Green's functions

Consider the 2nd order linear inhomogeneous ODE

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u(t) = f(t).$$

Of course, in practice we'll only deal with the two particular types of 2nd order ODEs we discussed last week, but let me keep the discussion more general, since it works for any 2nd order linear ODE. We want to find $u(t)$ for all $t > 0$, given the initial conditions $u(t = 0) = u_0$, $\frac{du}{dt}|_{t=0} = v_0$.

Let us assume that the two linearly-independent solutions $u_1(t), u_2(t)$ of the homogeneous equation are known (we've discussed what these are for the special kinds of equations we will need to solve). Then, we know that the general solution of the inhomogeneous equation is:

$$u(t) = C_1u_1(t) + C_2u_2(t) + u_p(t)$$

where $u_p(t)$ is any particular solution of the inhomogeneous equation. After we find $u_p(t)$, we can use the initial conditions to find C_1 and C_2.

Two ways to find $u_p(t)$ that we've discussed in class are guessing and the variation of parameters. There is nothing wrong with either, except that every time a new $f(t)$ is given, we have to try another guess or go into all the work required by the variation of parameters – we have to redo the whole calculation again to get the new u_p. This becomes quite difficult, especially if $f(t)$ is not a simple function. The idea behind the Green's function is to find a general expression that tells us what $u_p(t)$ is for any $f(t)$ that we care to use. We still need to do is one calculation (to find the Green's function), but once we have it, we can find $u_p(t)$ for any $f(t)$ without much further work.

Before launching into how this works, let me point out that sometimes this solution is shown as a sort of math "trick" related to a certain special way to do variation of parameters – for instance, this is how it's presented in the 6th edition of the textbook, pages 20-23. I want to try to do a bit of a better job in explaining why this works, and how should we think about the meaning of this solution, as physicists. This should help us figure out how to generalize this idea, because we will use it later for PDEs.

The main idea is to "decompose" $f(t)$ as a sum of simple functions, for which we know the particular solutions. Remember that if $f(t) = f_1(t)+f_2(t)$, then $u_p(t) = u_{p_1}(t)+u_{p_2}(t)$, where $u_{p_1}(t)$ is the particular solution for $f_1(t)$, etc. Of course, this would be true if we "broke" $f(t)$ in any number of pieces, so long as we could find the corresponding $u_p(t)$ for each piece. Now, the way we do this "breaking" is with the Dirac function: remember that we can write:

$$f(t) = \int_0^\infty d\tau\delta(t-\tau)f(\tau)$$

which means that $f(t)$ is a sum of short "kicks" (described by the δ-function), and so that the kick applied at τ has the strength $f(\tau)$. Of course, "sum" here is really an integral, because the time τ is a continuous variable. The reason why the integral is from 0 to ∞ is that I am only interested in times $t \geq 0$ – in this sorts of problems we don't care what happened in the past, the question is always what happens after the initial moment $t = 0$.

So, the idea is that if we can find the particular solution for a kick $\delta(t-\tau)$, i.e. a kick applied at time τ, then we're done – we need only "sum" over the particular contributions from all the kicks that contributed to our $f(t)$.

The Green's function $G(t, \tau)$ is the solution of the inhomogeneous equation

$$\frac{d^2G(t,\tau)}{dt^2} + k(t)\frac{dG(t,\tau)}{dt} + p(t)G(t,\tau) = \delta(t-\tau) \tag{1}$$

In other words, it tells us what is a particular solution is we apply a single kick of strength 1 at the time $t = \tau$ – exactly what we need. It has two arguments because, of course, the solution will be different if we kick at different times τ, so we need to keep track of τ as well.

Finally, G is not just any particular solution of this inhomogeneous equation, but we will ASK that it satisfies the initial conditions $G(t = 0, \tau) = 0$, $\frac{dG}{dt}|_{t=0} = 0$. Note the very special form of these initial conditions – they look like the ones for the general solution, but they are both HOMOGENEOUS (i.e., $= 0$)! The reason for this is that I don't want to have to recalculate the Green's function every time I change the values of u_0, v_0 – as you'll see below, the homog. part of the solution can take care of u_0 and v_0.

Once we have this particular solution, we know that for any arbitrary sequence of kicks that makes up $f(t)$, the particular solution must be:

$$u_p(t) = \int_0^\infty d\tau G(t,\tau) f(\tau)$$

(if you're not quite sure about this, plug this u_p in the ODE and check that indeed it satisfies Eq. (**??**)).

