An Introduction to Anyons

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In this paper, we introduce anyons, particles that pick up a non-trivial phase under exchange. We start by considering the topological conditions that allow anyons to be created and construct the braid group, which defines anyonic exchange statistics. We proceed to discuss abelian and non-abelian exchange representations and review cyons, fibonacci anyons, and Ising anyons as illustrative examples. Finally, we briefly examine solved condensed matter models in which anyons are produced and review the exchange statistics of these systems.

I. INTRODUCTION

It is well known that in three-dimensions, there are two types of particles; bosons are symmetric under exchange and fermions are anti-symmetric under exchange. However, in two-dimensions, it is possible to have particles that are neither anti-symmetric nor symmetric under exchange. These particles are known as anyons. It has been shown that the exotic statistics of anyons can be used to encode information for topological quantum computation [1], and it is believed that they can be used in the construction of superconductors [2]. Earlier this year, abelian anyons were directly observed in a two-dimensional electronic gas collider [3] and in an electronic Fabry-Perot intereferometer [4], showing that these particles actually exist – they are not mere mathematical oddities.

II. ANYON EXCHANGE STATISTICS

A. Two Particle Exchange

Consider a system of two indistinguishable particles with hard-core repulsion at the positions r_1 and r_2 described by the wave function $\psi(r_1, r_2)$. We define the exchange of two particles as the arbitrary adiabatic transport of each particle to the position that was previously occupied by the other particle. Because the particles are indistinguishable, the probability density of the system must be unchanged, so the wave-function must be unchanged up to accumulating some phase $e^{i\theta}$. Suppose we exchange the particles once, and then exchange the particles again. Notice that we have moved the particles in a closed loop about each other back to their initial positions. In three dimensions, such an "exchange loop" is topologically equivalent to the trivial loop (which is no loop at all), so the wave function must be invariant under this double exchange. This means the $e^{2i\theta}$ phase accumulated by the exchanges must satisfy the constraint:

$$\psi(r_1, r_2) = e^{2i\theta}\psi(r_1, r_2) \tag{1}$$

so $\theta = n\pi$ for some integer n. When n is an even integer, the wave function is invariant under exchange, which gives us boson exchange statistics. When n is an odd integer, the wave function is multiplied by a negative sign under exchange, which gives us fermi statistics.

In two-dimensions, however, not all exchange loops are topologically equivalent to the trivial loop, so double exchanges do not restore the initial system. Then, we do not have any constraint on the phase θ under two exchanges, and θ could be any value. We say that particles that accumulate a factor of θ obey this exchange rule follow θ-statistics. Bosons obey 0-statistics, fermions obey π-statistics, and we refer to particles obeying θ-statistics for $\theta \neq 0, \pi$ as anyons [5]. The interested reader can consult the appendix for a rigorous version of this argument.

One important consequence of θ being any value is that it is not generally true that $e^{i\theta} = e^{-i\theta}$. We must then distinguish between "counterclockwise" exchanges and "clockwise" exchanges, which correspond to the accumulated phases θ and $-\theta$ respectively.

B. The Braid Group

Consider a system of n particles that follow θ-statistics with the positions $r_1, ..., r_n \in \mathbb{R}^2$ in the center of mass frame, described by the wave function $\psi(r_1, ..., r_n)$. Suppose now that we perform some arbitrary exchange of these particles such that the particles have the final positions $r_{P(1)}, ..., r_{P(n)}$, where P is some permutation of n particles. We can perform this exchange through the composition of exchanges of numerically adjacent particles, each of which may be a clockwise exchange or a counterclockwise exchange. In light of this, we define σ_i and σ_i^{-1} for $1 \leqslant i \leqslant n-1$ to be the counterclockwise and clockwise exchanges respectively of the i^{th} and $(i+1)^{\text{th}}$ particle. Notice that a clockwise exchange of two particles undoes the effect of a counterclockwise exchange, so they are indeed inverses [5].

The exchange of two disjoint pairs of particle should not affect one another, so the order in which those exchanges occur does not matter. This adds the constraint that $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geqslant 2$. We also add the mathematically imposed Yang Baxter constraint, that $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, which is necessary for a valid commutator structure [6].

Using our construction of the σ_i, we see that exchange in a system of n-particles with θ-statistics has the same structure as the group generated by $\sigma_1, .., \sigma_n$. This group is known as the Braid group B_n. It is not generally true that $\sigma_i^2 = 1$ for each of the σ_i, which means that the braid group B_n is an infinite group. Notice that if we repeat this construction in three dimensions, we obtain the same result with the additional topological constraint that $\sigma_i^2 = 1$. We recognize this as being the symmetric group S_n, which agrees with our understanding that particle exchange in three dimensions is equivalent to permuting the particles [6]. A reader interested in a useful visual representation of the braid group can consult the appendix.

To understand how the braid statistics impact the quantum mechanical behavior of an anyonic system, we must determine the action of the braid group on the Hilbert space $\mathbb{H}$. This requires us to determine a way to write the elements of the braid group as operators on the Hilbert space that leave probabilities unchanged, that is we need a unitary representation $\pi : B_n \longrightarrow \mathcal{U}(\mathbb{H})$, where $\mathcal{U}(\mathbb{H})$ denotes the space of unitary operators acting on the Hilbert space.

