Fourier Series – summary

Motivation: sometimes it is convenient to express complicated functions in terms of simple ones. An example is the Taylor expansion, which allows us to write any (suitably well behaved) function as a sum of simple powers of x for $x \sim x_0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left. \frac{d^n f}{dx^n} \right|_{x=x_0}.$$

For instance, this is useful if we need to do an integral of f over an interval centered at x_0 – the rhs is trivial to integrate, whereas f itself may be very difficult to deal with directly.

The question of interest to us is: if f is a periodic function, can we express it in terms of simple functions, and what are those functions?

DEF: $f(x)$ is periodic if a period L exists such that $f(x+L) = f(x)$ for any x.

Notes:

(i) of course that $2L, 3L, ..., -L, -2L, ...$ are also periods of this function, but we call "the period" the minimum possible interval over which the function repeats itself, and that is what I will mean by L from now on.

(ii) if we know the values of the function on any interval of length L, i.e. for $x \in [x_0, x_0 + L)$, then we know the function everywhere – we just need to repeat this pattern. That is what makes a function periodic.

Traditionally, one chooses $x_0 = 0$ or $x_0 = -L/2$ as the beginning of this interval, but it can be any value x_0 whatsoever. For the time being, let's be a bit general and work with any x_0. The textbook always takes $x_0 = -L/2$, which is not always the smartest possible choice. By the way, the textbook also always chooses either $L = 2\pi$, or $L = 2a$ (and most formulas are in terms of this a). I will use the formulas with L, because otherwise people tend to forget that a is not the whole period, only half of it, and this can lead to problems.

If $f(x)$ is periodic and we want to express it in terms of simpler functions, it's clear that we should better choose simple *periodic* functions (not polynomials). The simplest examples are $\sin(\theta), \cos(\theta)$, which have the period 2π.

As we will show in class, if we want to make these functions periodic with period L, we much work with $\sin\left(\frac{2\pi n}{L}x\right)$ and $\cos\left(\frac{2\pi n}{L}x\right)$, where n is any integer – these have L periodicity.

DEF: If $f(x)$ is a periodic function of period L, we *associate* to it a *Fourier series*:

$$f(x) \leftrightarrow a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{L}x\right) + b_n \sin\left(\frac{2\pi n}{L}x\right)\right] = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L}x}$$

The second term is called the *complex Fourier series* (for obvious reasons). One can use either form, with the sin/cos or with the exponentials, depending on which is more convenient. We will derive the formulas for both, but we will mostly use the sin/cos formulas simply because we will mostly look at real quantities, and it will make sense to work with real functions.

Three questions now need to be settled. We will do so in class, after which I will add here the results we found.

Q1: what is the link between the a's, the b's and the c's? In other words, if I know all c's how do I calculate the a's and the b's from them, and vice-versa, so that the second equality holds for any value of x?

Q2: if we are given an expression for $f(x)$, what values of a's and b's (or of c's) should we associate with its Fourier series, i.e. how do we choose those numbers?

Q3: when is $f(x)$ actually equal to its Fourier series? (needless to say, this is what we're hoping for).

In class we'll work through answering Q1 and Q2, and I'll then add the answers here. This is what is roughly covered in sections 1.1, 1.2 and 1.10, although we will do things a bit more generally. Q3 is a long theorem, so I will tell you when the equality holds (and argue why it makes sense to be so, and expect you to know the answer) and you can see the details of the proof in the textbook, in section 1.7, if you wish to.

You've all figured out the answer to Q1:

$$a_0 = c_0$$

$$a_n = c_n + c_{-n}, n = 1, 2, ...$$

$$b_n = i(c_n - c_{-n}), n = 1, 2, ...$$

To answer Q2, we went in 3 steps. First, we proved the identity:

$$\frac{1}{L}\int_{x_0}^{x_0+L} dx e^{i\frac{2\pi(n-m)x}{L}} = \delta_{n,m}$$

where $\delta_{n,m}$ is the Kronecker symbol, equal to 1 if $n = m$ and 0 otherwise. This follows from doing the integral and using the fact that for $n \neq m$, the resulting function is periodic so its values at the upper limit and lower limit precisely cancel each other out.

