Green's functions

Consider the 2nd order linear inhomogeneous ODE

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u(t) = f(t).$$

Of course, in practice we'll only deal with the two particular types of 2nd order ODEs we discussed last week, but let me keep the discussion more general, since it works for any 2nd order linear ODE. We want to find $u(t)$ for all $t > 0$, given the initial conditions $u(t = 0) = u_0$, $\frac{du}{dt}|_{t=0} = v_0$.

Let us assume that the two linearly-independent solutions $u_1(t), u_2(t)$ of the homogeneous equation are known (we've discussed what these are for the special kinds of equations we will need to solve). Then, we know that the general solution of the inhomogeneous equation is:

$$u(t) = C_1u_1(t) + C_2u_2(t) + u_p(t)$$

where $u_p(t)$ is any particular solution of the inhomogeneous equation. After we find $u_p(t)$, we can use the initial conditions to find C_1 and C_2.

Two ways to find $u_p(t)$ that we've discussed in class are guessing and the variation of parameters. There is nothing wrong with either, except that every time a new $f(t)$ is given, we have to try another guess or go into all the work required by the variation of parameters – we have to redo the whole calculation again to get the new u_p. This becomes quite difficult, especially if $f(t)$ is not a simple function. The idea behind the Green's function is to find a general expression that tells us what $u_p(t)$ is for any $f(t)$ that we care to use. We still need to do is one calculation (to find the Green's function), but once we have it, we can find $u_p(t)$ for any $f(t)$ without much further work.

Before launching into how this works, let me point out that sometimes this solution is shown as a sort of math "trick" related to a certain special way to do variation of parameters – for instance, this is how it's presented in the 6th edition of the textbook, pages 20-23. I want to try to do a bit of a better job in explaining why this works, and how should we think about the meaning of this solution, as physicists. This should help us figure out how to generalize this idea, because we will use it later for PDEs.

The main idea is to "decompose" $f(t)$ as a sum of simple functions, for which we know the particular solutions. Remember that if $f(t) = f_1(t)+f_2(t)$, then $u_p(t) = u_{p_1}(t)+u_{p_2}(t)$, where $u_{p_1}(t)$ is the particular solution for $f_1(t)$, etc. Of course, this would be true if we "broke" $f(t)$ in any number of pieces, so long as we could find the corresponding $u_p(t)$ for each piece. Now, the way we do this "breaking" is with the Dirac function: remember that we can write:

$$f(t) = \int_0^\infty d\tau\delta(t-\tau)f(\tau)$$

which means that $f(t)$ is a sum of short "kicks" (described by the δ-function), and so that the kick applied at τ has the strength $f(\tau)$. Of course, "sum" here is really an integral, because the time τ is a continuous variable. The reason why the integral is from 0 to ∞ is that I am only interested in times $t \geq 0$ – in this sorts of problems we don't care what happened in the past, the question is always what happens after the initial moment $t = 0$.

So, the idea is that if we can find the particular solution for a kick $\delta(t - \tau)$, i.e. a kick applied at time τ, then we're done – we need only "sum" over the particular contributions from all the kicks that contributed to our $f(t)$.

The Green's function $G(t, \tau)$ is the solution of the inhomogeneous equation

$$\frac{d^2G(t,\tau)}{dt^2} + k(t)\frac{dG(t,\tau)}{dt} + p(t)G(t,\tau) = \delta(t-\tau) \tag{1}$$

In other words, it tells us what is a particular solution is we apply a single kick of strength 1 at the time $t = \tau$ – exactly what we need. It has two arguments because, of course, the solution will be different if we kick at different times τ, so we need to keep track of τ as well.

Finally, G is not just any particular solution of this inhomogeneous equation, but we will ASK that it satisfies the initial conditions $G(t = 0, \tau) = 0$, $\frac{dG}{dt}|_{t=0} = 0$. Note the very special form of these initial conditions – they look like the ones for the general solution, but they are both HOMOGENEOUS (i.e., = 0)! The reason for this is that I don't want to have to recalculate the Green's function every time I change the values of u_0, v_0 – as you'll see below, the homog. part of the solution can take care of u_0 and v_0.

Once we have this particular solution, we know that for any arbitrary sequence of kicks that makes up $f(t)$, the particular solution must be:

$$u_p(t) = \int_0^\infty d\tau G(t,\tau) f(\tau)$$

(if you're not quite sure about this, plug this u_p in the ODE and check that indeed it satisfies Eq. (1)).

