

Fourier Transform – a quick introduction

So far, we have looked at Fourier *series* for periodic functions with period L : $f(x) = f(x + L)$, $(\forall)x$. We argued that if the function is reasonably well behaved (continuous with at most a finite number of jump discontinuities inside a period, and such that $\int_{x_0}^{x_0+L} dx |f(x)| < \infty$, i.e., the integral over one period is finite; etc) then everywhere where the function is continuous, it is equal to its Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2n\pi x}{L} + b_n \sin \frac{2n\pi x}{L} \right] = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2n\pi x}{L}}$$

where $a_0 = \frac{1}{L} \int_{x_0}^{x_0+L} dx f(x)$ and for any $n = 1, 2, \dots$ we have

$$a_n = \frac{2}{L} \int_{x_0}^{x_0+L} dx f(x) \cos \frac{2n\pi x}{L}; \quad b_n = \frac{2}{L} \int_{x_0}^{x_0+L} dx f(x) \sin \frac{2n\pi x}{L};$$

Or, for the complex Fourier series, for any $n = \dots, -2, -1, 0, 1, 2, \dots$ we have:

$$c_n = \frac{1}{L} \int_{x_0}^{x_0+L} dx f(x) e^{-i \frac{2n\pi x}{L}}$$

At the points where $f(x)$ has a discontinuity, we know that the Fourier series converges to $\frac{1}{2}(f(x_-) + f(x_+))$.

This is all nice and beautiful, and we'll use these Fourier series a lot in the rest of the course.

However, you might be wondering "But what happens if $L \rightarrow \infty$?" Note that if the period becomes infinite, then somehow this should apply to almost any function since it never really needs to "repeat" itself (but it still has to be continuous or at most with a finite number of jump discontinuities, and so that $\int_{-\infty}^{\infty} dx |f(x)| < \infty$. This last condition becomes important to keep in mind, because the period now extends over all real numbers. That means that, for example, we cannot use this for $f(x) = x$, even though it's continuous everywhere. In fact, after some thinking you should convince yourselves that $f(x)$ needs to go to zero reasonably fast, as $x \rightarrow \pm\infty$, if this condition is to be obeyed). But, if these conditions hold, somehow we should be able to extend the properties listed above to such functions.

What we get in this limit is known as the *Fourier transform*. In the textbook, it is done in both versions, both with sin and cos, and for the complex version. I have never encountered the sin/cos version in the past 20 years of working as a physicist, whereas I run into the complex Fourier transform pretty much every day. So I am going to focus on that, because I'd be very willing to bet that it is the one that you'll encounter outside this course, too.

So the question is what happens to:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2n\pi x}{L}}$$

$$c_n = \frac{1}{L} \int_{x_0}^{x_0+L} dx f(x) e^{-i \frac{2n\pi x}{L}} = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) e^{-i \frac{2n\pi x}{L}}$$

as we let $L \rightarrow \infty$?

Let me start by defining

$$k_n = \frac{2n\pi}{L}$$

If L is a length, then k_n has units of a wave-number, which is why I call it k . It takes equally spaced values

$$k_n = n\Delta k$$

where if $L \rightarrow \infty$, then

$$\Delta k = \frac{2\pi}{L} \rightarrow 0$$

i.e. these values get more and more closed spaced as we let L increase, and eventually become continuous.

Ok, so let's start with the second equality, and rewrite it as:

$$Lc_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) e^{-i \frac{2n\pi x}{L}} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) e^{-ik_n x}$$

On the right hand side, I can take the limit $L \rightarrow \infty$ which will just make the integral run from $-\infty \rightarrow +\infty$. This is a well-defined quantity, so the left-hand side must be a well-defined quantity, too. I'll give this a new name:

$$\text{Definition: } \tilde{f}(k_n) = \lim_{L \rightarrow \infty} [Lc_n] = \int_{-\infty}^{\infty} dx f(x) e^{-ik_n x}$$

However, note that $k_n \Rightarrow k$ becomes a continuous variable, because the spacing $\Delta k \rightarrow 0$ in $k_n = n\Delta k$. So instead of a discrete series of values $\tilde{f}(k_n)$ which replace the coefficients c_n for the discrete integers n , in the limit $L \rightarrow \infty$ we get a continuous function of a continuous variable k :

$$\text{Definition: } \tilde{f}(k) = \int_{-\infty}^{\infty} dx f(x) e^{-ikx}$$

Now, we have to write $f(x)$ in terms of this $\tilde{f}(k)$, by taking the $L \rightarrow \infty$ limit in the first equality. First, I'll rewrite it in terms of the new variables:

$$f(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} [Lc_n] e^{ik_n x} = \frac{1}{L} \sum_{n=-\infty}^{\infty} \tilde{f}(k_n) e^{ik_n x} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \Delta k \tilde{f}(k_n) e^{ik_n x}$$

But, in the limit $L \rightarrow \infty$, I recognize the sum to go to an integral over the (now) continuous variable k . This is illustrated graphically below, for a still finite L which means still discrete values k_n . The sum on the right is the shaded area = sum of areas of rectangles of width Δk and height $\tilde{f}(k_n) e^{ik_n x}$. As $\Delta k \rightarrow 0$, this converges towards the area below the curve $\tilde{f}(k) e^{ikx}$, i.e. towards its integral.

