Entrainment

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Abstract

The driven Van der Pol oscillator displays entrainment, quasiperiodicity, and chaos. The characteristics of these different modes are discussed as well as the transitions between the modes. Entrainment in collections of coupled oscillators is discussed with reference to synchronization of firefly flashing within a swarm.

Introduction

One of the first recorded observations of a nonlinear effect in dynamics was made by Christiaan Huygens in 1665. He wrote in a letter about observing two pendulum clocks that he had made hanging on a wall. He noticed that they swung at the same frequency and were always in antiphase. He tried perturbing one of the pendulums but it eventually went back into antiphase with the other and they again swung at the same frequency. He took one of the clocks and put it on the opposite wall of the room, and the clocks gradually fell out of step. When the pendulums were "coupled" through contact with the same wall, they eventually fell in step. This phenomenon is called entrainment.

Entrainment is a property of coupled oscillators. The simplest case occurs when the coupling is only one-way; the frequency of one oscillator is constant, and entrainment occurs when the other matches this frequency. For example, if one of Huygens' pendulum bobs had infinite mass, its motion would be unaffected by the other pendulum. One can model this situation as a forced single oscillator.

Entrainment of this type is a common phenomenon in nature. It is also a useful control mechanism for some systems. For example, a pacemaker entrains the nerve cells of a heart to prevent heart attacks. A powerful triode generator, a now antique device that produces a periodically alternating electric current, can be entrained by a less powerful triode generator that has a more precise frequency. Circadian rhythms are an example of how the day light cycle drives sleeping cycles in plants and animals.

I will investigate the forced Van der Pol equation as an example of this one-way coupling. Later, I will discuss a system of many coupled oscillators, a swarm of fireflies.

Part I: Van der Pol oscillator and Driven Entrainment

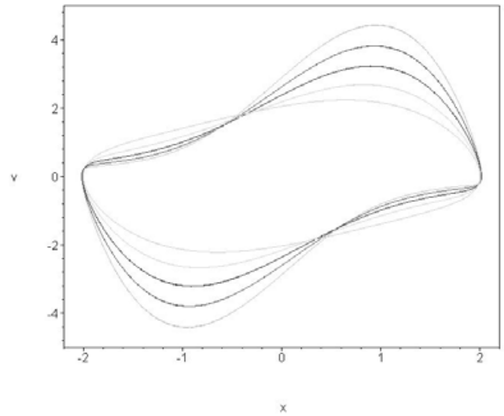
The Van der Pol equation is

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0$$

or, in the first order form,

$$\frac{dx}{dt} = y\ , \ \frac{dy}{dt} = \varepsilon(1 - x^2)y - x$$

When x is smaller than 1, then the system experiences negative damping and its amplitude increases. For x greater than 1, damping causes the amplitude to decrease. So the entire phase plane is a basin of attraction for the stable limit cycle. When ε is small, approximate perturbation methods show that the limit cycle is circular. For larger ε, numerical solutions show the limit cycles in figure 1.



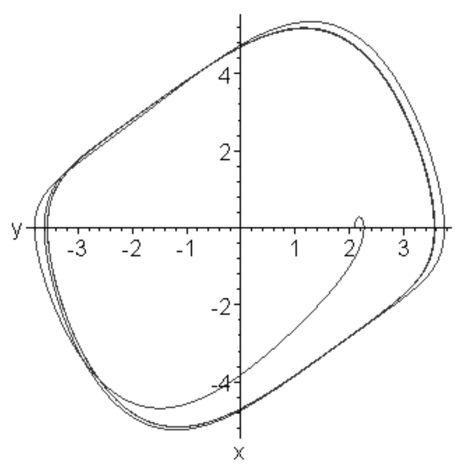
Because the Van der Pol equation exhibits oscillation no matter what its initial conditions, it is called self-exciting.

Figure 1: Limit Cycles for Van der Pol Oscillator with ε=0.5, 1, 1.5, 2, 2.5 going from the most circular curve to the curviest.