Because of the initial conditions satisfied by G, it follows that $u_p(0) = 0$, $\frac{du_p}{dt}|_{t=0} = 0$. So the general solution of our equation is:

$$u(t) = C_1 u_1(t) + C_2 u_2(t) + \int_0^\infty d\tau G(t,\tau) f(\tau)$$

where C_1, C_2 must be chosen such that:

$$u_0 = u(t=0) = C_1 u_1(0) + C_2 u_2(0) + u_p(0) = C_1 u_1(0) + C_2 u_2(0)$$

$$v_0 = \frac{du}{dt}|_{t=0} = C_1 \frac{du_1}{dt}|_{t=0} + C_2 \frac{du_2}{dt}|_{t=0} + \frac{du_p}{dt}|_{t=0} = C_1 \frac{du_1}{dt}|_{t=0} + C_2 \frac{du_2}{dt}|_{t=0}$$

This explains why we chose those initial conditions for $G(t,\tau)$ – this way we can adjust C_1 and C_2 to take care of the actual initial conditions, and we don't need to recalculate G and therefore u_p if we change the initial conditions – all that is needed is to adjust C_1, C_2 accordingly.

In summary, once we know $G(t,\tau)$, we only need to do the integral $\int_0^\infty d\tau G(t,\tau) f(\tau)$ for the function $f(t)$ of interest, and also find C_1 and C_2, and we're done. And this works for any $f(t)$, all we need is to do one integral.

Before calculating $G(t,\tau)$, let's see what general conditions it must satisfy.

1. $G(t,\tau)$ must be continuous at all t, because we expect $u(t)$, and therefore $u_p(t)$ to be continuous at all times. For example, if this equation comes from applying Newton's second law, then u describes the location of some object. Clearly, then, it must change continuously in time. In particular, $G(t,\tau)$ must be continuous when we apply the kick:

$$G(t = \tau_-, \tau) = G(t = \tau_+, \tau)$$

where $\tau_\pm$ are times infinitesimally close to τ (just before and just after).

2. However, since we apply this singular kick at $t = \tau$, we expect something discontinuous to happen to G there. Indeed, it turns out that the derivative is discontinuous:

$$\frac{dG}{dt}|_{t=\tau+} - \frac{dG}{dt}|_{t=\tau-} = 1 \tag{2}$$

in other words, the derivative of $G(t,\tau)$ has a jump of precisely 1 at $t = \tau$ when the kick is applied. This comes directly from Eq. (1) if we integrate it from $\tau - \epsilon$ to $\tau + \epsilon$, and we let $\epsilon \to 0$. What we get is:

$$\int_{\tau-\epsilon}^{\tau+\epsilon} dt \left[\frac{d^2 G(t,\tau)}{dt^2} + k(t) \frac{dG(t,\tau)}{dt} + p(t) G(t,\tau) \right] = \int_{\tau-\epsilon}^{\tau+\epsilon} dt \delta(t-\tau) = 1$$

(the second equality is just the value of the integral on the rhs, see delta functions). On the lhs, we have three terms. Let me take them from the end. When $\epsilon \to 0$,

$$\int_{\tau-\epsilon}^{\tau+\epsilon} dt p(t) G(t,\tau) \to 0$$

because the integrand is a continuous function, and we're shrinking the integration interval to zero. Similarly, after integrating by parts, we find:

$$\int_{\tau-\epsilon}^{\tau+\epsilon} dt k(t) \frac{dG(t,\tau)}{dt} = [k(t) G(t,\tau)]_{t=\tau-\epsilon}^{t=\tau+\epsilon} - \int_{\tau-\epsilon}^{\tau+\epsilon} dt \frac{dk}{dt} G(t,\tau) = 0$$

because again we're dealing only with continuous functions in the limit where the integration interval goes to zero. Finally:

$$\int_{\tau-\epsilon}^{\tau+\epsilon} dt \frac{d^2 G(t,\tau)}{dt^2} = \frac{dG}{dt}|_{t=\tau-\epsilon}^{t=\tau+\epsilon} = \frac{dG}{dt}|_{t=\tau+} - \frac{dG}{dt}|_{t=\tau-}$$

and Eq. (2) follows directly.