III. ABELIAN ANYONS

A. The One-Dimensional Representation of the Braid Group

Perhaps the simplest non-trivial representation of the braid group B^n is the one-dimensional representation:

$$\pi(\sigma_k) = e^{i\theta} \tag{2}$$

It is easy to see that this representation is a unitary homomorphism that has all of the properties required of the generators. Furthermore, the $\pi(\sigma_k)$ mutually commute, which tells us that the resultant statistics are abelian. Anyons described by these statistics are called abelion anyons.

We can write an arbitrary exchange of particles $(r_1, ..., r_n) \mapsto (r_{P(1)}, ..., r_{P(n)})$ as some element $b \in B_n$. Writing this in terms of the generators, we have:

$$b = \prod_{k=1}^{n} \sigma_k^{m_k} \tag{3}$$

for $m_1, ..., m_n \in \mathbb{Z}$. Applying the the representation π, we find that b operates on the Hilbert space as the following operator:

$$\pi(b) = e^{i(m_1+\cdots+m_n)\theta} \tag{4}$$

The exchanged wave-function is given by applying the operator $\pi(b)$ to the wave function, so we have:

$$\psi(r_{P(1)}, ..., r_{P(n)}) = e^{i(m_1+\cdots+m_n)\theta}\psi(r_1, ..., r_n) \tag{5}$$

Notice that $m_1 + \cdots m_n$ is an effective degree of the exchange; each m_k is the number of counter-clockwise exchanges of the k and $k+1$ minus the number of clockwise exchanges of these particles required to go from the initial state to the final state.

Next, consider a system of two particles, each of which are composite bound states of n abelian anyons that obey θ-statistics. A counter-clockwise exchange σ of these particles involves n^2 counter-clockwise anyon exchanges – each anyon in one bound state has a counter-clockwise exchange with each anyon in the other bound state. Then, it follows that:

$$\pi(\sigma) = e^{in^2\theta} \tag{6}$$

which tells us that the composite particles obey $n^2\theta$-statistics. Similarly, we see that the counter-clockwise exchange of two bound state composed of one anyon that obeys θ-statistics and one anyon that obeys $(-\theta)$-statistics accumulates no total phase. Such a bound state, denoted $\mathbf{I}$, is known the vacuum, and it is equivalent to having no particle at all.

Now, consider two bound states of anyons that follow θ-statistics composed of α and β anyons respectively. Using the same argument as we did with for a particle composed of n-anyons, the particle made up of the combination, or fusion $\alpha * \beta$, of the two bound states will obey $(\alpha + \beta)^2$ statistics. This rule, the fusion rule for Abelian anyons, is written as [5]:

$$\alpha^2\theta * \beta^2\theta = (\alpha + \beta)^2\theta \tag{7}$$

B. Physical Realization – Cyons

Despite the fact that particles constrained to two dimensions are able to have anyon statistics, electrons, atoms, and photons constrained to two dimensions retain the same fermionic/bosonic statistics that they obey in three dimensions. However, it is possible to construct systems with quasiparticles that obey anyon statistics.

Consider an infinite and very thin solenoid along the z-axis with magnetic flux Φ along with a spinless particle of charge q and mass m orbiting the solenoid at a radius r constrained to some plane perpendicular to the solenoid. The electric field $\mathbf{E}$ experienced by the charged particle is:

$$\mathbf{E} = -\frac{\dot{\Phi}}{2\pi|\mathbf{r}|}\hat{\boldsymbol{\phi}} \tag{8}$$

Using Coulomb's Law and Newton's Equation of motion, we find that the contribution of the electric field to the angular momentum of the particle is:

$$\mathbf{J}_E = \frac{q\Phi}{2\pi c}\hat{\boldsymbol{\phi}} \tag{9}$$

The canonical angular momentum $\mathbf{J}_C = \mathbf{r} \times (\mathbf{p} + e\mathbf{A})$ of the particle is some integer multiple of $\hbar$. Then, using the fact that the canonical angular momentum is the sum of the electronic angular momentum $\mathbf{J}_E$ and the kinetic angular momentum $\mathbf{J}_K = \mathbf{r} \times \mathbf{p}$, we have:

$$\mathbf{J}_K = n\hbar\hat{\boldsymbol{\phi}} - \frac{q\Phi}{2\pi c}\hat{\boldsymbol{\phi}} \tag{10}$$

for $n \in \mathbb{Z}$. Now, consider the limit where $r \to 0$ and the solenoid become infinitely thin. The resulting quasiparticle is called a cyon. The spin S is defined to be the kinetic angular momentum at $n = 0$ divided by $\hbar$, so the cyon is a quasiparticle with mass m, charge q, and fractional spin [7]:

$$S = \frac{q\Phi}{2\pi\hbar c} \tag{11}$$

Consider a system of two cyons at the positions $\mathbf{r}_1$ and $\mathbf{r}_2$. In center of mass and relative coordinates $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the wave function describing this system (see the Appendix for a derivation) is:

