# Inhomogeneous PDE

The general idea, when we have an inhomogeneous linear PDE with (in general) inhomogeneous BC, is to split its solution into two parts, just as we did for inhomogeneous ODEs: $u = u_h + u_p$. The first term, $u_h$, is the solution of the *homogeneous* equation which satisfies the inhomogeneous boundary conditions (plus the initial conditions, if the time is a variable) of the full problem. We have already learned how to obtain this solution for all the equations of interest to us.

Then, $u_p$ must be a solution of the *inhomogeneous* equation, and satisfy homogeneous BC (plus homogeneous initial conditions, if time is a variable) because $u_h$ has "taken care" of any inhomogeneous parts in the BC and IC. So now we need to figure out how to find this solution – we will use Green's functions for this, like we did for ODEs. Note that in principle "guessing" is also fine, but more difficult for these more complicated equations. The textbook shows a solution based on guessing for the Poisson equation, if the inhomogeneous term is a constant (ie., for a constant density of charge). Usually one has to deal with problems that are more complicated than that, so I'll show you the full solution. I'm sure you'll run into precisely such types of problems in your E&M course.

I will show you how this works in two cases – the 2D Poisson equation (time is not a variable), and the 1D heat equation (time is a variable). Once you understand the general idea, the generalizations to other equations should be straightforward.

## 0.1 2D Poisson equation

Suppose that we are asked to find the electric potential in a rectangular region of space $x \in [0, a]; y \in [0, b]$ where there is a known density of charge $\rho(x, y)$ (I will absorb the $-\frac{1}{\epsilon_0}$ factor into it, for simplicity of notation). In other words, we need to solve for:

$$\nabla^2 u(x, y) = \rho(x, y) \tag{1}$$

Let's say the boundary conditions are of Dirichlet type, $u(0, y) = f_1(y); u(a, y) = f_2(y); u(x, 0) = g_1(x), u(x, b) = g_2(x)$ where the f's and g's are known functions.

As discussed above, we search for $u(x, y) = u_h(x, y)+u_p(x, y)$. Here, $u_h$ satisfies the hom. equation:

$$\nabla^2 u_h(x, y) = 0$$

and the full BC: $u_h(0, y) = f_1(y); u_h(a, y) = f_2(y); u_h(x, 0) = g_1(x), u_h(x, b) = g_2(x)$. We know how to solve for this, by writing it as a sum of simpler problems which have hom. BC on opposite sides (like we did for the steady-state solution in the 2D rectangle. Note that mathematically that problem for $u_s$ and this problem for $u_h$ are identical). For each of these simpler problems we can then use the separation of variables to find the solution, and the total $u_h$ is the sum of these solutions.

Next, we need to find the particular solution $u_p = u - u_h$. The equation satisfied by this is

$$\nabla^2 u_p(x, y) = \nabla^2 u(x, y) - \nabla^2 u_h(x, y) = \rho(x, y)$$

i.e., it is indeed the full inhomogeneous equation. However, all its BC are homogeneous, because $u_p(0, y) = u(0, y) - u_h(0, y) = 0$ and so are $u_p(a, y) = u_p(x, 0) = u_p(x, b) = 0$. This is a direct parallel with what we did for ODEs. There we also left $u_h$ to take care of any inhomogeneous BCs or ICs, and we defined the Green's function such that the BCs or ICs for $u_p$ are homogeneous.

Note : if the BCs for $u$ are changed – for instance, suppose that on the $x = 0$ edge we are given the electric field $\frac{\partial u}{\partial x}|_{x=0} = e_1(y)$ – then we require that $\frac{\partial u_h}{\partial x}|_{x=0} = e_1(y)$, and $\frac{\partial u_p}{\partial x}|_{x=0} = 0$. So $u_p$ has BCs of the same type as the full solution, except they are always homogeneous. In the following, I will assume Dirichlet-type BCs like those specified above.