3. As I said, we ask for the simplest possible initial conditions $G(t=0,\tau)=0$, $\frac{dG}{dt}|_{t=0}=0$. The reason for this is that we do not want the Green's function to depend on the initial conditions u_0, v_0 of the equation – if this was the case, then anytime we changed the initial conditions we would have to recalculate G for the new initial conditions. Asking that $G(t=0,\tau)=0$, $\frac{dG}{dt}|_{t=0}=0$ means that $G(t,\tau)$ always has the same expression, and we let C_1 and C_2 adjust so as to take care of u_0, v_0.

So let's find $G(t,\tau)$. Let us consider first the interval $0<t<\tau$. Because in this interval $\delta(t-\tau)=0$, here the equation for G is:

$$\frac{d^2G(t,\tau)}{dt^2}+k(t)\frac{dG(t,\tau)}{dt}+p(t)G(t,\tau)=0$$

in other words here G is a solution of the homogeneous equation, so it must be of the general form:

$$\text{if } 0<t<\tau, \text{ then} G(t,\tau)=a_1u_1(t)+a_2u_2(t)$$

Similarly, for $t>\tau$, again $\delta(t-\tau)=0$ and the equation becomes homogeneous, so we must have:

$$\text{if } t>\tau, \text{ then} G(t,\tau)=b_1u_1(t)+b_2u_2(t)$$

All that is left is to find a_1, a_2, b_1, b_2 and we're done. For this, we use the 4 conditions we have. Let's start with the initial conditions. The time $t=0<\tau$, so we must have:

$$G(t=0,\tau)=a_1u_1(0)+a_2u_2(0)=0$$

and

$$\frac{dG}{dt}(t=0,\tau)=a_1\frac{du_1}{dt}(0)+a_2\frac{du_2}{dt}(0)=0$$

Because u_1, u_2 are linearly independent solutions, their Wronskian $W(t)=u_1\frac{du_2}{dt}-u_2\frac{du_1}{dt}\neq 0$ for any time, therefore also for $t=0$. As a result, the only solution of those two equations is $a_1=a_2=0$. Nice and simple. So we find that $G(t,\tau)=0$ if $t<\tau$. Now we use conditions 1 and 2 to find $G(t,\tau)$ for $t>\tau$. First:

$$G(t=\tau_-,\tau)=G(t=\tau_+,\tau)\rightarrow 0=b_1u_1(\tau)+b_2u_2(\tau)$$

because the functions u_1, u_2 are continuous. Also,

$$\frac{dG}{dt}|_{t=\tau+}-\frac{dG}{dt}|_{t=\tau-}=1\rightarrow b_1\frac{du_1}{dt}(\tau)+b_2\frac{du_2}{dt}(\tau)=1$$

Since the Wronskian is again guaranteed to be non-zero, the solution of this system of coupled equations is:

$$b_1=-\frac{u_2(\tau)}{W(\tau)}; b_2=\frac{u_1(\tau)}{W(\tau)}$$

So the conclusion is that the Green's function for this problem is:

$$G(t,\tau)=\begin{cases} 0 & \text{if } 0<t<\tau \\ \frac{u_1(\tau)u_2(t)-u_2(\tau)u_1(t)}{W(\tau)} & \text{if } \tau<t \end{cases}$$

and we basically know it if we know u_1 and u_2 (which we need to calculate in any event).

Let me make some comments.

1. As I said, in the textbook this formula is derived as a special case of the "variation of parameters" solution, and then is called a Green's function. There is nothing wrong with that derivation as such, except it is not very clear how to extend that procedure to equations with boundary conditions (which is what we will do next). The derivation I have here is much more general and we will go through precisely the same steps to find the Green's functions if we are given boundary conditions, instead of initial conditions. Of course, in that case the solution for G will change, but we can find it just as easily.