$$\Psi(\mathbf{R}, \mathbf{r}) = \sum_{\ell, \mathbf{k}_R} c_{\ell,\mathbf{k}_R} e^{i\mathbf{k}_R\cdot\mathbf{R}} J_\ell(k_r r) e^{i(\ell - q\Phi/2\pi)\phi} \tag{12}$$

for the constants $c_{\ell,\mathbf{k}_R}$ which are such that the wave-function is normalized, where $\mathbf{r} = (r, \phi)$, and where J_ℓ is the ℓ^{th} bessel function of the first kind. Using this result, we see that:

$$\Psi(\mathbf{R}, r, \phi + 2\pi) = \Psi(\mathbf{R}, r, \phi)e^{-iq\Phi} \tag{13}$$

The rotation $\phi \mapsto \phi + 2\pi$ of the relative coordinates is two counter-clockwise exchanges of the cyons. Then, if $q\Phi \neq a\pi$ for some even integer a, the cyons accumulate a non-trivial phase under exchange and are subsequently anyons with $(q\Phi/2)$-statistics [8].

IV. NON-ABELIAN ANYONS

A. Non-Abelian Representations

In dimensions greater than one, operator representations of the braid group are not generally abelian, and we can obtain non-abelian braiding statistics. Consider a degenerate set of $m \geqslant 2$ states at the positions $r_1, ..., r_m$ with the orthonormal basis $\psi_1, ..., \psi_m$. Let $\mathbf{\Pi} : B_n \longrightarrow U(m)$ be some representation of B_n. We see that $\mathbf{\Pi}(\sigma_k)$ is an $m \times m$ unitary matrix, called the braiding matrix, and that performing the exchange σ_k transforms the basis states in the following way:

$$\psi_i = \sum_j \Big[\mathbf{\Pi}(\sigma_k)\Big]_{ij} \psi_j \tag{14}$$

where $[\mathbf{\Pi}(\sigma_k)]_{ij}$ denotes the ij^{th} entry of the braiding matrix. This equation tells us that instead of just adding a phase to the wave function, the exchange creates non-trivial rotations in the many-particle Hilbert space [5].

Each species of anyon has a particular type of exchange statistic dependent on its topology, so we often refer to the identity of the anyon as its topological charge. Composite particles made up of non-abelian anyons do not combine uniquely; their fusion is a quantum superposition of states with different topological charges. We subsequently define the fusion $a * b$ of a particle with topological charge a and a particle with topological b to be [9]:

$$a * b = \sum_c N_{ab}^c c \tag{15}$$

where the sum over c denotes a sum over all of the topological charges of the species. The number N_{ab}^c is a non-negative integer equal the number of different ways we can combine the topological charges a and b to obtain topological charge c, and each method of fusing a and b into c is known as a fusion channel. The fusion algebra must satisfy the relations $N_{ab}^c = N_{ba}^c$ and $N_{a\bar{a}}^{\mathbf{I}} = 1$ where a and $\bar{a}$ are anti-particles. A non-abelian anyon theory is often stated by specifying N_{ab}^c for the particles in the system.

For abelian anyons, the fusion product is unique, so:

$$a * b = \sum_c N_{ab}^c c = \sum_c \delta_{cc'} c \tag{16}$$

for some topological charge c'. We can define the topological charge of a composite abelian anyon with statistics $k^2\theta$ to be k, so $c' = a + b$. Then, the general anyon fusion rule for abelian anyons is [5]:

$$a * b = \sum_c \delta_{a+b,c} c \tag{17}$$

Anyons a and b in some anyonic system that obey this rule are abelian anyons, so they only pick up a phase under exchange.

B. Fibonacci Anyons

Consider a system composed of anyons with two topological charges $\mathbf{I}$ and τ that have the fusion rules [10]:

$$\chi * \mathbf{I} = \chi \qquad\qquad \tau * \tau = \mathbf{I} + \tau \tag{18}$$

where $\chi = \mathbf{I}, \tau$. Anyons obeying this relation are known as Fibonacci anyons. Physically, we see that τ is its own anti-particle, but that fusing two τ particles can also yield a τ particle. We can re-write the fusion of two τ particles as:

$$\tau * \tau = f_1 \mathbf{I} + f_2 \tau \tag{19}$$

where f_k denotes the k^{th} number in the Fibonacci sequence. Taking the fusion of this linear combination, using this as a base case for induction, we see that:

$$\mathop{\ast}_{k=1}^{m} \tau = f_{m-1} \tau * \mathbf{I} + f_m \tau * \tau = f_m \mathbf{I} + f_{m+1} \tau \tag{20}$$