Entrainment can be observed in the forced Van der Pol equation:

$$\frac{dx}{dt} = y\,,\ \ \frac{dy}{dt} = \varepsilon(1 - x^2)y - x + f\cos(\omega t)$$

The limit cycle of this system appears similar to others, but by examining the time series we can confirm that the system is oscillating at the drive frequency.



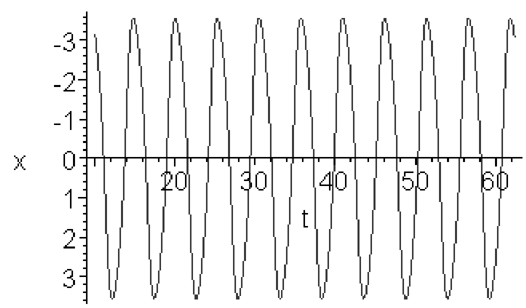
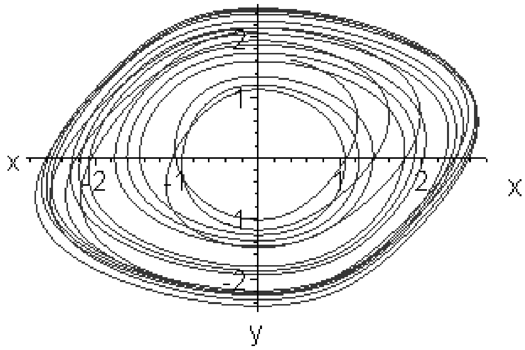


Figure 2: Entrainment: Limit Cycle for driven Van der Pol Oscillator with ε=0.25, f=3, ω=1.2

Figure 3: Entrainment: Time series for driven Van der Pol Oscillator with ε=0.25, f=3, ω=1.2

In figure 3, there are ten cycles in 52.4 time units, giving an angular frequency of ω=1.2, confirming that the system is entrained at the driving frequency.

A related phenomenon to entrainment is quasiperiodicity. This occurs when the amplitude of the forcing function is fairly low. The natural frequency of the system competes with the forcing frequency in determining the frequency of the system, and the result is that the frequency and amplitude varies with time. The phase plain trajectory is confined to an annular region.



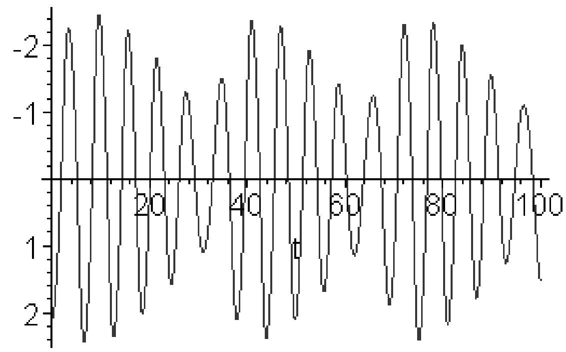


Figure 4: Quasiperiodicity: Phase trajectory for driven Van der Pol Oscillator with ε=0.25, f=0.4, ω=1.2

Figure 5: Quasiperiodicity: Time series for driven Van der Pol Oscillator with ε=0.25, f=0.4, ω=1.2

It is plain that the amplitude is changing, but to see the evidence for competing frequencies, it is best to look at a power spectrum.

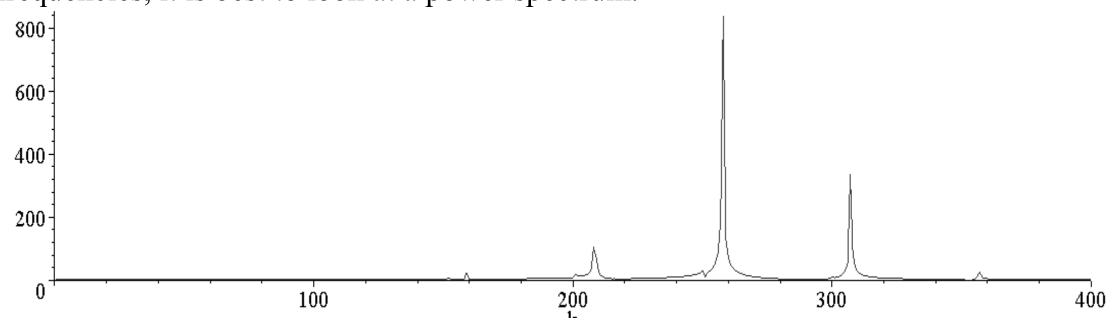


Figure 6: Quasiperiodicity: Power spectrum for driven Van der Pol Oscillator with ε=0.25, f=3, ω=1.2. On x-axis, 256 corresponds to a frequency of 1.