Because of the initial conditions satisfied by G, it follows that $u_p(0) = 0$, $\frac{du_p}{dt}|_{t=0} = 0$. So the general solution of our equation is:

$$u(t) = C_1 u_1(t) + C_2 u_2(t) + \int_0^\infty d\tau G(t,\tau) f(\tau)$$

where C_1, C_2 must be chosen such that:

$$u_0 = u(t=0) = C_1 u_1(0) + C_2 u_2(0) + u_p(0) = C_1 u_1(0) + C_2 u_2(0)$$

$$v_0 = \frac{du}{dt}|_{t=0} = C_1 \frac{du_1}{dt}|_{t=0} + C_2 \frac{du_2}{dt}|_{t=0} + \frac{du_p}{dt}|_{t=0} = C_1 \frac{du_1}{dt}|_{t=0} + C_2 \frac{du_2}{dt}|_{t=0}$$

This explains why we chose those initial conditions for $G(t,\tau)$ – this way we can adjust C_1 and C_2 to take care of the actual initial conditions, and we don't need to recalculate G and therefore u_p if we change the initial conditions – all that is needed is to adjust C_1, C_2 accordingly.

In summary, once we know $G(t,\tau)$, we only need to do the integral $\int_0^\infty d\tau G(t,\tau) f(\tau)$ for the function $f(t)$ of interest, and also find C_1 and C_2, and we're done. And this works for any $f(t)$, all we need is to do one integral.

Before calculating $G(t,\tau)$, let's see what general conditions it must satisfy.

1. $G(t,\tau)$ must be continuous at all t, because we expect $u(t)$, and therefore $u_p(t)$ to be continuous at all times. For example, if this equation comes from applying Newton's second law, then u describes the location of some object. Clearly, then, it must change continuously in time. In particular, $G(t,\tau)$ must be continuous when we apply the kick:

$$G(t = \tau_-, \tau) = G(t = \tau_+, \tau)$$

where $\tau_\pm$ are times infinitesimally close to τ (just before and just after).

2. However, since we apply this singular kick at $t = \tau$, we expect something discontinuous to happen to G there. Indeed, it turns out that the derivative is discontinuous:

$$\frac{dG}{dt}|_{t=\tau+} - \frac{dG}{dt}|_{t=\tau-} = 1 \tag{2}$$

in other words, the derivative of $G(t,\tau)$ has a jump of precisely 1 at $t = \tau$ when the kick is applied. This comes directly from Eq. (**??**) if we integrate it from $\tau - \epsilon$ to $\tau + \epsilon$, and we let $\epsilon \to 0$. What we get is:

$$\int_{\tau-\epsilon}^{\tau+\epsilon} dt \left[\frac{d^2 G(t,\tau)}{dt^2} + k(t) \frac{dG(t,\tau)}{dt} + p(t) G(t,\tau) \right] = \int_{\tau-\epsilon}^{\tau+\epsilon} dt \delta(t-\tau) = 1$$

(the second equality is just the value of the integral on the rhs, see delta functions). On the lhs, we have three terms. Let me take them from the end. When $\epsilon \to 0$,

$$\int_{\tau-\epsilon}^{\tau+\epsilon} dt p(t) G(t,\tau) \to 0$$

because the integrand is a continuous function, and we're shrinking the integration interval to zero. Similarly, after integrating by parts, we find:

$$\int_{\tau-\epsilon}^{\tau+\epsilon} dt k(t) \frac{dG(t,\tau)}{dt} = [k(t) G(t,\tau)]_{t=\tau-\epsilon}^{t=\tau+\epsilon} - \int_{\tau-\epsilon}^{\tau+\epsilon} dt \frac{dk}{dt} G(t,\tau) = 0$$

because again we're dealing only with continuous functions in the limit where the integration interval goes to zero. Finally:

$$\int_{\tau-\epsilon}^{\tau+\epsilon} dt \frac{d^2 G(t,\tau)}{dt^2} = \frac{dG}{dt}|_{t=\tau-\epsilon}^{t=\tau+\epsilon} = \frac{dG}{dt}|_{t=\tau+} - \frac{dG}{dt}|_{t=\tau-}$$

and Eq. (**??**) follows directly.

3. As I said, we ask for the simplest possible initial conditions $G(t=0,\tau)=0$, $\frac{dG}{dt}|_{t=0}=0$. The reason for this is that we do not want the Green's function to depend on the initial conditions u_0, v_0 of the equation – if this was the case, then anytime we changed the initial conditions we would have to recalculate G for the new initial conditions. Asking that $G(t=0,\tau)=0$, $\frac{dG}{dt}|_{t=0}=0$ means that $G(t,\tau)$ always has the same expression, and we let C_1 and C_2 adjust so as to take care of u_0, v_0.