The dimension of the fusion product $\ast_{k=1}^m \tau$ is equal to the total number of fusion channels available, which we see is $f_m + f_{m+1}$. Adding an additional τ particle to the fusion product increases the number of fusion channels to $f_{m+1} + f_{m+2}$. Then, the dimension of a single τ particle is the ratio $(f_{m+1} + f_{m+2})/(f_{m+1} + f_m)$ in the limit where m becomes very large. We see that:

$$\dim(\tau) = \lim_{m \to \infty} \frac{f_{m+1} + f_{m+2}}{f_{m+1} + f_m} = \varphi \tag{21}$$

where φ is the golden ratio $(1 + \sqrt{5})/2$. Notice further that:

$$\mathop{\ast}_{k=1}^{m} \mathbf{I} = \mathbf{I} \tag{22}$$

Then, introducing an additional vacuum particle does not change the number of fusion channels, so $\dim(\mathbf{I}) = 1$. Using our result for the dimension of τ, $\dim(\tau * \tau) = \varphi^2$. The subspace corresponding to τ production has dimension φ and the subspace corresponding to $\mathbf{I}$ production has dimension 1. Then, the probabilities that a τ particle or $\mathbf{I}$ particle will be produced, denoted $P(\tau)$ and $P(\mathbf{I})$ respectively, are [10]:

$$P(\tau) = \frac{1}{\varphi} \qquad\qquad P(\mathbf{I}) = \frac{1}{\varphi^2} \tag{23}$$

It has been shown that the braiding of Fibonacci anyons allows for universal topological quantum computation, though this result is far beyond the scope of our discussion [1].

C. Ising Anyons

Consider a system composed of anyons with three topological charges, $\mathbf{I}$, σ, and ψ that have the fusion rules [9]:

$$\begin{aligned} \sigma * \sigma &= \mathbf{I} + \psi & \qquad \psi * \psi &= \mathbf{I} \\ \mathbf{I} * \chi &= \chi & \qquad \sigma * \psi &= \sigma \end{aligned} \tag{24}$$

for $\chi = \mathbf{I}, \sigma, \psi$. Anyons obeying these rules are known as Ising anyons. We immediately see that $\mathbf{I}$ and ψ obey Equation 17 and are then abelian anyons with dimension 1. Consider the following grouping of the fusion of an infinite number of σ particles:

$$\lim_{m\to\infty} \mathop{\ast}_{k=1}^{m} \sigma = \lim_{m\to\infty} \mathop{\ast}_{k=1}^{m} \sigma * \sigma = \lim_{m\to\infty} \mathop{\ast}_{k=1}^{m} (\mathbf{I} + \psi) \tag{25}$$

Notice that $\mathbf{I}$ and ψ have dimension 1, so $\mathbf{I}+\psi$ has dimension 2. This tells us that each factor of $\sigma * \sigma$ has dimension 2, so $\dim(\sigma) = \sqrt{2}$. Then, we see that the fusion $\sigma * \sigma$ has a 1/2 probability of either result.

V. SOLVABLE ANYON MODELS

In this section, we shall discuss two solvable models in which anyons arise, one resulting in the emergence of abelian anyons and the other resulting in the emergence of non-abelian anyons. Each model is worthy of being the focus of an entire project on its own, so, for brevity, we shall quote a number of results without proof. Details can be found in the referenced material.

A. The Toric Code on a Square Lattice

Consider a two-dimensional $N \times N$ square lattice with periodic boundary conditions. Suppose that we place spins

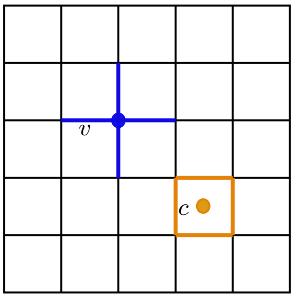

FIG. 1: This is a diagram to demonstrate how we have defined A_v and B_c in the toric code. The point v denotes a vertex, and we take the product of spin operators on the blue lines to obtain A_v. The point c denotes a center point, and we take the product of spin operators on the orange lines to obtain B_c.

on each of the edges of the lattice. All spin components on different edges mutually commute, and the spin components on a given edge α obey the anti-commutation rule $\{\sigma_{i\alpha}, \sigma_{i\beta}\} = 2\delta_{\alpha\beta}$. The Hamiltonian $\mathcal{H}$ of this system is given by [1]:

$$\mathcal{H} = -\sum_{v \in V} A_v - \sum_{c \in C} B_c \tag{26}$$

where V is the set of all vertices and C is the set of all centers of a square of four adjacent lattice points, and where we have defined the vertex operators A_v and plaquette operators B_c to be :

$$A_v = \prod_{j \text{ edge of } v} \sigma_j^x \qquad\qquad B_c = \prod_{j \text{ nearest to } c} \sigma_j^z \tag{27}$$