Ideally, we'd like to find a Green's function $G(x, y; \xi, \chi)$ such that:

$$u_p(x, y) = \int_0^a d\xi \int_0^b d\chi G(x, y; \xi, \chi)\rho(\xi, \chi)$$

This is just a straightforward generalization of the ODE Green's function, and it's easy to show by substituting into the equation satistied by $u_p$ that $G$ must satisfy the inhomog. PDE:

$$\nabla^2 G(x, y; \xi, \chi) = \frac{\partial^2}{\partial x^2} G(x, y; \xi, \chi) + \frac{\partial^2}{\partial y^2} G(x, y; \xi, \chi) = \delta(x - \xi)\delta(y - \chi)$$

and all the homog. BC. In other words, $G(x, y; \xi, \chi)$ is the electric potential at point $(x, y)$ created by a unit point charge placed at $(\xi, \chi)$, if all the BC are homogeneous (you can see from the link between $u_p$ and $G$ that $u_p$ will vanish on all the edges if and only if $G$ vanishes on all the edges; and more generally that the BCs for $G$ are identical to those for $u_p$, and thus homogeneous). Physically, the Green's function has precisely the same meaning like for ODEs, except now we have a 2D problem, not a 1D one. The generalization to 3D problems is hopefully fairly obvious, as well.

With this meaning of $G$, the equation for $u_p$ is just the superposition principle, telling us that the total potential at $x, y$, i.e. $u_p(x, y)$, is the "sum" (integral) of the potentials created at this point by charges $\rho(\xi, \chi)$ located at $\xi, \chi$. To find the total potential we have to "sum" over all the possible locations of these charges, in other words everywhere inside the region of interest.

The question is how to find this $G$. Here's the logic. In all problems we've solved so far, we always ended up expressing our solution as a linear combination of eigenfunctions – so that's probably what will happen in this case as well. In other words, we expect to find:

$$u_p(x, y) = \sum_\alpha c_\alpha \phi_\alpha(x, y).$$

where $\alpha$ is a complete set of indexes for the complete set of eigenfunctions $\phi_\alpha(x, y)$. The question is how to choose which eigenfunctions $\phi_\alpha(x, y)$ to use? To decide this, remember that the equation we're trying to solve is:

$$\rho(x, y) = \nabla^2 u_p(x, y) = \sum_\alpha c_\alpha \nabla^2 \phi_\alpha(x, y).$$

So we should choose the eigenfunctions $\phi_\alpha(x, y)$ such that the expressions $\nabla^2\phi_\alpha(x, y)$ are as simple as possible, but this must also be an eigenproblem, because we know that these are eigenfunctions. The answer is to choose the eigenfunctions to be the solutions of the eigenproblem:

$$\nabla^2\phi(x, y) = \lambda\phi(x, y) \tag{2}$$

where the eigenvalues $\lambda$ are such that the homogeneous BC are satisfied, $\phi(0, y) = \phi(a, y) = \phi(x, 0) = \phi(x, b) = 0$ (because each $\phi_\alpha$ must satisfy the proper homogeneous BCs if we want their sum $u_p$ to also satisfy them). Again, for a problem with a different type of BCs, the BCs for $\phi$ change accordingly, but remain homogeneous.

Solving this eigenproblem is easy, because this is a homog. PDE with homog. BCs. We know that we can directly use separation of variables $\phi(x, y) = X(x)Y(y)$ which leads to:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

in other words

$$\frac{d^2X}{dx^2} = -AX; \quad \frac{d^2Y}{dy^2} = -BY; \quad \lambda = -(A + B)$$

with BCs now giving $X(0) = X(a) = Y(0) = Y(b) = 0$. We've solved these many times, as we know we find $A_n = \left(\frac{n\pi}{a}\right)^2, n = 1, 2, ...$ and $X_n(x) = \sin\frac{n\pi x}{a}$; and $B_m = \left(\frac{m\pi}{b}\right)^2, m = 1, 2, ...$ and $Y_m(y) = \sin\frac{m\pi y}{b}$.