The large peak corresponds to $\omega_o = 1$. The peak to its right corresponds to $\omega = 1.2$. There are other peaks corresponding $\Delta\omega=0.2$.

The forcing amplitude for the transition from quasiperiodicity to entrainment can be calculated numerically. Figure 7 is the same principle as a bifurcation diagram. For a given forcing amplitude, the forced Van der Pol equations are numerically solved, and after time is allowed for transients to disappear the angular velocity of the system is measured whenever the phase of the driving force repeats, i.e. whenever $t=2*\pi/\omega$. If the system has a period equal to the driving frequency, a single point appears for that forcing amplitude. If a discrete number of points appear, then the system is oscillating at that multiple of the period of the driving force. Figure 7 shows that entrainment occurs for $f > 0.75$ for these particular parameters.

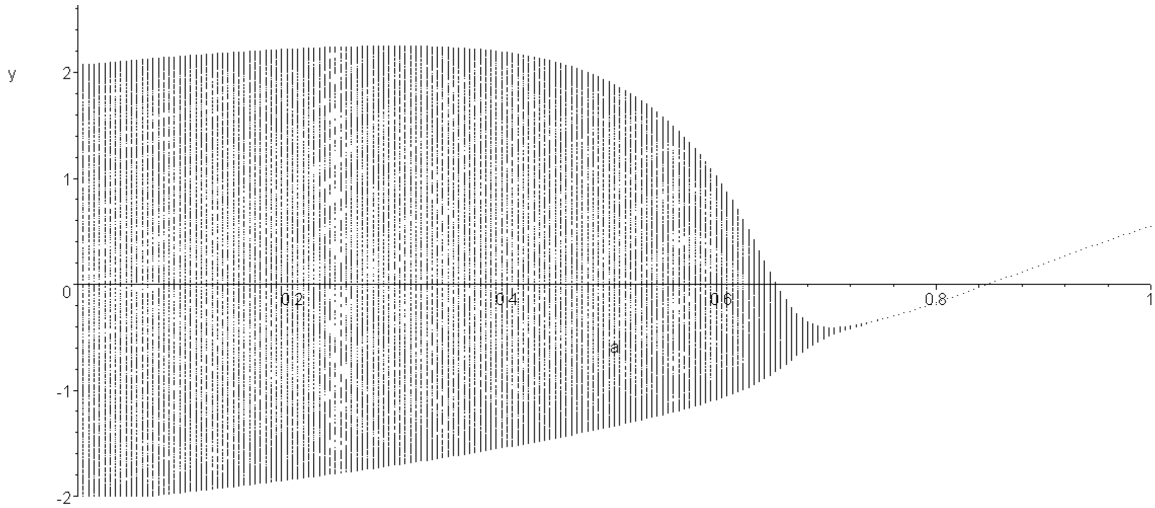


Figure 7: Measurements of the angular velocity (y) taken when the phase of the driving force is zero. ε=0.25, ω=1.2. The x-axis is the forcing amplitude, f.

For frequencies less than the transition frequency, the system isn't oscillating at the drive frequency. The ratio of the total period for the system to repeat (approximately 40 seconds in figure 5) to the driving period is an irrational number, as indicated by the Poincaré diagram, which is a continuous curve.

