Jaynes-Cummings Model

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Abstract- This paper presents Jaynes-Cummings model, the simplest model that describes light-matter interaction. Light-matter interaction is an overlap between condensed matter physics and quantum optics. Thus, this model consists of atomic, electromagnetic field and interaction Hamiltonians. Besides, it describes briefly the experimental realization of light-matter interaction in the strong coupling regime.

I. Introduction

Jaynes-Cummings model was developed in 1963 by Edwin Jaynes and Fred Cummings, to describe the quantum mechanical behavior of an atom interacting with a single mode radiation of a quantized electromagnetic field. The two systems exchange a quanta; the atom absorbs a photon and moves to the excited state or emits a photon and moves down to the ground state. The uniqueness of this model is that it relies on the quantum treatment of the electromagnetic field, therefore the electromagnetic field must be quantized in a defined volume [1,2].

Here, we introduce the atomic, electromagnetic field and interaction Hamiltonian and discuss a few special cases. As well as its experimental realization and the strong coupling concept.

II. Atom Hamiltonian

In this model, the atom is approximated as a two-level system. It is described by a two-dimensional state space spanned by the two energy eigenstates: ground state $|g\rangle$ and excited state $|e\rangle$, with energy eigenvalues E_g and E_e respectively as shown in figure 1.

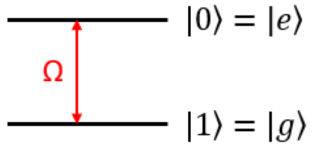

Figure 1: two-level atom

The Hamiltonian of the two-level atom in the energy representation is:

$$H_{atom} = E_g|g\rangle\langle g| + E_e|e\rangle\langle e| = (E_g + E_e)I = \frac{1}{2}(E_g + E_g)I + \frac{1}{2}(E_g - E_g)\sigma_z \quad (1)$$

Where I is the unitary matrix and $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ denotes the Hermitian Pauli operator.

The transition frequency of the atom is determined by the atom internal structure and is fixed for a given material; it is given as:

$$\omega_{01} = \frac{\Omega}{\hbar} \quad (2)$$

Where $\Omega = E_e - E_g$ denotes the energy gap. By shifting the zero energy to $E_e + E_g$, the atomic Hamiltonian can be written as:

$$\hat{H}_{atom} = \frac{\Omega}{2}\sigma_z \quad (3)$$

II. Electromagnetic Field Hamiltonian[2]

Classically, solving Maxwell equations with Coulomb Gauge ($\nabla \cdot A = 0$) would give the wave equation that can be solved as plane waves given by the following vector potential:

$$\boldsymbol{A}_{k,\alpha} = \epsilon_{k,\alpha} A_{k,\alpha} e^{i(kr - \omega_k t)} \quad (4)$$

Where $\epsilon_{k,\alpha}$ denotes the polarization vector of the radiation field with wavevector k and polarization α, $A_{k,\alpha}$ complex amplitude and ω_k frequency of the radiation field.

The total energy of radiation field inside a box with periodic boundary conditions and volume $V = L^3$ is:

$$H = \sum_{k,\alpha} \varepsilon_0 V {\omega_k}^2 [A_{k,\alpha} {A_{k,\alpha}}^* + {A_{k,\alpha}}^* A_{k,\alpha}] \quad (5)$$

It can be simplified as a sum of energies in each individual radiation mode described by (k, α):

$$H = \sum_{k,\alpha} E_{k,\alpha} \quad (6)$$

Where:

$$E_{k,\alpha} = \varepsilon_0 V {\omega_k}^2 [A_{k,\alpha} {A_{k,\alpha}}^* + {A_{k,\alpha}}^* A_{k,\alpha}] \quad (7)$$

This classical electromagnetic field can be quantized by associating a quantum harmonic oscillator to each radiation mode (k, α). Under these terms, the photon is defined as one elementary excitation of the quantum harmonic oscillator associated with a specific radiation mode. Therefore, photons can be created and annihilated through the ladder operators:

$$\hat{a}_k |n_k\rangle = \sqrt{n_k} |n_k - 1\rangle \quad (8)$$

$$\hat{a}_k{}^{\dagger}|n_k\rangle = \sqrt{n_k + 1}|n_k + 1\rangle \quad (9)$$

The number states $|n_k\rangle$ are known also as Fock states with precisely defined photon numbers. Figure 2 shows a quantum harmonic oscillator with Fock states.