So the eigenvalues and eigenfunctions are characterized by two positive integers $\alpha = (n, m)$, and we have $\phi_{nm}(x, y) = X_n(x)Y_m(y)$ and $\lambda_{nm} = -(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2})$. Note that if we had different BC (von Neumann, or mixed) this would change the specific forms of $X_n, Y_m$ and the specific eigenvalues to something else. But in all cases, because these are Sturm-Liouville equations, these eigenfunctions are guaranteed to be orthonormal and complete. Therefore, the particular solution can indeed be written as:

$$u_p(x, y) = \sum_{n,m} c_{nm}\phi_{nm}(x, y).$$

It certainly satisfies the BC, because each individual $\phi_{nm}$ does. The final thing we need to do now, is to find the coefficients $c_{nm}$ so that $u_p$ satisfies the inhomg. PDE.

Putting this guess into our equation, we find after using Eq. (2) that:

$$\nabla^2 u_p(x, y) = \sum_{n,m} c_{nm}\nabla^2\phi_{nm}(x, y) = \sum_{n,m} c_{nm}\lambda_{nm}\phi_{nm}(x, y)$$

so we must have:

$$\sum_{n,m} c_{nm}\lambda_{nm}\phi_{nm}(x, y) = \rho(x, y)$$

Now we use the orthogonality of the eigenfunctions to find (note that here the weight is unity, see discussion for Sturm-Liouville equations)

$$c_{nm}\lambda_{nm} = \frac{\int_0^a d\xi \int_0^b d\chi\rho(\xi, \chi)\phi_{nm}(\xi, \chi)}{\int_0^a d\xi \int_0^b d\chi[\phi_{nm}(\xi, \chi)]^2}$$

I know you are used to writing these as integrals over $x$ and $y$. You will see in a bit why I prefer to change the name of the integration variables to $\xi, \chi$ – these are definite integrals so it makes no difference what names we use for the integration variables. The denominators are some numbers that are independent of the charge density $\rho$. For simplicity, let me denote them by

$$\langle\phi_{nm}, \phi_{nm}\rangle = \int_0^a d\xi \int_0^b d\chi[\phi_{nm}(\xi, \chi)]^2 = \int_0^a d\xi[X_n(\xi)]^2 \int_0^b d\chi[Y_m(\chi)|^2$$

(you will encounter similar notation in advanced courses of quantum mechanics). For our simple sine solutions obtained for the Dirichlet boundary conditions, these numbers are $\langle\phi_{nm}, \phi_{nm}\rangle = \frac{ab}{4}$.

Now we substitute these values for $c_{nm}$ in $u_p$:

$$u_p(x, y) = \sum_{n,m} \frac{1}{\lambda_{nm}} \frac{\int_0^a d\xi \int_0^b d\chi\rho(\xi, \chi)\phi_{nm}(\xi, \chi)}{\langle\phi_{nm}, \phi_{nm}\rangle}\phi_{nm}(x, y)$$

and now you see why I needed to save $x, y$ and couldn't use them as as the dummy variables inside the integrals. But remember that we're looking for:

$$u_p(x, y) = \int_0^a d\xi \int_0^b d\chi G(x, y; \xi, \chi)\rho(\xi, \chi)$$

so by comparing the two, the Green's function is:

$$G(x, y; \xi, \chi) = \sum_{n,m} \frac{\phi_{nm}(x, y)\phi_{nm}(\xi, \chi)}{\lambda_{nm}\langle\phi_{nm}, \phi_{nm}\rangle}$$

And there it is: we have the Green's functions in terms of eigenvalues and eigenfunctions, which are already known. We can now calculate $u_p(x,y)$ for any desired $\rho(x,y)$. For other types of BCs, the particular expressions for $\lambda_{nm}, \phi_{nm}$ will change, but the general expression for $G$ stays the same.

