\boldsymbol{r'}$. We refer the reader to [7] for such a discussion

[5] sorry Mona!

and introduce the superconducting order parameter

$$\Delta(\boldsymbol{r}) = V \left\langle c_{\uparrow\boldsymbol{r}} c_{\downarrow\boldsymbol{r}} \right\rangle. \tag{14}$$

We can now write down the mean field approximated BdG Hamiltonian as:

$$\begin{aligned} \mathcal{H}_{\mathrm{BdG}} = \int d^d r \Big[& h_0^{\sigma\sigma'}(\boldsymbol{r}) c_{\sigma r}^\dagger c_{\sigma'\boldsymbol{r}} \\ & + \left(\Delta(\boldsymbol{r}) c_{\uparrow\boldsymbol{r}}^\dagger c_{\downarrow\boldsymbol{r}}^\dagger + \text{h.c.} \right) - \frac{1}{V} |\Delta(\boldsymbol{r})|^2 \Big] \end{aligned} \tag{15}$$

The idea is to find the eigenstates of the above approximated Hamiltonian in terms of the parameter $\Delta(\boldsymbol{r})$ and check for self consistency of equation (14). In order to write the Hamiltonian in a matrix form we define the *Nambu spinor*

$$\hat{\Psi}_r = \begin{pmatrix} c_{\uparrow r} \\ c_{\downarrow r} \\ c_{\downarrow r}^\dagger \\ -c_{\uparrow r}^\dagger \end{pmatrix} \equiv \begin{pmatrix} \hat{\psi}_r \\ i\sigma^y \hat{\psi}_r^* \end{pmatrix} \tag{16}$$

using which we can write

$$\mathcal{H}_{\mathrm{BdG}} = \int d^d r \left[\hat{\Psi}_r^\dagger H_{\mathrm{BdG}}(r) \hat{\Psi}_r - \frac{1}{V} |\Delta(r)|^2 \right] \tag{17}$$

where

$$H_{\mathrm{BdG}}(\boldsymbol{r}) = \begin{pmatrix} h_0(\boldsymbol{r}) & \Delta(\boldsymbol{r}) \\ \Delta^*(\boldsymbol{r}) & -\sigma^y h_0^*(\boldsymbol{r}) \sigma^y \end{pmatrix}. \tag{18}$$

Here $h_0(\boldsymbol{r})$ and $\Delta(\boldsymbol{r})$ are to be treated as 2×2 matrices. We can effectively diagonalise (17) by finding a eigenspinor $\Phi_n(\boldsymbol{r}) = [u_{n\uparrow}(\boldsymbol{r}), u_{n\downarrow}(\boldsymbol{r}), v_{n\uparrow}(\boldsymbol{r}), v_{n\downarrow}(\boldsymbol{r})]$ that satisfies

$$H_{\mathrm{BdG}}(\boldsymbol{r}) = E_n \Phi_n(\boldsymbol{r}) \tag{19}$$

and an annihilator operator

$$\gamma_n = \int \mathrm{d}^d \boldsymbol{r} \Phi_n^\dagger(\boldsymbol{r}) \hat{\Psi}_{\boldsymbol{r}} \tag{20}$$

Using equations (19) and (20) we can write equation (17) as

$$\mathcal{H}_{BdG} = \sum_{n|E_n>0} E_n \gamma_n^\dagger \gamma_n + E_g \tag{21}$$

where E_g is the ground state energy. At this point we must note that by using the Nambu spinor to write (17) we expanded the Hilbert space to accommodate the Δ term. Thus only half of its independent solutions are actually physical, as they have to respect the constraint that the last two components of the Nambu spinor are related to the first two. Thus we sum only over the positive energy states to avoid double-counting.

The structure of the Nambu spinor makes the connection to Majorana fermions clear. It can be seen from (16) that $\hat{\Psi}_{\boldsymbol{r}}$ satisfies the Majorana condition,

$$\hat{\Psi}_{\boldsymbol{r}} = C \hat{\Psi}_{\boldsymbol{r}}^* \tag{22}$$

automatically and it is required because of the mathematical structure of the quantised Hamiltonian $\mathcal{H}_{BdG}$ in (17). Hence the eigenstates of this system, which correspond to particles created by the $\gamma_n^\dagger$ defined by (20) are superpositions of Majorana states.