Using this, we found the values of c_n if we are to have any hope that:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n x}{L}}.$$

For this, we noticed that:

$$c_m = \sum_{n=-\infty}^{\infty} c_n \delta_{n,m} = \frac{1}{L}\int_{x_0}^{x_0+L} dx e^{-i\frac{2\pi m x}{L}} \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n x}{L}} = \frac{1}{L}\int_{x_0}^{x_0+L} dx e^{-i\frac{2\pi m x}{L}} f(x)$$

Finally, we combined this with Q1 to find:

$$a_0 = \frac{1}{L}\int_{x_0}^{x_0+L} dx f(x)$$

in other words, a_0 is just the average value of $f(x)$ over one period; and for any $n \geq 1$, we found:

$$a_n = \frac{2}{L}\int_{x_0}^{x_0+L} dx f(x) \cos\frac{2\pi n x}{L}$$

$$b_n = \frac{2}{L}\int_{x_0}^{x_0+L} dx f(x) \sin\frac{2\pi n x}{L}$$

The textbook derives these formulas quite differently, but of course they arrive at the same answers. I expect you to know these last equations from now on. As we discussed in class, if the function $f(x)$ is even, then all $b_n = 0$ because the sin are odd functions. Viceversa, if $f(x)$ is odd, then all $a_n = 0$, because the cosines are even.

Finally, we can now answer Q3. Even if we use these values for the a's and b's (or the c's, if we work with the complex Fourier series), can we be sure that $f(x)$ is equal to its Fourier series?

The conditions for this to hold are called the Dirichlet conditions – we've discussed them in class, so I quickly summarize them here. The proof that this indeed so is in the textbook.

The Dirichlet conditions:

If: (i) $f(x)$ is periodic with period L,

(ii) $f(x)$ is continuous everywhere, or at most has a finite number of jump discontinuities per period (i.e., discontinuities where the value of the function changes by a finite amount);

(iii) $f(x)$ has a finite number of minima and maxima per period; and

(iv) $\int_{x_0}^{x_0+L} dx |f(x)|$ is finite,

then $f(x)$ is equal to its Fourier series everywhere where it is continuous. At the jump discontinuities, the value of the Fourier series is $\frac{1}{2}(f(x_+) + f(x_-))$, i.e. the average value at the jump.

To conclude, if I give you a "well-behaved" periodic function $f(x)$ (i.e., one that satisfies Dirichlet's conditions) and ask you for its Fourier series, then you have to do those integrals to find the various a's and b's, and that's that.

In this course, as we discussed in class, the sort of problems we will need to solve is of the type where we are given a (typically not periodic function) $f(x)$, and we are asked either to find the coefficients a_n such that:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a}, \text{ for all } x \in [0, a]$$

OR to find the coefficients b_n so that:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a}, \text{ for all } x \in [0, a].$$

Let me discuss the first case. Let me define $f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{a}$, for all x. This function is called **the even periodic extension**. What do we know about it?

(i) it is periodic, with period $L = 2a$ (this last part comes from looking at the argument of the cos. Remember that we have $\frac{2n\pi x}{L}$ in a Fourier series);

(ii) it is an even function, because it's the sum of even functions.

(iii) For $x \in [0, a]$, $f_e(x) = f(x)$ is known.

These 3 pieces of information allow us to find this periodic function. We know that it is $f(x)$ for $x \in [0, a]$ and because it is even, $f_e(x) = f(-x)$ for $x \in [-a, 0]$. Because the period $L = 2a$, this means we know $f_e(x)$ everywhere and we can find its Fourier series with the usual formulae and then we are sure that for $x \in [0, a]$, $f(x) = f_e(x)$ is indeed that Fourier series (at all points where $f(x)$ is continuous, of course).

Using $L = 2a$ and $x_0 = -a$ (always do this for such problems!), we have

$$a_0 = \frac{1}{2a} \int_{-a}^{a} f_e(x) = \frac{1}{a} \int_{0}^{a} dx f(x)$$

because f_e is even so we can halve the integration interval, and then use the fact that for $x \in [0, a]$, $f_e(x) = f(x)$. Similarly:

$$a_n = \frac{2}{2a} \int_{-a}^{a} f_e(x) \cos \frac{2n\pi x}{2a} = \frac{2}{a} \int_{0}^{a} dx f(x) \cos \frac{n\pi x}{a}$$

and so we have to do these integrals and this is the answer.

In the second case, we need to construct the **odd periodic extension** $f_o(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a}$, for all . The arguments go just like before, except now this periodic function is odd, so only its b_n coefficients are non-zero. Using the expression for b_n and the fact that the total integrand is even (the product of two odd functions is an even function, and here both f_o and the *sin* are odd), we can again halve the integration interval to find:

$$b_n = \frac{2}{a} \int_{0}^{a} dx f(x) \sin \frac{n\pi x}{a}$$

A more detailed discussion is in the Fourier series Maple worksheet – save that on your computer and open it with Maple. Let me know if you have any trouble with this.