Our definitions of A_v and B_v are shown in Figure 1. We can show that A_v and B_c have the eigenvalues ± 1 for each v and c. In addition, we have the constraint that:

$$\prod_{v \in V} A_v = \prod_{c \in C} B_c = 1 \tag{28}$$

Some simple calculations show that $[A_v, A_s] = [B_c, B_p] = [A_v, B_c] = 0$, so all of the terms in the Hamiltonian mutually commute. We define ω_c of some configuration of spins s closest to the center c to be:

$$\omega_c(s) = \prod_{j \text{ nearest to } c} s_j^z \tag{29}$$

where s_j^z is the eigenvalue of the spin σ_j^z. Notice that as the eigenvalues of σ^z are ± 1, we must have $\omega_c(s) = \pm 1$. We say that when $\omega_c(s) = -1$, the configuration is a vortex, and when $\omega_c(s) = 1$, we say that there is no flux. Notice that $-\sum_c B_c$ is minimized when $B_c = 1$ for each c. In addition, we see that in the z-basis, A_v flips each of the spins upon which it acts, which leaves the ω_c of the centers nearest to the vertex invariant. Similarly, we can define vortices ω_v corresponding to the vertexes that add energy to the term $-\sum_v A_v$ to be:

$$\omega_v(s) = \prod_{j \text{ edge of } v} s_j^x \tag{30}$$

that are analogues of the ω_c. Then, the ground state $|\Psi_{\text{GS}}\rangle$ is a linear combination of configurations with no vortices.

We shall now determine the behavior of excitations of this system. Intuitively, these excitations should involve the production of a vortex at some center/vertex by flipping the appropriate spin at some edge. However, we see that this will also produce another vortex at the other center/vertex that borders that edge. Notice that we can eliminate the vertex at this second center by flipping a different spin that borders this center/vertex, but this will similarly create another vortex. Continuing this procedure, we find that we can create a chain of spin flips with vortices at the ends of the chain. We subsequently define the excitations S^z and S^x to be [1]:

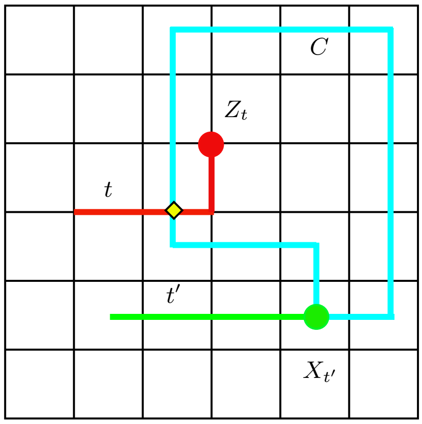

FIG. 2: This is a diagram of the excitations on the toric code. The red line t is a countour made up of a chain of spin flips that passes through vertices, and the green line t' is a contour made up of a chain of spin flips passing through centers. The quasiparticles Z_t and $X_{t'}$ are the endpoints of the contours t and t' respectively, and the cyan line C is a closed loop of spin flips that represents adiabatically transporting $X_{t'}$ around Z_t. We have also marked the intersection of C and t that causes the anyonic exchange statistics.

$$S^z(t) = \prod_{j \in t} \sigma_j^z \qquad\qquad S^x(t') = \prod_{j \in t'} \sigma_j^x \tag{31}$$

for t being a contour passing through vertices and t' being a contour passing through centers. See Figure 2 for a diagram of this. The operator S^z trivially commutes with each of the B_c, and it commutes with each of the A_v except those at the end points, because it inverts the sign of A_v there. Similarly, S^x commutes with each of the A_v and with each of the B_c, except at the end points. This tells us that the energy of each excitation comes entirely from the endpoints, so, provided that the contour does not have any topological winding about the boundary, different contours with the same endpoints are equivalent. Thus, we can consider the endpoints of the contour to be independent quasiparticles [11].

Let $X_{t'}$ and Z_t denote an endpoint of an S^x contour t' and S^z contour t respectively. Suppose, we adiabatically move an $X_{t'}$ quasiparticle counterclockwise about a Z_t particle. This is equivalent adding a closed loop C to the end of a t' contour that crosses the t contour. See Figure 2 for a diagram of this procedure. The state of the system Ψ as result of adding the contour C to the S^x chain is:

$$\Psi = S^x(c) S^z(t) S^x(t') \,|\Psi_{\text{GS}}\rangle \tag{32}$$

Notice that c and t' cross at one point, and that the operators acting there anticommute. Then:

$$\Psi = -S^z(t) S^x(c) S^x(t') \,|\Psi_{\text{GS}}\rangle = -S^z(t) S^x(t') S^x(c) \,|\Psi_{\text{GS}}\rangle \tag{33}$$

However, we see that the only action of C on the system is to introduce two vortices at some center, which is equivalent

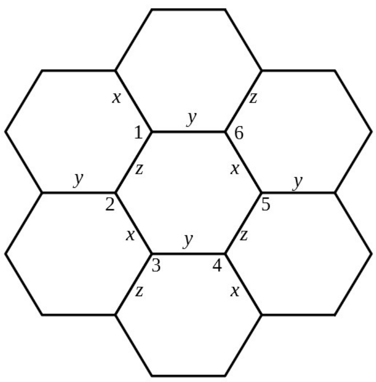

FIG. 3: This a diagram of the Kitaev Honeycomb Lattice showing the labeling scheme for the edges x, y, and z and for the vertices in the plaquette operator.

to not changing the system at all. Then:

$$\Psi = -S^z(t)S^x(t')\left|\Psi_{\text{GS}}\right\rangle \tag{34}$$

Then, adiabatically moving $X_{t'}$ about Z_t caused the state to accumulate a phase of $e^{-i\pi}$. This process is equivalent to making two of the $X_{t'}$ and Z_t particles, so we see that the phase accumulated by these particles under exchange is $e^{i\pi/2}$. Notice that adiabatically moving an $X_{t'_1}$ quasiparticle about an $X_{t'_2}$ quasiparticle or a $Z_{t'_1}$ quasiparticle about a $Z_{t'_2}$ quasiparticle leaves the system invariant, as all of the operators of the in the excitations commute. Then, we see that the $X_{t'}$ and Z_t quasiparticles exchange as relative, abelian anyons, which are known as semions [1].