However, whether the particle created by $\gamma_n^\dagger$ itself is a Majorana fermion, depends on the nature of the stationary state $\Phi_n(\boldsymbol{r})$. For arbitrary $\Phi_n(\boldsymbol{r})$, $\gamma_n^\dagger \neq \gamma_n$ Fortunately in case of *Majorana zero modes*(MZMs), the stationary state itself satisfies the Majorana condition which leads to the corresponding annihilation operator to be Hermitian. MZMs are a special case of Majorana fermions that occur at zero energy, i.e $E_n = 0$. By taking the complex conjugate of (19) and multiplying the charge conjugation matrix C, we can see that $\Phi^c(\boldsymbol{r}) = C\Phi^*(\boldsymbol{r})$ is a stationary state with energy $-E_n$. Now consider a state with zero energy:

$$H_{BdG}\Phi_0'(\boldsymbol{r}) = 0 \tag{23}$$

By charge conjugation symmetry, $\Phi_0'^c(\boldsymbol{r})$ also has 0 energy. Thus we can define a state $\Phi_0(\boldsymbol{r}) = \frac{1}{\sqrt{2}}(\Phi_0'^c(\boldsymbol{r}) + \Phi_0'(\boldsymbol{r}))$ and it must have zero energy too. Moreover, it is clear that $\Phi_0(\boldsymbol{r}) = \Phi_0^c(\boldsymbol{r})$ which can be written in terms of individual components,

$$\begin{pmatrix} u_{0\uparrow\boldsymbol{r}} \\ u_{0\downarrow\boldsymbol{r}} \\ v_{0\uparrow\boldsymbol{r}} \\ v_{0\downarrow\boldsymbol{r}} \end{pmatrix} = \begin{pmatrix} -v^*_{0\downarrow\boldsymbol{r}} \\ v^*_{0\uparrow\boldsymbol{r}} \\ u^*_{0\downarrow\boldsymbol{r}} \\ -u^*_{0\uparrow\boldsymbol{r}} \end{pmatrix} \tag{24}$$

The Bogoliubov annihilator corresponding to this state can be found by plugging $\Phi_0(\boldsymbol{r})$ into (20) which gives us

$$\hat{\psi}_0 = i\int d^dr\left[u^*_{0\uparrow}(\boldsymbol{r})c_{\boldsymbol{r}\uparrow} + u^*_{0\downarrow}(\boldsymbol{r})c_{\boldsymbol{r}\downarrow} - v^*_{0\uparrow}(\boldsymbol{r})c^\dagger_{\boldsymbol{r}\downarrow} + v^*_{0\downarrow}(\boldsymbol{r})c^\dagger_{\boldsymbol{r}\uparrow}\right] \tag{25}$$

Clearly, $\hat{\psi}_0 = \hat{\psi}_0^\dagger$ and hence it represents a Majorana fermion.

Interestingly it turns out that as long as there is a single zero energy mode separated from other states through a finite gap, it is topologically protected[3]. This is because the zero mode cannot acquire a nonzero energy E by any continuous deformation of the Hamiltonian that does not close this gap. If it did by the charge conjugation symmetry another mode with energy $-E$ will appear. The Unitary evolution of states however does not permit one mode to transform into two modes.

IV. TOY MODELS

The discussion in the previous section sufficiently motivates how Majorana zero modes might arise in topological superconductors. The important question that remains to be answered is how to design a Hamiltonian and subsequently an experimental setup that would allow a single unpaired MZM to be observed. In an attempt to solve this problem Kitaev proposed an exactly soluble one dimensional model system consisting of spinless fermions [8] that serves as a useful paradigm for experimental search of MZMs. Due to the fact that the model comprises of spinless fermions, it had been initially viewed as somewhat un-physical. However, it has been realized more recently that in the presence of spin-orbit coupling and the Zeeman field real electrons can in fact behave essentially like spinless fermions.

We consider spinless fermions hopping between the sites of a 1D lattice described by the Hamiltonian:

$$\mathcal{H} = \sum_j \left[-t\left(c_j^\dagger c_{j+1} + \text{h.c.}\right) - \mu\left(c_j^\dagger c_j - \frac{1}{2}\right) + \left(\Delta c_j^\dagger c_{j+1}^\dagger + \text{h.c.}\right)\right] \tag{26}$$

where Δ represents the nearest-neighbor pairing amplitude. We must notice the close similarity between the Bogoliubov-DeGennes Hamiltonian in (15) and the Kitaev Hamiltonian. This suggests that Δ is the analogous to SC order with spinless fermions. Henceforth we shall assume for the sake of simplicity that Δ is real and consider a chain with N sites and open boundary conditions. We now transform the Hamiltonian (26) into the Majorana basis using (9) to obtain,

$$\mathcal{H} = \frac{i}{2}\sum_j \left[-\mu\gamma_{j,1}\gamma_{j,2} + (t+\Delta)\gamma_{j,2}\gamma_{j+1,1} + (-t+\Delta)\gamma_{j,1}\gamma_{j+1,2}\right]$$

It turns out that this Hamiltonian describes two physically distinct phases. First, consider the case $\Delta = t = 0$. The Hamiltonian becomes

$$\mathcal{H} = \frac{i}{2}(-\mu)\sum_j \gamma_{j,1}\gamma_{j,2} = -\mu\sum_j\left(c_j^\dagger c_j - \frac{1}{2}\right)$$

The ground state consists of all fermion states at site j either occupied ($\mu > 0$) or empty ($\mu < 0$) and this is clearly a topologically trivial phase.

