Fourier Series

Motivation: sometimes it is convenient to express complicated functions in terms of simple ones. An example is the Taylor expansion, which allows us to write any (suitably well behaved) function as a sum of simple powers of x for $x \sim x_0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \left. \frac{d^n f}{dx^n} \right|_{x=x_0} .$$

For instance, this is useful if we need to do an integral of f over an interval centered at x_0 – the rhs is trivial to integrate, whereas f itself may be very difficult to deal with directly.

The question of interest to us is: if f is a periodic function, can we express it in terms of simple functions, and what are those functions?

DEF: $f(x)$ is periodic if a period L exists such that $f(x+L) = f(x)$ for any x.

Notes:

(i) of course that $2L, 3L, ..., -L, -2L, ...$ are also periods of this function, but we call "the period" the minimum possible interval over which the function repeats itself, and that is what I will mean by L from now on.

(ii) if we know the values of the function on any interval of length L, i.e. for $x \in [x_0, x_0 + L)$, then we know the function everywhere – we just need to repeat this pattern. That is what makes a function periodic.

Traditionally, one chooses $x_0 = 0$ or $x_0 = -L/2$ as the beginning of this interval, but it can be any value x_0 whatsoever. For the time being, let's be a bit general and work with any x_0. The textbook always takes $x_0 = -L/2$, which is not always the smartest possible choice. By the way, the textbook also always chooses either $L = 2\pi$, or $L = 2a$ (and most formulas are in terms of this a). I will use the formulas with L, because otherwise people tend to forget that a is not the whole period, only half of it, and this can lead to problems.

If $f(x)$ is periodic and we want to express it in terms of simpler functions, it's clear that we should better choose simple *periodic* functions (not polynomials). The simplest examples are $\sin(\theta), \cos(\theta)$, which have the period 2π.

As we will show in class, if we want to make these functions periodic with period L, we much work with $\sin\left(\frac{2\pi n}{L}x\right)$ and $\cos\left(\frac{2\pi n}{L}x\right)$, where n is any integer – these have L periodicity.

DEF: If $f(x)$ is a periodic function of period L, we *associate* to it a *Fourier series*:

$$f(x) \leftrightarrow a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{L}x\right) + b_n \sin\left(\frac{2\pi n}{L}x\right) \right] = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L}x}$$

The second term is called the *complex Fourier series* (for obvious reasons). One can use either form, with the sin/cos or with the exponentials, depending on which is more convenient. We will derive the formulas for both, but we will mostly use the sin/cos formulas simply because we will mostly look at real quantities, and it will make sense to work with real functions.

Three questions now need to be settled. We will do so in class, after which I will add here the results we found.

Q1: what is the link between the a's, the b's and the c's? In other words, if I know all c's how do I calculate the a's and the b's from them, and vice-versa, so that the second equality holds for any value of x?

Q2: if we are given an expression for $f(x)$, what values of a's and b's (or of c's) should we associate with its Fourier series, i.e. how do we choose those numbers?

Q3: when is $f(x)$ actually equal to its Fourier series? (needless to say, this is what we're hoping for).

In class we'll work through answering Q1 and Q2, and I'll then add the answers here. This is what is roughly covered in sections 1.1, 1.2 and 1.10, although we will do things a bit more generally. Q3 is a long theorem, so I will tell you when the equality holds (and argue why it makes sense to be so, and expect you to know the answer) and you can see the details of the proof in the textbook, in section 1.7, if you wish to.