B. The Kitaev Honeycomb Model

Consider a planar honeycomb lattice with spin 1/2 particles at each vertex. The spin components at different vertices commute, and spin indices at the same vertex obey the anticommutation relations $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. We label the three edges leaving a vertex to be x, y, and z, as shown in Figure 3, and we label the two vertices connected by an edge α to be i_α and j_α. We consider the nearest neighbor Hamiltonian $\mathcal{H}$ given by [11]:

$$\mathcal{H} = -J_x\sum_x \sigma^x_{i_x}\sigma^x_{j_x} - J_y\sum_y \sigma^y_{i_y}\sigma^y_{j_y} - J_z\sum_z \sigma^z_{i_z}\sigma^z_{j_z} \tag{35}$$

for the coupling constants J_x, J_y, and J_z. We define the plaquette operator W_p for some hexagon h of lattice points:

$$W_p = \sigma^x_1\sigma^y_2\sigma^z_3\sigma^x_4\sigma^y_5\sigma^z_6 \tag{36}$$

where we enumerate the vertices starting at a point on a hexagon at the end of an x-edge and proceeding counterclockwise around the hexagon as shown in Figure 3. Recognizing that distinct W_p operators share either 0 or 2 vertices, we see that each of the W_p commute with each other, and a brief computation shows that the W_p commute with the Hamiltonian as well. Then, we can simultaneously diagonalize each of the W_p along with the Hamiltonian. We find that W_p has the eigenvalues $W_p = \pm 1$, and that the ground state lies in the region where $W_p = 1$ for each of the hexagons. In this region, a Jordan-Wigner transformation can be used to Hamiltonian convert the Hamiltonian into the form [12]:

$$\mathcal{H} = \sum_k \epsilon_k a_k^\dagger a_k + \frac{i}{2}\sum_k \Delta_k\left(a_k^\dagger a_{-k}^\dagger + a_{-k}a_k\right) \tag{37}$$

where we have defined the parameters:

$$\epsilon_k = 2J_z - 2J_z\cos k_x 2J_y\cos k_y \tag{38}$$
$$\Delta_k = 2J_x\sin k_x + 2J_y\sin k_y \tag{39}$$

We subsequently find that the quasiparticle energy $E(k)$ is of the form:

$$E(k) = \sqrt{\epsilon_k^2 + \Delta_k^2} \tag{40}$$

There are two distinct behaviors depending on the values of the J_a. If $J_a > J_b + J_c$ for one combination of a, b, c, all distinct, the quasiparticle spectrum has a gap. This is called the A-phase. When this condition is not met for any of the J_a, there is no gap. This region is called the B-phase [12].

Similar to the toric code, we add excitations by changing W_p from 1 to -1 for some hexgan, introducing a vortex on one of the hexagon. However, these vortices can only be added in pairs. The excitation operators are [11]:

$$S_v^z = e^{-i\pi\sigma_v^z/2} \qquad S_{uv}^{xy} = e^{i\pi\sigma_u^x}e^{i\pi\sigma_v^y} \tag{41}$$

where v is some vertex and where u is some vertex adjacent to v. The first operator flips σ^x and σ^y at v, which changes the value of W_p for the two hexagons with the z-boundary coming out of v as an edge. We shall denote such an excitation as an a excitation. The second excitation operator changes changes a single spin at u and v, which flips the W_p that have only one of u and v as vertices. We denote these kinds of excitations as b excitations.

In the A phase, the Honeycomb system maps directly to the toric code, and we find that these excitations are semions. In the B-phase, however, we find that the combination of two u vortices annihilate, the combination of a u and a v vortex yields a v vortex, and that the combination of two v vortices annihilate with frequency 1/2 and produce a u vortex with frequency 1/2. Notice that these are precisely the fusion rules for Ising anyons, strongly indicating that these vortexes are non-abelian Ising anyons [13].

VI. CONCLUSION

In conclusion, we have demonstrated that in two dimensions, it is possible to have particles called anyons that accumulate a non-trivial phase under exchange. We have then determined the structure of the exchange statistics of these particles, which we found is described by the braid group B_n. We proceeded to define a one-dimensional abelian representation of the braid group as unitary Hilbert space operators, and found the fusion rules for abelian anyons, which are anyons described by this representation. Next, we discussed how abelian anyons can be realized as cyons, the combination of a charged particle and a magnetic flux. We also explored higher-dimensional non-abelian representations of the braid group, and defined the fusion rules for non-abelian anyons, anyons described by these non-abelian representations. We used this formalism to discuss Fibonacci and Ising anyons, two examples of non-abelian anyons. We concluded our work with the toric code and the Kitaev Honeycomb Lattice, two solvable condensed matter models in which anyons emerge.

