# Boundary conditions

So far we have discussed 2nd order ODEs where the variable is time $t \geq 0$, and therefore the natural choice is to specify the **initial conditions** for the solution of interest, i.e. $u(t=0) = u_0$, $\frac{du}{dt}|_{t=0} = v_0$.

However, we will also encounter situations where the variable is a position such as $x$, for instance when we want to calculate how an electric potential or a temperature changes with position. As we'll see, this will lead to 2nd order ODEs for $u(x)$ just like the ones we had for $u(t)$, and which we will solve similarly to get the general solutions. After all, from a math point of view it makes no difference what we call the variable.

The main difference between dealing with $u(t)$ and $u(x)$ comes from the conditions we specify to get a unique solution from the general solution of the ODE. As I just mentioned, for problems depending on time we give the initial conditions – this makes sense to us as physicists, because we can measure the state of the system at the initial time, i.e. $u(t=0)$, $\frac{du}{dt}|_{t=0}$ and we're solving the ODE to find out what will be $u(t)$ at any later time.

For problems where the variable is $x$, we need to specify what interval $a \leq x \leq b$ we're interested in (for instance, in what region of space do we want to calculate the electric potential, or the temperature). In most cases it will be a finite interval, but it could also be semi-infinite or infinite. For such problems we specify two **boundary conditions** (one at each end) in order to select a unique solution from the general solution of the ODEs. These 2 conditions replace the initial conditions that we had in time-dependent 2nd order ODEs.

We have 3 possible options for such boundary conditions (BC):

(i) we can specify $u(a) = u_a, u(b) = u_b$, i.e. what is the value of the function at the two ends. These are known as **Dirichlet conditions**;

(ii) we can specify $\frac{du}{dx}|_{x=a} = v_a$, $\frac{du}{dx}|_{x=b} = v_b$, i.e. the value of the derivative of the function at the two ends. These are known as **van Neumann conditions**;

(iii) we can also have **mixed BC**, where we know either $u(a) = u_a$, $\frac{du}{dx}|_{x=b} = v_b$ OR $\frac{du}{dx}|_{x=a} = v_a, u(b) = u_b$.

The reason for this will become clear when we'll discuss specific equations and we'll see that these options correspond to various situations that we can implement in an experiment. For instance, if $u(x)$ is the temperature of a rod that extends between $a$ and $b$, then we could keep the left end of the rod at a fixed temperature $T_a$ and therefore in such case we use Dirichlet BC $u(a) = T_a$. As we will see, if $u(x)$ is a temperature, then its derivative is proportional to the heat current that flows at $x$. If, say, the left end is isolated then no heat current can flow through that end, so in this case we must ask that $\frac{du}{dx}|_{x=a} = 0$. It turns out that we can only use one such condition for each end: either we fix the temperature (and then the heat current will adjust itself to be whatever is needed to keep the temperature at the set value), or we set the heat current (and then the temperature adjusts itself accordingly).* We can do this at each end, so for each boundary (end) we need to specify what is the correct condition for the set-up we're interested in.

In today's class we'll explore together how switching from IC to BC changes how we calculate the Green's functions for a 2nd order ODE.

* In fact, there are also situations where the heat current depends on the temperature, in which case the BC will involve both the value of $u$ and of its derivative, for that end. But let's not worry about such complications right now.