If you have to solve a 3D Poisson equation, then there will be 3 sets of eigenvalues $\phi_{n,m,k}(x,y,z) = X_n(x)Y_n(y)Z_n(z)$ where $\lambda_{nmk} = -A_n - B_m - C_k$ is the sum of eigenvalues for each simple ODE; and going through the same steps, you'll find:

$$G(x,y,z;\xi,\chi,\zeta) = \sum_{n,m,k} \frac{\phi_{nmk}(x,y,z)\phi_{nmk}(\xi,\chi,\zeta)}{\lambda_{nmk}\langle\phi_{nmk},\phi_{nmk}\rangle}$$

to be the potential at $(x,y,z)$ if a point charge is located at $(\xi,\chi,\zeta)$, and the proper hom. BC are satisfied on all outside surfaces. Again, we can calculate $u_p(x,y,z)$ for any density of charge $\rho(x,y,z)$ by just integrating over all space where charges are placed:

$$u_p(x,y,z) = \int_0^a d\xi \int_0^b d\chi \int_0^c d\zeta G(x,y,z;\xi,\chi,\zeta)\rho(\xi,\chi,\zeta)$$

Of course, let's not forget that for the full solution, we then need to add $u_h$ to $u_p$ to get the full $u$.

In other coordinates things change accordingly, for instance in 2D, if we choose to work with $\rho,\theta$ instead of $x,y$, then we need to look for the eigenvalues $\nabla^2\phi(\rho,\theta) = \lambda\phi(\rho,\theta)$ with the proper hom. BC, where now $\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\theta^2}$ (this is what the Laplacean looks in polar coordinates, I may give this to you as a homework to check). So this will change the expressions of the eigenfunctions, but the final expression for $G$ is of the same type as above.

### 0.2 1D heat equation

Let us now consider an example where time is also a variable. Consider the inhomogeneous equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{\kappa}\frac{\partial u(x,t)}{\partial t} = \rho(x,t)$$

where $x \in [0,a]$ and we have the IC $u(x,t=0) = f(x)$ and the BCs $u(0,t) = T_L, u(a,t) = T_R$.

Again, we split the problem in 2 parts, $u(x,t) = u_h(x,t) + u_p(x,t)$. The homog. solution satisfies the homog. equation, and the IC and BC of the full problem:

$$\frac{\partial^2 u_h(x,t)}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u_h(x,t)}{\partial t}$$

where $u_h(x,t=0) = f(x)$ and $u_h(0,t) = T_L, u_h(a,t) = T_R$. We know how to solve for $u_h$ in 2 steps, first the steady-state solution and then the transient solution and then $u_h = u_s + u_t$, so I won't say anymore about it.

The second part $u_p(x,t) = u - u_h$ satisfies the full inhomogeneous equation:

$$\frac{\partial^2 u_p(x,t)}{\partial x^2} - \frac{1}{\kappa}\frac{\partial u_p(x,t)}{\partial t} = \rho(x,t)$$

but homog. IC and BC: $u_p(x,t=0) = 0$ and $u_p(0,t) = u_p(a,t) = 0$.

Again, we're aiming to find a Green's function such that:

$$u_p(x,t) = \int_0^a d\xi \int_0^\infty d\tau G(x,t;\xi,\tau)\rho(\xi,\tau)$$

Note that the time interval runs to infinity, as that's the interval of time we are concerned with.

Putting this guess into the equation satisfied by $u_p$, we find that the equation satisfied by the Green's function must be (as you hopefully expect):

$$\frac{\partial^2 G(x,t;\xi,\tau)}{\partial x^2} - \frac{1}{\kappa}\frac{\partial G(x,t;\xi,\tau)}{\partial t} = \delta(x-\xi)\delta(t-\tau)$$

The approach to finding $G$ is just as before: again, we expect that $u_p(x,t) = \sum_n a_n(t)X_n(x)$, where $X_n(x)$ are a complete set of eigenfunctions. Note that because $t$ is a variable, the coefficients must depend on it. We know that each $X_n(x)$ must satisfy the homog. BC, so that their sum $u_p$ also does. Still, we have the freedom to choose for which eigenproblem are these $X_n(x)$ the eigenfunctions – of course, we will do this so as to simplify the spatial derivatives as much as possible.