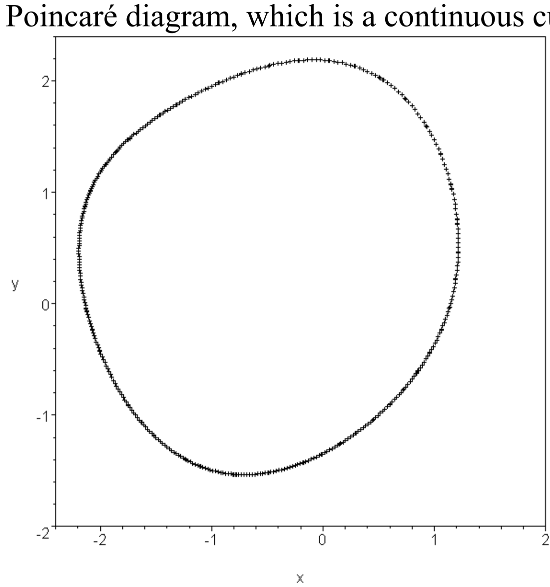
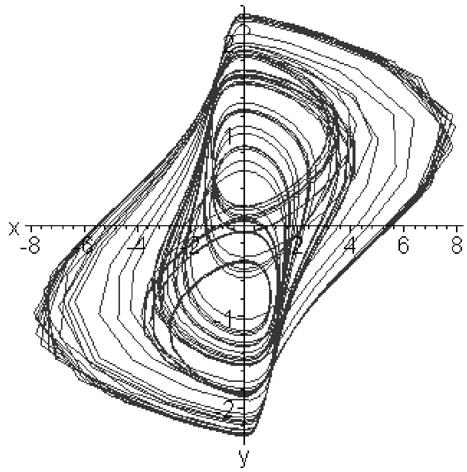


Figure 8: Quasiperiodicity: Poincaré section for driven Van der Pol Oscillator with ε=0.25, f=0.4, ω=1.2

Chaos in the Van der Pol Oscillator

Chaos can occur in the Van der Pol oscillator, although this occurs for only a small set of parameters. The Lyapunov exponent is largest (positive Lyapunov exponent indicates chaotic motion) when ε=3, ω=1.788 and f=5 [1]. The fractal nature of the Poincaré plot in figure 11 indicates that the motion is chaotic.



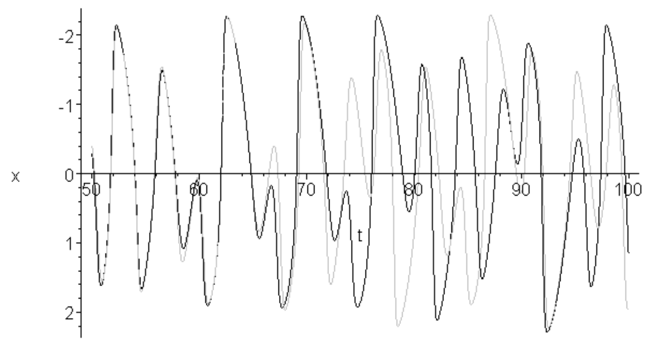
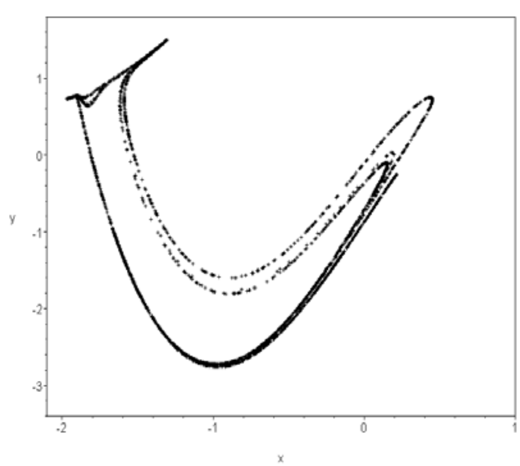


Figure 9: Chaos: Phase trajectory for driven Van der Pol Oscillator with ε=3, f=5, ω=1.788

Figure 10: Chaos: Time series for driven Van der Pol Oscillator with ε=3, f=5, ω=1.788. Both have initial velocity of zero. Initial position of black line is 2.05, of gray is 2.09