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VII. APPENDIX

A. The Topology of Exchange

Consider a system of two, indistinguishable particles with hard-core repulsion at the positions $r_1, r_2 \in \mathbb{R}^n$ in the center of mass frame. Suppose that the system is described by the wave function $\psi(r_1, r_2)$. The relative configurations of these particles in space can be categorized by the vector $v = r_1 - r_2$, where $v \neq 0$ due to the hard-core repulsion. In addition, because the particles are indistinguishable, the configurations defined by v and $-v$ are the same. Thus, we make the identification $v \sim -v$, which is equivalent to taking the quotient of the space by $\mathbb{Z}_2$. Then, we see that the configuration space is $(\mathbb{R}^n \backslash \{0\})/\mathbb{Z}_2 = (\mathbb{S}^{n-1} \times \mathbb{R})/\mathbb{Z}_2 = \mathbb{P}^{n-1} \times \mathbb{R}^+$, where $\mathbb{S}^{n-1}$ is the $n-1$-sphere and where $\mathbb{P}$ is the real projective plane [14]

Now, suppose that we exchange the two particles. The configuration vector v continuously transforms into the associated vector $-v$, and, in the process, traces out a closed loop γ in configuration space. The trajectories taken by the particles are arbitrary continuous paths, so the exchange defined by γ is equivalent to any exchange defined by a loop that can be continuously deformed into γ. Thus, particle exchange has the same structure as the homotopy classes of loops in X, which is the fundamental group $\pi_1(\mathbb{P}^{n-1} \times \mathbb{R}^+) = \pi_1(\mathbb{P}^{n-1}) \times \pi_1(\mathbb{R}^+) = \pi_1(\mathbb{P}^{n-1})$ [14].

In three spatial dimensions, the configuration space is $\mathbb{P}^2 \times \mathbb{R}^+$, which has the fundamental group $\mathbb{Z}_2$. Then, because $\gamma^2 = 1$ for each $\gamma \in \mathbb{Z}_2$, we see that exchanging the two particles twice leaves the system invariant. Then, the phase θ accumulated by the wave-function under exchange must satisfy [5]:

$$\psi(r_1, r_2)e^{2i\theta} = \psi(r_1, r_2) \tag{42}$$

This imposes the condition that $\theta = 0, \pi$, which are the bosonic and and fermionic exchange statistics respectively.

In two spatial dimensions, however, the configuration space is $\mathbb{P}^1 \times \mathbb{R}^+$, which has the fundamental group $\mathbb{Z}$. Then, we see that exchanging the particles twice in succession does not necessarily restore the system to its original state, and there is no constraint on the phase θ accumulated by the wave-function upon particle exchange.

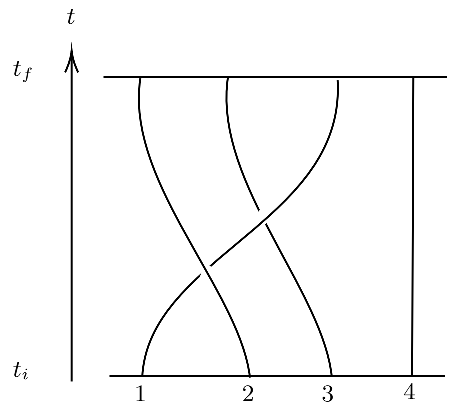

FIG. 4: Here we see an example of a braid diagram with four braids. We have plotted the time t along the vertical axis with initial time t_i and final time t_i. We see that this is the composition of a clockwise crossing of the first and second particles and a counterclockwise crossing of the second and third particles. This corresponds to the B_4 element $\sigma_1^{-1}\sigma_2$.

B. Braid Diagrams

The braid group B_n has a useful diagrammatic representation that we shall briefly introduce. These diagrams are known as braid diagrams; an example of a such a diagram is shown in Figure 4. Consider a system of n identical anyonic particles with hard-core repulsion at the positions $r_1, .., r_n$. We denote each position r_k as the site i. We (discretely) plot the site along the horizontal axis and plot time along the vertical axis, and graph the trajectory of the particle, called a braid, as it adiabatically moves between sites. Because we are representing particle exchange, each particle must be at some site at the initial and final times. Then, at the initial and final times, there must be one braid at each site. In addition, the braids can only move forward in time, and due to the hard-core repulsion no two braids can intersect. Braids can, however, cross each other, which corresponds to an exchange of the particles.