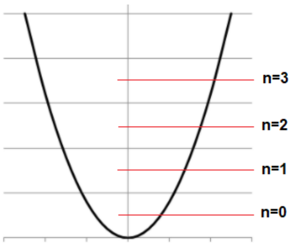

Figure 2: quantum harmonic oscillator

Consequently, the total energy of a quantified radiation field is the sum of all the quantum harmonic oscillators associated with radiation modes k:

$$\hat{H}_{field} = \sum_k \hat{H}_k \quad (10)$$

Where:

$$\hat{H}_k = \frac{1}{2}\hbar\omega_k(\hat{a}_k\hat{a}^{\dagger}{}_k + \hat{a}^{\dagger}{}_k\hat{a}_k) \quad (11)$$

For simplicity, the polarization component α is taken as constant.

Comparing the classical and quantum total energy of the radiation field gives an analogy between the classical complex amplitude and its conjugate to the creation and anhelation operators:

$$A_k \to \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\hat{a}_k \quad (12)$$

$$A_k{}^* \to \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\hat{a}^{\dagger}{}_k \quad (13)$$

Using the commutation relation of $[\hat{a}_k, \hat{a}^{\dagger}{}_k] = 1$, we can rewrite the general electromagnetic field Hamiltonian:

$$\hat{H}_{field} = \sum_k \hbar\omega_k\,(\hat{a}^{\dagger}{}_k\hat{a}_k + \frac{1}{2}) \quad (14)$$

III. Interaction Hamiltonian

The light-matter interaction occurs between the quantized electric field operator of a single mode radiation and the dipole moment of a single atom and is given as:

$$\hat{H}_{int} = -\hat{d}\cdot\hat{E}(t) \quad (15)$$

Where $\hat{E}(t)$ is the electric field operator and $\hat{d}$ is the dipole moment operator.

The dipole moment can be expressed in terms of the atomic ladder operators:

$$\hat{d} = I\hat{d}I = \sum_{i,j}|i\rangle\langle i|\,\hat{d}|j\rangle\langle j| = d_{10}(\sigma_+ + \sigma_-) \quad (16)$$

Where σ_+ and σ_- denotes the atomic ladder operators:

$$\sigma_+ = |e\rangle\langle g| \quad (17)$$

$$\sigma_- = |e\rangle\langle g| \quad (18)$$

Using the previous description of the quantized electromagnetic field, the electric field operator (in Schrodinger picture) can be written as:

$$\hat{E}(r) = \sum_k \epsilon_k\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}(a^{\dagger}{}_k e^{ikr} + a_k e^{-ikr}) \quad (19)$$

Therefore, the interaction Hamiltonian becomes:

$$\hat{H}_{int} = \sum_k g_k\hbar(a^{\dagger}{}_k e^{ikr} + a_k e^{-ikr})(\sigma_+ + \sigma_-) \quad (20)$$

Where g_k denotes the atom-light coupling constant, it depends on the mode frequency, modal volume and dipole moment:

$$g_k = \sqrt{\frac{\omega_k}{2\varepsilon_0 V}}d_{10}\epsilon_k \quad (21)$$

A simplified presentation of the interaction Hamiltonian at $r = 0$:

$$\hat{H}_{int} = \sum_k g_k\hbar(a^{\dagger}{}_k + a_k)(\sigma_+ + \sigma_-) \quad (22)$$

Due to energy conservation inside the cavity, the only terms that survive are the combination of creation and annihilation operators:

$$\hat{H}_{int} = \sum_k g_k\hbar(a_k\sigma_+ + a^{\dagger}{}_k\sigma_-) \quad (23)$$

The first term describes the annihilation of a photon (absorption) and creation of an atomic excitation while the second term describes the creation of a photon (emission) and annihilation of an atomic excitation.

Consequently, the electromagnetic field in mode k and the atom cam be described by joined states:

$$|n_k, i\rangle = |n_k\rangle\otimes|i\rangle, i = 0,1 \quad (24)$$

The transition rates of creation and annihilation of a photon (which equivalent to annihilation and creation of an atomic excitation) can be expressed in terms of an absorption matrix element:

$$\langle n_k - 1,0|\hat{H}_{int}|n_k, 1\rangle = g_k\hbar\sqrt{n_k} \quad (25)$$

And an emission matrix element:

$$\langle n_k + 1,1|\hat{H}_{int}|n_k, 0\rangle = g_k \hbar\sqrt{n_k + 1} \tag{26}$$

It should be noted that the transition rates are stronger if there are already photons in the same mode k, i.e. $n_k > 1$. This is known as Bosonic enhancement of the absorption and emission transition rates.