2. The formula we derived here is quite easy to understand if we think in physical terms. Suppose that $u(t)$ is the location of some object of mass $m=1$, and the ODE is Newton's second law: maybe there is some drag (the term

proportional to du/dt) and maybe some elastic force (the term proportional to u) and so $f(t)$ describes whatever other external force is applied at time t. Then, as discussed, $G(t,\tau)$ will be the location of the object if it starts at the origin and at rest (see initial conditions for G), and if we "kick" it with $f(t) \to \delta(t-\tau)$ at the time τ. Obviously, before we kick it the object will remain at rest at the origin, which explains why $G(t,\tau) = 0$ if $t < \tau$. After the kick, the object will move as described by $G(t,\tau), t > \tau$. For $t \to \tau_+$, i.e. just when it starts to move, it is still at the origin where it was all the time until $t = \tau$ – that's condition 1. But because it was kicked with this very short but intense force, its speed jumps from 0 to 1 (that's condition 2 – remember that applying a force changes the momentum of an object. For this "unit" of kick-force we apply, the speed increases by one "unit" as well). What the method does, then, is to say that any external force can be thought of as a sequence of kicks with various strengths. Because the equation is linear, to find the whole solution we simply need to sum the contributions due to each kick (superposition principle).

We will spend some time practicing this in class, and comparing it against the other two methods.

After you're comfortable with this, we will derive together the Green's function for ODEs where we are given boundary conditions – think variable x (space) instead of t (time), and that we're interested in a finite region of space $a \le x \le b$ (for example, we might want to know the temperature of a rod which is located between a and b). Then, we might be given the temperature at each end:

$$u(x=a) = T_L, u(x=b) = T_R.$$

Conditions like this, which specify the value of the unknown, are known as Dirichlet conditions.

Or we could be told what the derivatives of u are at the ends of the rod (as we'll see in a bit, this derivative is proportional to how much heat flows into/out of the rod, and maybe that's what we control, not the temperatures as such – for instance, for an isolated end, no heat can flow out and the derivative is zero). Such conditions are known as von Neumann conditions:

$$\frac{du}{dx}|_{x=a} = h_L, \frac{du}{dx}|_{x=b} = h_R.$$

Or we could have a mixed bag, where we know u at one end and du/dx at the other one. To each of these situations will correspond a different Green's function; but once we know that Green's function, we can solve any inhomogeneous ODE with that type of boundary conditions without any further trouble.

How to derive the proper Green's functions for such ODEs with boundary conditions (=BC) is the topic of the in-class activity 6. I will summarize here the results most of you arrived at by the end of the class. I hope that those of you who have not quite finished will try to do so at home, before looking at this solution.

So we want to find the Green's function $G(x,\xi)$ that will allow us to find a particular solution of the form $u_p(x) = \int_a^b d\xi G(x,\xi) f(\xi)$ for the ODE $\frac{d^2u}{dx^2} + k(x)\frac{du}{dx} + p(x)u = f(x)$ with the BC $u(x=a) = u_a, u(x=b) = u_b$.

The first step is to find what equation must be satisfied by $G(x,\xi)$. Since $\frac{du_p(x)}{dx} = \int_a^b d\xi \frac{dG(x,\xi)}{dx} f(\xi)$ and similarly for the second derivative, putting this into the ODE and grouping the terms leads to:

$$\int_a^b d\xi f(\xi) \left[\frac{d^2G(x,\xi)}{dx^2} + k(x)\frac{dG(x,\xi)}{dx} + p(x)G(x,\xi)\right] = f(x) = \int_a^b d\xi f(\xi)\delta(x-\xi)$$

where the second equation is true for any $x \in [a,b]$ (see Dirac functions). For this to hold for any $f(x)$, it follows that we must have:

$$\frac{d^2G(x,\xi)}{dx^2} + k(x)\frac{dG(x,\xi)}{dx} + p(x)G(x,\xi) = \delta(x-\xi)$$

In other words, just like we found for $G(t,\tau)$, the Green's function satisfies *precisely* the same ODE as $u(x)$, but the inhomogeneous part is a δ-function. This should not be surprising, since if we changed $x \to t$, this would be identical to the time-dependent ODE, so naturally we should get the same result.

2. If this was a Poisson equation, then we know that $u(x)$ would be the electric potential at x due to a distribution of charges described by $f(x)$. Since G satisfies the same equation, it follows that $G(x,\xi)$ is the electric potential at x due to a distribution of charge $\delta(x-\xi)$ – i.e. a unit charge located at ξ.

So we can see the logic here: if we can figure out the potential due to any point charge (i.e., G) then using the superposition principle, the total potential due to all charges will be the "sum" (integral, really) of contributions from each charge – that's precisely the relationship between u_p and G.