So let's find $G(t,\tau)$. Let us consider first the interval $0<t<\tau$. Because in this interval $\delta(t-\tau)=0$, here the equation for G is:

$$\frac{d^2G(t,\tau)}{dt^2}+k(t)\frac{dG(t,\tau)}{dt}+p(t)G(t,\tau)=0$$

in other words here G is a solution of the homogeneous equation, so it must be of the general form:

$$\text{if } 0<t<\tau, \text{ then} G(t,\tau)=a_1u_1(t)+a_2u_2(t)$$

Similarly, for $t>\tau$, again $\delta(t-\tau)=0$ and the equation becomes homogeneous, so we must have:

$$\text{if } t>\tau, \text{ then} G(t,\tau)=b_1u_1(t)+b_2u_2(t)$$

All that is left is to find a_1, a_2, b_1, b_2 and we're done. For this, we use the 4 conditions we have. Let's start with the initial conditions. The time $t=0<\tau$, so we must have:

$$G(t=0,\tau)=a_1u_1(0)+a_2u_2(0)=0$$

and

$$\frac{dG}{dt}(t=0,\tau)=a_1\frac{du_1}{dt}(0)+a_2\frac{du_2}{dt}(0)=0$$

Because u_1, u_2 are linearly independent solutions, their Wronskian $W(t)=u_1\frac{du_2}{dt}-u_2\frac{du_1}{dt}\neq 0$ for any time, therefore also for $t=0$. As a result, the only solution of those two equations is $a_1=a_2=0$. Nice and simple. So we find that $G(t,\tau)=0$ if $t<\tau$. Now we use conditions 1 and 2 to find $G(t,\tau)$ for $t>\tau$. First:

$$G(t=\tau_-,\tau)=G(t=\tau_+,\tau)\to 0=b_1u_1(\tau)+b_2u_2(\tau)$$

because the functions u_1, u_2 are continuous. Also,

$$\frac{dG}{dt}|_{t=\tau+}-\frac{dG}{dt}|_{t=\tau-}=1\to b_1\frac{du_1}{dt}(\tau)+b_2\frac{du_2}{dt}(\tau)=1$$

Since the Wronskian is again guaranteed to be non-zero, the solution of this system of coupled equations is:

$$b_1=-\frac{u_2(\tau)}{W(\tau)}; b_2=\frac{u_1(\tau)}{W(\tau)}$$

So the conclusion is that the Green's function for this problem is:

$$G(t,\tau)=\begin{cases} 0 & \text{if } 0<t<\tau \\ \frac{u_1(\tau)u_2(t)-u_2(\tau)u_1(t)}{W(\tau)} & \text{if } \tau<t \end{cases}$$

and we basically know it if we know u_1 and u_2 (which we need to calculate in any event).

Let me make some comments.

1. As I said, in the textbook this formula is derived as a special case of the "variation of parameters" solution, and then is called a Green's function. There is nothing wrong with that derivation as such, except it is not very clear how to extend that procedure to equations with boundary conditions (which is what we will do next). The derivation I have here is much more general and we will go through precisely the same steps to find the Green's functions if we are given boundary conditions, instead of initial conditions. Of course, in that case the solution for G will change, but we can find it just as easily.

2. The formula we derived here is quite easy to understand if we think in physical terms. Suppose that $u(t)$ is the location of some object of mass $m=1$, and the ODE is Newton's second law: maybe there is some drag (the term

proportional to du/dt) and maybe some elastic force (the term proportional to u) and so $f(t)$ describes whatever other external force is applied at time t. Then, as discussed, $G(t,\tau)$ will be the location of the object if it starts at the origin and at rest (see initial conditions for G), and if we "kick" it with $f(t) \to \delta(t-\tau)$ at the time τ. Obviously, before we kick it the object will remain at rest at the origin, which explains why $G(t,\tau) = 0$ if $t < \tau$. After the kick, the object will move as described by $G(t,\tau), t > \tau$. For $t \to \tau_+$, i.e. just when it starts to move, it is still at the origin where it was all the time until $t = \tau$ – that's condition 1. But because it was kicked with this very short but intense force, its speed jumps from 0 to 1 (that's condition 2 – remember that applying a force changes the momentum of an object. For this "unit" of kick-force we apply, the speed increases by one "unit" as well). What the method does, then, is to say that any external force can be thought of as a sequence of kicks with various strengths. Because the equation is linear, to find the whole solution we simply need to sum the contributions due to each kick (superposition principle).

We will spend some time practicing this in class, and comparing it against the other two methods.

After you're comfortable with this, we will derive together the Green's function for ODEs where we are given boundary conditions – think variable x (space) instead of t (time), and that we're interested in a finite region of space $a \le x \le b$ (for example, we might want to know the temperature of a rod which is located between a and b). Then, we might be given the temperature at each end:

$$u(x=a) = T_L, u(x=b) = T_R.$$

Conditions like this, which specify the value of the unknown, are known as Dirichlet conditions.

Or we could be told what the derivatives of u are at the ends of the rod (as we'll see in a bit, this derivative is proportional to how much heat flows into/out of the rod, and maybe that's what we control, not the temperatures as such – for instance, for an isolated end, no heat can flow out and the derivative is zero). Such conditions are known as von Neumann conditions:

$$\frac{du}{dx}|_{x=a} = h_L, \frac{du}{dx}|_{x=b} = h_R.$$

Or we could have a mixed bag, where we know u at one end and du/dx at the other one. To each of these situations will correspond a different Green's function; but once we know that Green's function, we can solve any inhomogeneous ODE with that type of boundary conditions without any further trouble.