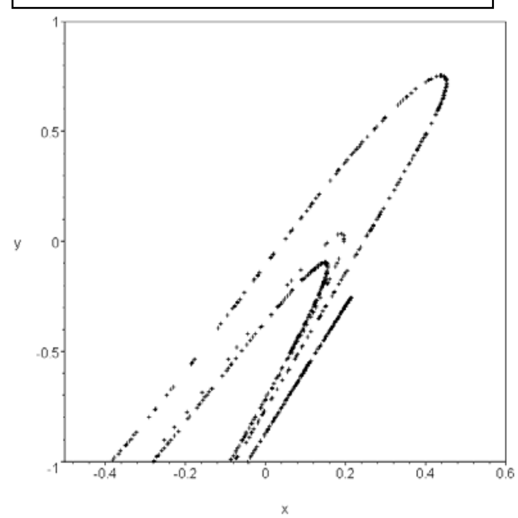
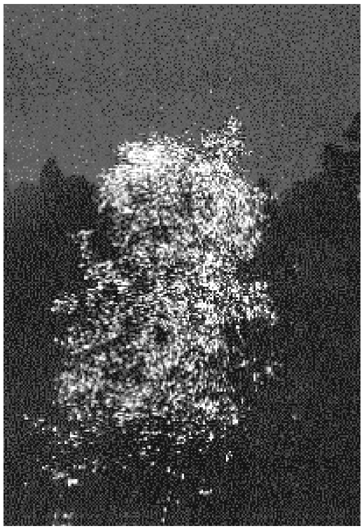


Figure 11 a: Chaos: The Poincaré section for driven Van der Pol Oscillator with ε=3, f=5, ω=1.788. Figure 11 b: Detail of the fractal nature of the section.

Part II: Fireflies and Coupled Entrainment

In nature, systems where the effect of the coupling is on both oscillators are much more common. For example, when someone is jogging, they swing their arms in time with their legs. It feels unnatural to have your arms swing at a different rate. When multiple oscillators are present, entrainment can still occur. Depending on how the oscillators are coupled, more complex patterns can occur as well.

One of the most interesting examples of a large system of coupled oscillators are the firefly swarms of south Asia. At nightfall, large numbers of fireflies gather in trees. At first, the timing of their flickering is random and scattered, but as time passes the fireflies become more and more synchronized until all of them flash in unison.



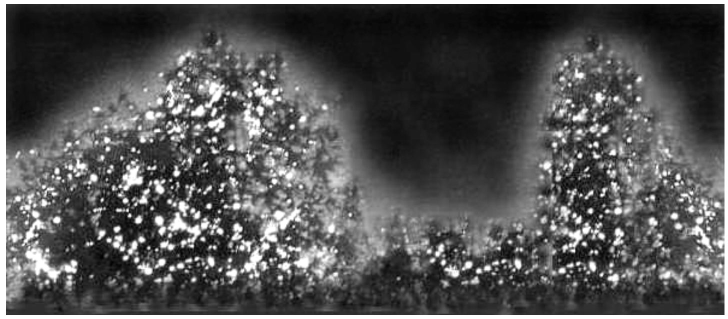


Figure 12: Trees lit with fireflies.

The simplest model of this phenomenon is a collection of a large number of identical oscillators with randomized starting values from 0 to a threshold value. These oscillators each increase linearly in time until they reach the threshold value, at which point they "fire" and their value returns to zero. When one oscillator fires, the value of all other oscillators jumps up by some constant amount. This may push some other oscillators over the threshold value, causing them to fire, and so on. Computer simulations show that the oscillators always end up flashing in unison [2].

More complicated models can take into account varying frequencies of the oscillators. This occurs in the fireflies; there are large frequency differences between different species, and it is observed that different species of firefly do not end up flashing in unison because of these differences. Another complication is giving each oscillator a radius of effect, outside of which the firing of the oscillator has no effect on the others. Depending on the spatial distribution of fireflies, different patterns can occur. If the flies are arranged in a ring, waves of flashing can circle around.

The behaviour of groups of fireflies turns out to be very complex. It is remarkable that through entrainment, a firefly on one side of a tree will flash at the same time as another firefly on the other side that it is not even aware of. Entrainment may be simple to observe and characterize compared to chaos, but its effect has deep implications for dynamics, electrical circuits, and the everyday life of all living things.

References

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