The Dirac delta function – a quick introduction

The Dirac delta function, i.e. $\delta(x)$, is a very useful object. Strictly speaking, it is not a function but a distribution - but that won't make any difference to us.

One of the simplest ways to try to picture what $\delta(x)$ looks like is to consider what happens to the piece-wise function

$$f_\eta(x) = \begin{cases} \frac{1}{\eta} & \text{, if } -\frac{\eta}{2} \le x \le \frac{\eta}{2} \\ 0 & \text{, otherwise} \end{cases}$$

if you let $\eta \to 0$. In other words, $\delta(x) = \lim_{\eta\to 0} f_\eta(x)$. This function is plotted below. As η decreases, it becomes "narrower" and "taller". However, no matter what η is, the area below this curve is precisely 1 (since this rectangle has width η and height $1/\eta$). Since the area below a function equals the integral of that function, it follows that:

$$\int_{-\infty}^{\infty} dx f_\eta(x) = 1$$

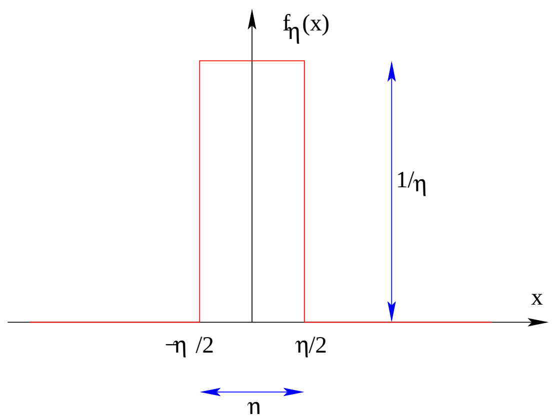

Figure 1: Sketch of $f_\eta(x)$. The function is zero everywhere except in a region of width η centered at 0, where it equals $1/\eta$. As a result, the integral of this function is 1. If η decreases, the function becomes more and more "pointy".

Putting these two facts together, we can basically say that

$$\delta(x) = \begin{cases} \infty & \text{, if } x = 0 \\ 0 & \text{, otherwise} \end{cases}$$

but such that

$$\int_{-\infty}^{\infty} dx \delta(x) = 1$$

This is by no means the only definition of $\delta(x)$. We can also get it as the limit of the continuous functions:

$$L_\eta(x) = \frac{1}{\pi}\frac{\eta}{x^2+\eta^2}$$

or

$$G_\eta(x) = \frac{1}{\sqrt{2\pi\eta^2}} e^{-\frac{x^2}{2\eta^2}}$$

The first is a so-called Lorentzian, the second is called a Gaussian. Chances are you've met them before, in some lab doing error analysis. Both take the highest value when $x = 0$ and decrease to zero as $|x| \to \infty$. The constants in front are chosen such that $\int_{-\infty}^{\infty} dx L_\eta(x) = \int_{-\infty}^{\infty} dx G_\eta(x) = 1$ for any value of η, so again, the area under each of these curves is precisely 1. Moreover, if you plot them (use Maple to get quick plots) you will see that, again, as η decreases, they become pointier too – more narrow but taller. In the limit $\eta \to 0$, they also equal $\delta(x)$. There are many other such examples of ways to define $\delta(x)$.

One more interesting definition, which we'll come back to in a few weeks (when we will understand it as well) is:

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}$$

Why this is so, as I said, will be clear later on. Just for now, you can see that if $x = 0$, the integral is indeed infinite, so $\delta(x = 0) = \infty$ as required. If $x \neq 0$, you should be able to convince yourselves that the integral equals zero (rewrite $e^{ikx} = \cos(kx) + i\sin(kx)$, then use the fact that sin is an odd function so its integral vanishes. And then plot $\cos(kx)$ and consider what area is below this curve if you integrate over all real values). Of course, we'd also need to show that if we use this formula, $\int_{-\infty}^{\infty} dx\delta(x) = 1$ – this will come soon. For the moment, take this as a curiosity.

Why is this strange function of any use? Well, consider any continuous function $g(x)$ and let's calculate what is $\int_{-\infty}^{\infty} dxg(x)\delta(x - x_0)$. We'll use the first definition of $\delta(x - x_0) = \lim_{\eta\to 0} f_\eta(x - x_0)$. Of course, $f_\eta(x - x_0)$ looks just like in Fig.1, except that now it is centered at x_0, not at the origin as before. So then:

$$\int_{-\infty}^{\infty} dxg(x)f_\eta(x - x_0) = \int_{x_0-\frac{\eta}{2}}^{x_0+\frac{\eta}{2}} dxg(x)\frac{1}{\eta} = \frac{1}{\eta}\int_{x_0-\frac{\eta}{2}}^{x_0+\frac{\eta}{2}} dxg(x) \approx \frac{1}{\eta}\eta g(x_0) = g(x_0)$$

where the first equality is because $f_\eta(x - x_0)$ vanishes for all values outside $[x_0 - \frac{\eta}{2}, x_0 + \frac{\eta}{2}]$ and equals $\frac{1}{\eta}$ inside this region. The $\approx$ sign becomes exact in the limit $\eta \to 0$. So since this is true for any small η, it follows that:

$$\int_{-\infty}^{\infty} dxg(x)\delta(x - x_0) = g(x_0)$$

You can think of the delta function like a "spike" that selects the value of g at its location. A graphic depiction of this is shown in the figure below.

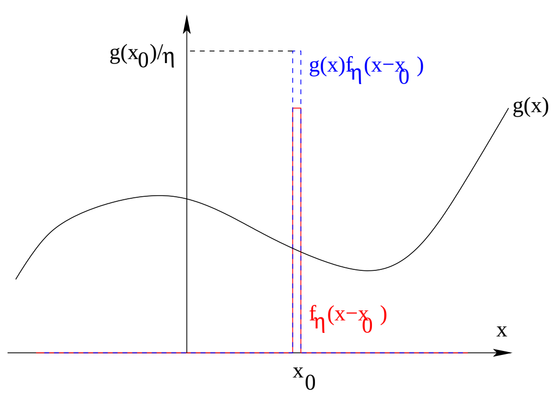

Figure 2: Sketch of some $g(x)$ (black curve), $f_\eta(x - x_0)$ (red curve, "spike" centered at x_0) and their product (blue dashed curve). The product is also a "spike" centered at x_0, of width η and of height $g(x_0)/\eta$.

Finally, it should be now apparent that if we integrate only over a finite interval, then:

$$\int_a^b dxg(x)\delta(x - x_0) = \begin{cases} g(x_0) & \text{, if } a < x_0 < b \\ 0 & \text{, otherwise} \end{cases}$$

We will use this quite a bit. Note that you can think of $\int_{-\infty}^{\infty} dx\delta(x) = 1$ as being a particular case of this more general identity, when $g(x) = 1$.