For this particular problem, we know that spatial part involves the 2nd derivative with respect to $x$, so we will choose the eigenproblem $\frac{d^2X}{dx^2} = \lambda X$. Together with the hom. BC $X(0) = 0, X(a) = 0$ we have $\lambda_n = -\left(\frac{n\pi}{a}\right)^2, X_n(x) = \sin\frac{n\pi x}{a}, n = 1, 2, ....$ So with this choice, we can now put the expression

$$u_p(x,t) = \sum_{n=1}^{\infty} a_n(t)X_n(x)$$

into the inhomog. PDE, to find:

$$\sum_{n=1}^{\infty}\left[-\left(\frac{n\pi}{a}\right)^2 a_n(t) - \frac{1}{\kappa}\frac{da_n(t)}{dt}\right]X_n(x) = f(x,t).$$

Because the $X_n(x)$ are orthogonal (and for this simple equation the weight is 1) we can find the coefficients of the expansion as usual:

$$-\left(\frac{n\pi}{a}\right)^2 a_n(t) - \frac{1}{\kappa}\frac{da_n(t)}{dt} = \frac{\int_0^a dx X_n(x)f(x,t)}{\langle X_n, X_n\rangle} = g_n(t)$$

where again I denoted $\langle X_n, X_n\rangle = \int_0^a dx[X_n(x)]^2$ – these are some numbers independent of $f(x,t)$. For our BC and their corresponding sine functions, $\langle X_n, X_n\rangle = \frac{a}{2}$, the usual value for Fourier series.

We can now solve for $a_n(t)$ – they satisfy simple inhomog. first order ODEs. The hom. solution is $e^{-\kappa k_n^2 t}$ so we use variation of parameters to find a particular solution $a_n(t) = v(t)e^{-\kappa k_n^2 t}$. After a bit of work, we find the general solution:

$$a_n(t) = e^{-\kappa k_n^2 t}\left[C_1 - \kappa\int_0^t d\tau e^{\kappa k_n^2\tau} g_n(\tau)\right] = e^{-\kappa k_n^2 t}\left[C_1 - \kappa\int_0^t d\tau e^{\kappa k_n^2\tau}\frac{\int_0^a d\xi X_n(\xi)f(\xi,\tau)}{\langle X_n, X_n\rangle}\right]$$

Remember that $u_p$ must also satisfy a homogeneous IC (the $u_h$ part took care of any inhomogeneity), i.e. $u_p(x, t=0) = 0 \to a_n(0) = 0 \to C_1 = 0$, and so we have our final solution:

$$u_p(x,t) = -\kappa\sum_{n=1}^{\infty} e^{-\kappa k_n^2 t}\int_0^t d\tau e^{\kappa k_n^2\tau}\frac{\int_0^a d\xi X_n(\xi)f(\xi,\tau)}{\langle X_n, X_n\rangle}X_n(x) = \int_0^t d\tau\int_0^a d\xi G(x,t;\xi,\tau)f(\xi,\tau)$$

where

$$G(x,t;\xi,\tau) = -\kappa\sum_{n=1}^{\infty} e^{-\kappa k_n^2(t-\tau)}\frac{X_n(x)X_n(\xi)}{\langle X_n, X_n\rangle} \text{ if } t > \tau$$

and $G = 0$ if $t < \tau$. Different BC will change the specific $X_n(x), k_n^2$ values, but the general expression remains the same.

Two final notes:

(1) although in the general expression of $u_p$ that we started from, we allowed the integral over the time $\tau$ to run to infinity, you can see that after solving the problem, we find that in practice the integral over $\tau$ only runs up to the time $t$ of interest. This is again a consequence of causality: the solution at time $t$ cannot depend on what happens after time $t$. This is directly analogous to what we found for Green's functions for time-dependent ODEs.

(2) if this was an inhomog. wave equation we would proceed similarly, however we would now have a second order inhomog. ODE for the $a_n(t)$ coefficients (because the wave eq. has $\partial^2 u/\partial t^2$, not $\partial u/\partial t$, like the heat equation). Not a problem, because we know how to solve 2nd order inhomog. ODES with constant coefficients! Of course, we now find 2 constants of integration but we also have 2 hom IC, so we can find them and then extract the Green's function. This will have a somewhat more complicated expression for the time-depending part. Since I do not expect that you will need to solve such an equation during your undergrad studies, I will stop here. But you are now fully equiped to actually solve such problems, if need be. They have all been reduced to simple ODEs and eigenproblems – each individual part of the calculation is quite simple, and you simply need to put together all these pieces.