3. The general solution of the ODE will be:

$$u(x) = C_1 u_1(x) + C_2 u_2(x) + \int_a^b d\xi G(x,\xi) f(\xi)$$

To find C_1, C_2, I need to use the BC:

$$u(x=a) = u_a = C_1 u_1(a) + C_2 u_2(a) + \int_a^b d\xi G(x=a,\xi) f(\xi)$$

$$u(x=b) = u_b = C_1 u_1(b) + C_2 u_2(b) + \int_a^b d\xi G(x=b,\xi) f(\xi)$$

The *simplest* choice (since it is independent of u_a, u_b, so it won't change if we change those values) is to ask:

$$G(x=a,\xi) = 0; G(x=b,\xi) = 0$$

in which case the integrals vanish, and we must solve $u_a = C_1 u_1(a) + C_2 u_2(a), u_b = C_1 u_1(b) + C_2 u_2(b)$ which is easy enough. So we ask for these simplest possible BC for G.

Note that if instead of $u(x=a) = x_a$, we were given at that end the von Neumann condition $\frac{du}{dx}|_{x=a} = v_a$, then we would need to solve:

$$v_a = C_1 \frac{du_1}{dx}|_{x=a} + C_2 \frac{du_2}{dx}|_{x=a} + \int_a^b d\xi \frac{dG(x,\xi)}{dx}|_{x=a} f(\xi)$$

So at this end, it now makes sense to ask that $\frac{dG(x,\xi)}{dx}|_{x=a} = 0$ as the simplest possible choice.

In conclusion, we will ask that G satisfies the same kind of BC like u, but always homogeneous (i.e., equal to 0): If u is given at an end, we ask $G = 0$ at that end. If $\frac{du}{dx}$ is given at an end, we ask that $\frac{dG}{dx} = 0$ there. We can treat similarly mixed conditions – if we're given the value of $\alpha u + \beta \frac{du}{dx}$ at an end, we ask that $\alpha G + \beta \frac{dG}{dx} = 0$ at that end. This means that even if the ODE is unchanged, the expression for G will be different for different BC. In terms of a Poisson equation, $G = 0$ means the electric *potential* created by the point charge is zero at the boundary, while $\frac{dG}{dx} = 0$ means that the electric *field* is zero at that boundary. So the two solutions will naturally be different, depending on what we ask to happen at the boundary.

4. Besides the two boundary conditions discussed at the previous point, we also have matching conditions for $x = \xi$ where we placed the point charge. First, G must be continuous because we expect u and therefore u_p to be continuous, so:

$$G(x=\xi_-,\xi) = G(x=\xi_+,\xi)$$

If this was a Poisson equation, it would just mean that we expect the electric potential to be continuous at all points, including where we placed the charge.

The second condition is that the derivative is discontinuous:

$$\frac{dG}{dx}|_{x=\xi_+} - \frac{dG}{dx}|_{x=\xi_-} = 1$$

For a mathematical derivation of this equality, you just need to follow the same steps we used for $G(t,\tau)$ – after all, since the ODEs are identical and the only change is $t \to x, \tau \to \xi$, then this condition (which is has nothing to do with the boundary conditions or initial conditions) should stay the same.

Physically, if G is an electric potential, then its derivative is (minus) the electric field. So this equation says that the electric field changes discontinuously from one side to the other side of the charge. But this is very reasonable, because we know that the electric field points towards the charge if the charge is negative, and away if it is positive. If we're in the later case, then $E < 0$ to the left of the charge, i.e. for $x < \xi$; and $E > 0$ to the right of the charge, for $x > \xi$. So indeed, the electric field changes discontinuously. The fact that the difference is precisely 1 is because we used a unit charge.

5. Now we consider the ODE:

$$\frac{d^2u}{dx^2} - \frac{1}{x}\frac{du}{dx} = f(x).$$

This ODE is of Cauchy-Euler form therefore the solutions are power laws, which after some work are found to be $u_1(x) = 1, u_2(x) = x^2$. We want to find $G(x,\xi)$ for $x \in [1,3]$. The equation satisfied by G is (see 1):

$$\frac{d^2G}{dx^2} - \frac{1}{x}\frac{dG}{dx} = \delta(x-\xi)$$

We'll solve this piecewise. If $1 \le x < \xi$, then $\delta(x-\xi) = 0$ (there is no charge in this interval) and the equation of G becomes:

$$\frac{d^2G}{dx^2} - \frac{1}{x}\frac{dG}{dx} = 0$$

which is a hom. ODE and so has the general solution $G(x,\xi) = a_1 + a_2x^2$.

