Definitions and important facts regarding ODEs and PDEs

These definitions and facts are scattered here and there in the textbook. I'm collecting them all together in one place, to make it easier for you. It is very important that you know what these words mean, since I will be using them without further explanations throughout the course. If you still have questions, ask me!

Definitions

1. *ODE= ordinary differential equation*: a differential equation whose unknown function depends on a single independent variable, eg $u(t) \rightarrow$ the equation only has derivatives with respect to t.

2. *PDE= partial differential equation*: a differential equation whose unknown function depends on two or more independent variables, eg $u(x,t) \rightarrow$ the equation has partial derivatives with respect to both t and x.

3. *The order of an ODE/PDE*: the highest derivative that appears in it.

4. *An ODE/PDE is homogeneous* if $u = 0$ is a solution of the ODE/PDE. An equation which is not homogeneous is said to be *inhomogeneous*.
For example, $\frac{du}{dt} + 3u = 2$ is inhomogeneous because $u = 0$ is not a valid solution.

5. *A boundary condition is homogeneous* if $u = 0$ satisfies it. A boundary condition which is not homogeneous is said to be *inhomogeneous*.
For example, "$u(x = 0,t) = 0$ at all t" is homogeneous, but "$u(x = 0,t) = 5t$ at all t" is not homogeneous.

6. *A homogeneous ODE/PDE is linear*: provided that for any u_1 and u_2 that are its solutions, then $\alpha u_1 + \beta u_2$ is also a solution for any constants α, β. Note: sometimes we improperly refer to an inhomogeneous ODE/PDE as being linear – what is meant is that if we kept only the homogeneous part, that one is linear.
For example: $\frac{d^2u}{dt^2} + u\frac{du}{dt} + tu = 0$ is not linear. Why? Suppose that u_1 and u_2 are solutions, in other words, $\frac{d^2u_1}{dt^2} + u_1\frac{du_1}{dt} + tu_1 = 0$, $\frac{d^2u_2}{dt^2} + u_2\frac{du_2}{dt} + tu_2 = 0$. Let's check if $u = \alpha u_1 + \beta u_2$ satisfies the equation. Putting the terms in and grouping them, we find:

$$\alpha\left(\frac{d^2u_1}{dt^2} + u_1\frac{du_1}{dt} + tu_1\right) + \beta\left(\frac{d^2u_2}{dt^2} + u_2\frac{du_2}{dt} + tu_2\right) + \alpha\beta\left(u_1\frac{du_2}{dt} + u_2\frac{du_1}{dt}\right) = 0$$

The first two parantheses are zero, but the 3rd is generally not. It follows that u is not automatically a solution, so this is a non-linear equation. The problem comes from the term $u\frac{du}{dt}$. After some thinking, you should agree that if all the terms in the ODE/PDE are either proportional to u or some derivative of u only, the equation is linear. Any product of such terms, like $u^2, u\frac{du}{dt}, \frac{\partial u}{\partial t}\frac{\partial u}{\partial x}, ...$ makes the equation non-linear. In this course we will study only linear ODE/PDEs, but you should remember that there's a huge class of non-linear equations that we are not learning anything about in PHYS312 – and for which the techniques that we will learn in this course do not work, because all these methods are based on the fact that the equations are linear.

6. Suppose that $u_1(t), ..., u_n(t)$ are solutions of a linear ODE. These solutions are said to be *linearly independent* if the equation $c_1u_1(t) + ... + c_nu_n(t) = 0$ is satisfied at all t if and only if $c_1 = c_2 = ... = c_n = 0$.
For example, take $u_1(t) = t, u_2(t) = e^t, u_3(t) = 3t$. Then, u_1 and u_2 are linearly independent (check!), but u_1 and u_3 are not. The equation $c_1u_1(t) + c_3u_3(t) = 0$ for all t has an infinite number of solutions $c_1 + 3c_2 = 0$, *eg* $c_1 = -3, c_2 = 1$, or $c_1 = 1, c_2 = -1/3$ and so on and so forth.

Very important facts about ODEs:

1. A linear ODE of order n has precisely n linearly independent solutions. There are many ways to choose these n solutions, but we are certain that there cannot be more than n of them.

2. If we know n linearly independent solutions $u_1(t), ..., u_n(t)$ of a nth order linear homogeneous ODE, then the *general solution* of this ODE has the form:

$$u(t) = c_1u_1(t) + ... + c_nu_n(t) = \sum_{k=1}^{n} c_ku_k(t)$$

To find a unique solution, we must be given an additional n conditions (generally known as initial conditions, if t is a time) which allow us to find the values of the constants $c_1, ..., c_n$.

3. The general solution of an inhomogeneous ODE has the general form:

$$u(t) = u_h(t) + u_p(t),$$

where $u_h(t)$ is the GENERAL solution of the homogeneous equation (and according to fact 2, is given by a linear combination $u_h(t) = \sum_{k=1}^{n} c_k u_k(t)$) and $u_p(t)$ is ANY particular solution of the inhomogeneous equation. We will discuss elsewhere and practice in class how to find $u_p(t)$.

Note: different textbooks have different names for these components (eg, our textbook uses $u_c(t)$ instead of $u_h(t)$). I will use u_h throughout the course to remind you that is for the **h**omogeneous part, and u_p to remind that this is a **p**articular solution of the inhomogeneous ODE.

If you are not sure why these facts are true, please consult your Math 215 notes. You can also talk to me, of course.