Notice that each braid can only exchange with a braid at an adjacent site. However, this is not a problem, as the braid group is generated by the exchange of particles at adjacent sites. See Figure 5 for diagrams corresponding to the braid generators. We say that a crossing is counterclockwise if the particle at position i crosses over the particle at position $i+1$ and say that a crossing is clockwise if the

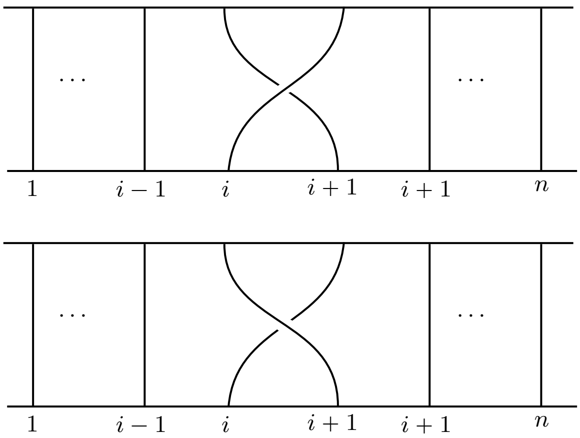

FIG. 5: In the top figure, we see that particle i crosses over the particle $i+1$. No other crossings occur This corresponds to the B_n counterclockwise generater σ_i. In the figure below is the same, except the crossing is clockwise. Then, this corresponds to the B_n inverse generator σ_i^{-1}.

particle at position i crosses under the particle at position $i+1$. These crossings correspond to counterclockwise and clockwise exchanges respectively. Multiplication of two group elements is denoted by "stacking" the two diagrams, that is if the first diagram ends at time t_1 and start the second diagram at this time t_1.

Two braid diagrams are said to be equivalent if the strings of one can be continuously deformed into the strings of the other without crossing two strings. We recognize that these as the statement that two exchange paths of particles are the same if they are topologically equivalent to each other.

Braid diagrams are a useful tool to visualize the braid group, and they demonstrate the topological nature of particle exchange. We can use braid diagrams to perform computations with the braid group, and they are particularly helpful with determining if two elements are equivalent [15].

C. Computing the Cyon Wave-function

Consider a system of two cyons at the positions $\mathbf{r}_1$ and $\mathbf{r}_2$. The wave function describing this systemwith conjugate momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ in some finite but arbitrarily large volume V. The Hamiltonian of this system $\mathcal{H}$ is given by:

$$\mathcal{H} = \frac{(\mathbf{p}_1 - \mathbf{A}_1)^2}{2m} + \frac{(\mathbf{p}_2 - \mathbf{A}_2)^2}{2m} \tag{43}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are the vector potentials given by:

$$\mathbf{A}_1 = \frac{\Phi}{2\pi} \frac{\hat{z} \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \tag{44}$$

$$\mathbf{A}_2 = \frac{\Phi}{2\pi} \frac{\hat{z} \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^2} \tag{45}$$

We shall now consider the system in terms of the center of mass coordinates $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and relative coordinates $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the Hamiltonian $\mathcal{H}$ is [16]:

$$\mathcal{H} = \frac{\mathbf{P}^2}{4m} + \frac{(\mathbf{p} - q\mathbf{A})^2}{m} \tag{46}$$

with $\mathbf{P}$ being the momentum conjugate to $\mathbf{R}$ and $\mathbf{p}$ being the momentum conjugate to $\mathbf{r}$, and where $\mathbf{A}$ is the relative vector potential given by:

$$\mathbf{A} = \frac{\Phi}{2\pi|\mathbf{r}|}\hat{\boldsymbol{\phi}} \tag{47}$$

We are able to separate the center of mass motion term and relative motion term in the Hamiltonian, so the wave-function of the system Ψ will be of the form:

$$\Psi(\mathbf{R}, \mathbf{r}) = \psi_R(\mathbf{R})\psi_r(\mathbf{r}) \tag{48}$$

where ψ_R is the wave-function corresponding to the first term in the Hamiltonian and ψ_r is the wave function corresponding to the second. We now consider the second term. Consider the gauge function Λ given by [8]:

$$\Lambda = \frac{\Phi\phi}{2\pi} \tag{49}$$

The vector potential $\mathbf{A}'$ after the gauge transformation corresponding to this gauge function is given by $\mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla}\Lambda = 0$. The gauge transformation also alters the derivatives in the Schrödinger equation, which causes the wave-function to transform in the following way [17]:

$$\psi_r'(\mathbf{r}) = e^{iq\Phi\phi/2\pi}\psi_r(\mathbf{r}) \tag{50}$$

In the primed gauge, the Hamiltonian $\mathcal{H}'$ is:

$$\mathcal{H}' = \frac{\mathbf{P}^2}{4m} + \frac{\mathbf{p}^2}{m} \tag{51}$$

We see that the wave functions ψ_R and ψ_r' are solutions to the free field Schrödinger equation. Using these known functions and using Equation 50 to transform back into the unprimed gauge, we find:

$$\Psi(\mathbf{R}, \mathbf{r}) = \sum_{\ell, \mathbf{k}_R} c_{\ell, \mathbf{k}_R} e^{i\mathbf{k}_R \cdot \mathbf{R}} J_\ell(k_r r) e^{i(\ell - q\Phi/2\pi)\phi} \tag{52}$$

for the constants $c_{\ell,\mathbf{k}_R}$ which are such that the wave-function is normalized, where $\mathbf{r} = (r, \phi)$, and where J_ℓ is the ℓ^{th} bessel function of the first kind.