However, thanks to the quantum treatment of the electrometric field, even if there are no photons in mode k, i.e. $n_k = 0$ the mode is in the vacuum state, there is still light-matter interaction!

The factor 1 in the emission matrix element (presented in red) represents the spontaneous emission of the atom that is triggered by the vacuum fluctuations of the radiation field. While n_k in the same term represents the normal stimulated emission.

IV. Jaynes-Cummings Hamiltonian

The Jaynes-Cummings Hamiltonian of a single mode radiation that interacts with a single atom is given as:

$$\hat{H}_{int} = \hbar\omega a^\dagger a + \frac{\Omega}{2}\sigma_z + g\hbar(a\sigma_+ + a^\dagger\sigma_-) \tag{27}$$

Let us switch to interaction picture to solve for time evolution:

$$|\Psi^{in}(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0 t}|\Psi(t)\rangle \tag{28}$$

Where $\hat{H}_0$ is the free Hamiltonian ($\hat{H}_{int}$ without the interaction term). The interaction Hamiltonian becomes:

$$\begin{aligned}\hat{H}_{int}^{in}(t) &= e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{H}_{int}e^{-\frac{i}{\hbar}\hat{H}_0 t} \\ &= g\hbar(a^\dagger\sigma_- e^{i\Delta t} + a\sigma_+ e^{-i\Delta t})\end{aligned} \tag{29}$$

Where $\Delta = \omega - \omega_{01}$ known as detuning.

The time evolution is given by Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}|\Psi^{in}(t)\rangle = \hat{H}_{int}^{in}(t)|\Psi^{in}(t)\rangle \tag{30}$$

For an arbitrary state of this system:

$$|\Psi^{in}(t)\rangle = \sum_n (c_{1,n}(t)|n, 1\rangle + c_{0,n}(t)|n, 0\rangle) \tag{31}$$

We need to solve for $c_{1,n}(t)$ and $c_{0,n}(t)$, given initialize condition of the light-atom system.

The states $|n_k + 1,1\rangle$ and $|n_k, 0\rangle$ are coupled, figure 3 shows the relation between these coupled states. Their coupling can be written in terms of differential equation:

$$\dot{c}_{1,n+1}(t) = -ig\sqrt{n+1}e^{i\Delta t}c_{0,n}(t) \tag{32}$$

$$\dot{c}_{0,n}(t) = -ig\sqrt{n+1}e^{-i\Delta t}c_{1,n+1}(t) \tag{33}$$

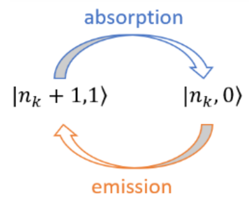

Figure 3: coupled states in an atom-light system

Special cases:

1. Initial conditions: n+1 photons, atom in ground state with resonant interaction $\Delta = 0$:

$$c_{1,n+1}(0) = 1 \tag{34}$$

The solution is:

$$c_{1,n+1}(t) = \cos(g\sqrt{n+1}t) \tag{35}$$

$$c_{0,n}(t) = -i\sin(g\sqrt{n+1}t) \tag{36}$$

Therefore, the probability for n+1 photons and atom in ground state is given as:

$$P_{1,n+1}(t) = |c_{1,n+1}(t)|^2 = \frac{1}{2}(1 + \cos(\Omega_n t)) \tag{37}$$

Where $\Omega_n = 2g\sqrt{n+1}$ called Rabi oscillation frequency; the atom oscillates between the ground and excited state. It is quantized.

If the detuning was not zero ($\Delta \neq 0$), the Rabi oscillation frequency becomes:

$$\Omega_n^\Delta = \sqrt{\Delta^2 + \Omega_n^2} = \sqrt{\Delta^2 + 4g^2(n+1)} \tag{38}$$

2. Initial conditions: 0 photons, atom in excited state with resonant interaction $\Delta = 0$:

$$c_{0,0}(0) = 1 \tag{39}$$

Therefore, the probability for 0 photon (vacuum state of the radiation field) and atom in excited state is given as:

$$P_{0,0}(t) = |c_{1,n+1}(t)|^2 = \frac{1}{2}(1 + \cos(\Omega_0 t)) \tag{40}$$

Where $\Omega_0 = 2g$ called vacuum Rabi oscillation; the atom oscillates between the ground and excited state continuously in vacuum.