If $\xi < x \le 3$, the equation again becomes homogeneous, so here we must have $G(x,\xi) = b_1 + b_2x^2$. All that's left to do is to find the 4 coefficients a_1, a_2, b_1, b_2 and we're done. For this we need 4 equations. We know that we have to ask for boundary conditions $G(x=1,\xi) = 0, G(x=3,\xi) = 0$ (see discussion at 3), so this gives 2 equations. We also have the 2 matching conditions, so this should work out.

At the left boundary:

$$G(x=1,\xi) = 0 \to a_1 + a_2 = 0 \to G(x,\xi) = a(x^2-1)$$

At the right boundary:

$$G(x=3,\xi) = 0 \to b_1 + b_2 9 = 0 \to G(x,\xi) = b(x^2-9)$$

So now we're down to 2 unknowns, a, and b. We have 2 more conditions. Continuity of G at $x=\xi$ means

$$a(\xi^2-1) = b(\xi^2-9)$$

and the jump in the derivative means that:

$$2b\xi - 2a\xi = 1$$

and after some work we find:

$$a = \frac{\xi^2-9}{16\xi}; \qquad b = \frac{\xi^2-1}{16\xi}.$$

So to conclude, the Green's function for this ODE with this type of BC is:

$$G(x,\xi) = \begin{cases} \frac{\xi^2-9}{16\xi}(x^2-1) & \text{, if } 1 \le x < \xi \\ \frac{\xi^2-1}{16\xi}(x^2-9) & \text{, if } \xi < x \le 3 \end{cases}$$

6. Since here $G(x,\xi)$ has the meaning of the electric potential at point x created by a charge placed at ξ, it is reasonable that it is non-zero on both sides of the charge.

7. Using the Green's function, we can now immediately get the particular solution:

$$u_p(x) = \int_1^3 d\xi f(\xi) G(x,\xi) = \int_1^x d\xi f(\xi) G(x,\xi) + \int_x^3 d\xi f(\xi) G(x,\xi)$$

In the first integral $x > \xi$, in the second $x < \xi$, so we have:

$$u_p(x) = \int_1^x d\xi f(\xi) \frac{\xi^2-1}{16\xi}(x^2-9) + \int_x^3 d\xi f(\xi) \frac{\xi^2-9}{16\xi}(x^2-1)$$

where we are given $f(\xi) = 16\xi$. With this:

$$u_p(x) = (x^2-9)\int_1^x d\xi(\xi^2-1) + (x^2-1)\int_x^3 d\xi(\xi^2-9)$$

$$= (x^2-9)\left[\frac{x^3-1}{3} - (x-1)\right] + (x^2-1)\left[\frac{27-x^3}{3} - 9(3-x)\right] = ... = \frac{16}{3}x^3 + 12 - \frac{52}{3}x^2$$

after simplifying. This looks reasonable – guessing would suggest something proportional to x^3 for u_p, and the last two terms are part of the homogenenous solution (whatever is needed to make sure u_p statisfies homogeneous BC).

To get the full solution $u(x) = C_1 + C_2 x^2 + u_p(x)$ we use the BC for u: $u(1) = 1, u(3) = 9$. Now you see how nice it is that we asked that $u_p(1) = 0, u_p(3) = 0$ (and therefore that G goes to zero at both ends) because we don't have to bother with it. All we have left is $C_1 + C_2 = 1, C_1 + 9C_2 = 9 \rightarrow C_1 = 0, C_2 = 1$, and

$$u(x) = x^2 + u_p(x) = \frac{16}{3}x^3 + 12 - \frac{49}{3}x^2$$

for the u_p we found above. Of course, for this simple function, you could have guess a u_p quite easily (try!) ... but the advantage here is that you can use this same G for any other $f(x)$ in this equation, as long as the BC stay of the same type (specify u at both ends).

Extra: In this case, we should choose $\frac{dG}{dx}|_{x=3} = 0$ at the right boundary. Since here $G(x, \xi) = b_1 + b_2 x^2 \rightarrow 2b_2 \cdot 3 = 0 \rightarrow b_2 = 0$ which makes the calculation even simpler than before. The other 3 equations are the same, but of course the overall solution will change.