Next we consider the case $\Delta = t$ and $\mu = 0$. Now the Hamiltonian takes the form

$$\mathcal{H} = it\sum_{j=1}^{N-1}\gamma_{j,2}\gamma_{j+1,1}$$

To find the ground state of this Hamiltonian we define a new set of fermionic operators that combine Majorana modes of neighbouring sites:

$$a_j = \frac{1}{2}\left(\gamma_{j,2} + i\gamma_{j+1,1}\right), \quad a_j^\dagger = \frac{1}{2}\left(\gamma_{j,2} - i\gamma_{j+1,1}\right)$$

for $j = 1, 2 \ldots N-1$. These live on nearest neighbor bonds of our 1D chain. In terms of these new fermions we have

$$\mathcal{H} = 2t\sum_{j=1}^{N-1}\left(a_j^\dagger a_j - \frac{1}{2}\right) \tag{27}$$

The ground state of this Hamiltonian for $t > 0$ is the vacuum state $|0\rangle$ with energy $E_g = -t(N-1)$.

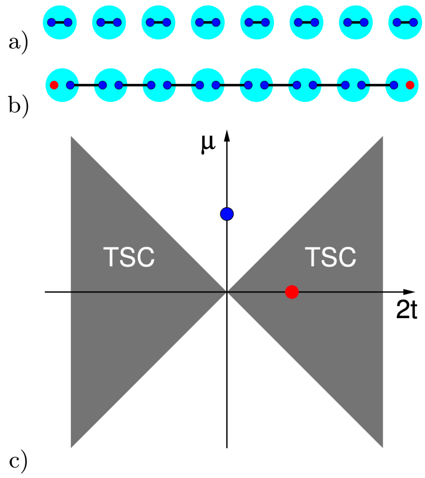

FIG. 1. Two phases of the Kitaev model, image taken from [3] a) This depicts the trivial phase, where Majoranas at each site can be thought of being bound into ordinary fermions. b) This depicts the topological phase Majoranas on neighbouring sites are bound, leaving 2 unpaired MZMs at the ends. c) The phase diagram of the Kitaev chain in $\mu - 2t$ plane; TSC denotes the topological phase and the unshaded part denotes the trivial phase.

Notice that in (27) the Majorana modes $\gamma_{1,1}$ and $\gamma_{N,2}$ do not occur. Thus the Majorana states localised at the ends of the chain $\gamma_{1,1}|0\rangle$ and $\gamma_{N,2}|0\rangle$ have ground state energy. Although together they constitute a single Dirac fermion, it is delocalised at the ends and thus can be considered effectively unpaired. Hence we have unpaired MZM modes. Clearly this phase is topologically relevant as there is a finite gap between the ground state and excited states. For general μ and t, we can solve the Hamiltonian (26) considering periodic boundary conditions and moving to the momentum space. We find that spectrum is,

$$E(q) = \pm\sqrt{(2t\cos q + \mu)^2 + (\Delta \sin q)^2} \qquad (28)$$

Except for the case $\mu = \pm 2t$, the spectrum above gives a gap between the two bands. These lines of zero gap define two regions in the $\mu - 2t$ plane, as shown in Fig.1. The topologically trivial phase where no MZMs occur is represented by the region $|\mu| > |2t|$ in the $\mu - 2t$ plane where as the second case where unpaired MZMs are supported is depicted by $|\mu| < |2t|$. At the boundary $\mu = \pm 2t$, a topological phase transition occurs.

It is useful to reflect upon the importance of the spinless assumption of the Kitaev Model. It ensures that a single zero-energy Majorana mode resides at each end of the chain in its topological phase. Having regular spin half fermions instead, merely doubles the degeneracy for every eigenstate of the Hamiltonian, so that when $|\mu| < 2|t|$ each end supports two Majorana zero-modes, or equivalently one ordinary fermionic zero-mode. A slightly more complicated but exactly solvable toy model also exists in two dimensions. It is a model of spinless 2D electron gas showing $p + ip$ superconductivity. It is in several ways a generalisation of the Kitaev model and has the same key features[9].

V. CONCLUSION

In this article, we begin by outlining the theory of how Majorana fermions arise from Dirac's famous theory of electrons. We then proceed to describe ideal systems where Majorana states could be found. Then we use the Bogoliubov-deGennes theory for superconductors to further demonstrate how excitations in superconductors naturally behave as Majorana fermions. Based on this foundation we introduce the concept of Majorana zero modes. We then describe the famous Kitaev model. The Kitaev model provides a paradigm that makes experimental realisation of MZMs more feasible. While an unambiguous observation of MZMs is yet to be observed in labs, in 2010 and 2011 a few remarkable papers [10–12] have shown how to map the Kitaev model into actual systems using semiconductor nano wires coupled with s-wave superconductors. They get over the key impediment in implementing the Kitaev model - suppressing the spin degree of freedom of electrons, by using a strong spin-orbit coupling along with Zeeman coupling. We refer the interested reader to explore the details of these propositions in [3, 9] and another Phys 502 project done in 2018 [13].

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