Figure 4 shows the probability $P_{0,0}(t)$ f an ideal atom, the atom oscillates between the ground and excited states every $\frac{2\pi}{\Omega_0}$.

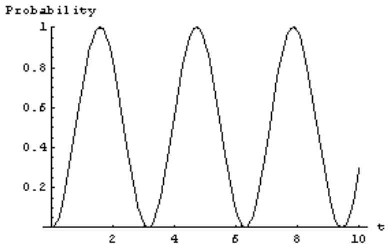

Figure 4: vacuum Rabi oscillations of an ideal two-level system

V. Experimental Realization

A single mode radiation can be achieved experimentally through an optical cavity that can be described as two reflecting mirrors. The atom can be trapped inside the cavity to achieve light-matter interaction. Figure 5 shows schematically a two-level atom trapped inside a cavity with modal volume V_0. The cavity can be described by three parameters: g, κ and γ that are atom-cavity coupling, photon decay rate from the cavity and non-resonant decay rate [3].

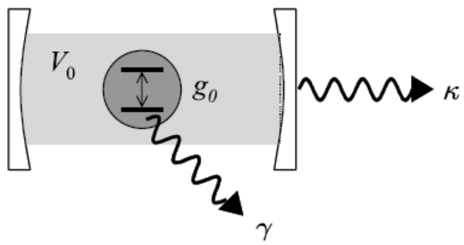

Figure 5: atom-light interaction inside an optical cavity

When the atomic transition frequency coincides with the one of the resonant modes of the cavity, the interaction between the atom and the radiation field will be strong. The resonance condition is achieved by tuning the cavity such that the frequency of the cavity mode matches with that of the atomic transition.

The interaction is said to be in the strong coupling regime if:

$$g \gg (\kappa, \gamma) \quad (41)$$

where (κ, γ) represents the larger of κ and γ. The strong coupling regime is known as cavity quantum electrodynamics (cQED).

The interaction is said to be in weak coupling regime if:

$$g \ll (\kappa, \gamma) \quad (42)$$

The cavity photon decay rate κ is governed by the properties of the cavity that determine its quality factor Q:

$$\kappa = \omega / Q \quad (43)$$

Thus, high Q values mean relatively small photon decay rate.

The non-resonant decay rate γ is determined by several factors. The atom could emit a photon of the resonant frequency in a direction that does not coincide with cavity mode or it could decay to other levels, emitting a photon of a different frequency that is not in resonance with the cavity or the electron could decay without emission of a photon. Both parameters γ and κ determines the cavity losses. These losses result in measuring damped vacuum Rabi oscillations in experiments, unlike figure 4 which represents an ideal two-level system.

In the strong coupling regime, the atom-photon interaction is faster than the irreversible process due to loss of photons out of the cavity mode, making the emission of the photon a reversible process in which the photon is re-absorbed by the atom before it is lost from the cavity.

The cavity can be characterized as well by the cooperativity parameter; the ratio of coherent coupling o incoherent coupling (losses):

$$C = \frac{g^2}{\kappa\gamma} \quad (44)$$

For a strong coupling, C should be as large as possible. That would allow us to observe, experimentally, decaying Rabi oscillations with a decay constant that depends on the cavity losses.

Experimentally, the strong coupling can be achieved by designing a cavity with a small modal volume and low losses, i.e. cavity with small V and high Q . This was demonstrated experimentally in different setups: superconducting mirrors for microwaves photons[4], optical Fabry-Perot cavities with cold atoms[5], on-chip microwave resonators with superconducting qubits that act as an artificial atom[6] and quantum dot in a photonic crystal cavity [7].

cQED field enabled studying the basic physics pf light-matter interaction and paved the way to realization of qubits (two-level systems) in different physical setups for quantum computing applications. In 2012, Serge Haroche and David Wineland won Nobel prize for physics for their work on controlling quantum system. cQED is a thriving field and best is yet to come.

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