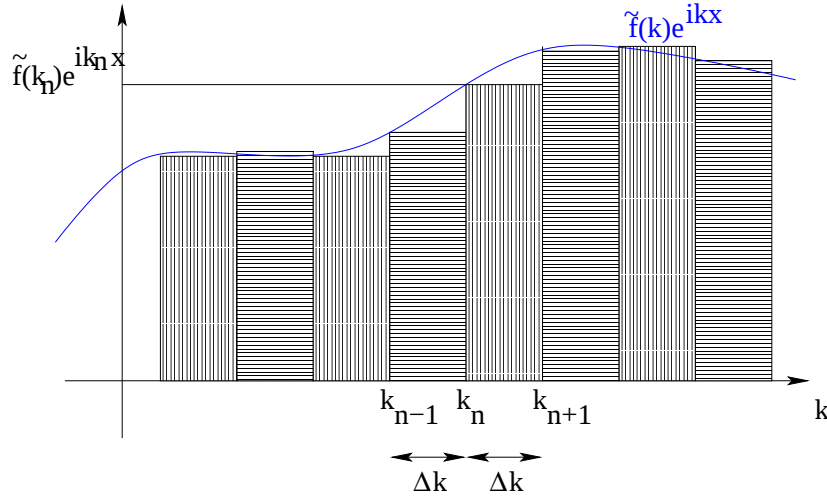


Figure 1: Geometrical proof for the limit used above.

Thus:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) e^{ikx}$$

where

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx f(x) e^{-ikx}$$

And this is the Fourier transform, which can be applied to any function $f(x)$ as long as it is continuous, etc (conditions listed above).

Let me show you first an example, and then I'll say something about conventions, and then I'll show one example of how one could use such Fourier transforms to solved ODEs.

Example:

Suppose I'm asked to find the Fourier transform of

$$f(x) = e^{\frac{-|x|}{a}} = \begin{cases} e^{\frac{-x}{a}} & , x > 0 \\ e^{\frac{x}{a}} & , x < 0 \end{cases}$$

where $a > 0$. Try plotting this. It looks like the figure shown below, the function is even and it decreases to zero on a characteristic scale controlled by a : the smaller a is, the faster the decrease. You should convince yourselves that it is continuous everywhere, no jumps, and moreover $\int_{-\infty}^{\infty} dx |f(x)|$ is finite only for such functions are the Fourier transform well defined.

What is the Fourier transform of this function? Well, we use the definition:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx f(x) e^{-ikx} = \int_{-\infty}^0 dx e^{\frac{x}{a}} e^{-ikx} + \int_0^{\infty} dx e^{-\frac{x}{a}} e^{-ikx} = \dots = \frac{1}{\frac{1}{a} - ik} + \frac{1}{\frac{1}{a} + ik} = \frac{\frac{2}{a}}{\frac{1}{a^2} + k^2}$$

This also happens to be a function centered at the origin $k = 0$, and decreasing to zero – but now with a characteristic scale $\frac{1}{a}$! In other words, the smaller a is, the more “peaked” $f(x)$ is, the “broader” $\tilde{f}(k)$ is, and viceversa.

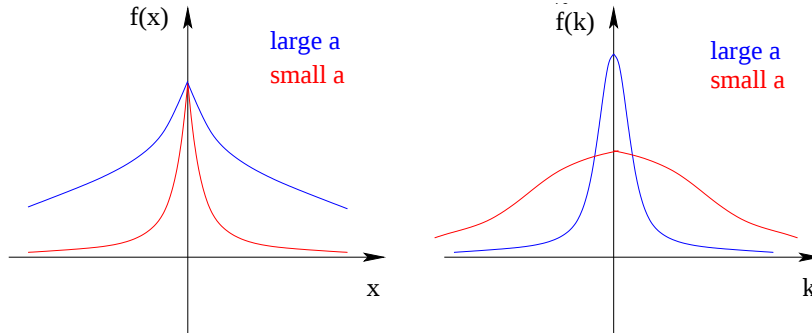


Figure 2: The function $f(x)$ and its Fourier transform

It turns out that this is a very general behavior – and it can be linked to Heisenberg’s uncertainty principle, as you’ll learn if you’ll take a quantum mechanics class. In other words, there are some pretty cool consequences. Unfortunately, this is not a physics class but a math one, so we’ll continue with the math.

First, let us derive an identity that I gave you for the Dirac delta function (remember that?) If I combine these two equations, it means that for any well-behaved function $f(x)$, we must have:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{\infty} dy f(y) e^{-iky}$$

where all I did was write the definition of $\tilde{f}(k)$ (I can’t use x because the name is taken, so I used y as the variable of integration). Now, I can switch the order of the two integrals, to find:

$$f(x) = \int_{-\infty}^{\infty} dy f(y) \left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-y)} \right]$$

and this must be true for any reasonable $f(x)$. It follows that what we have inside the parentheses must be yes, you got it, it must be the Dirac function, because:

$$f(x) = \int_{-\infty}^{\infty} dy f(y) \delta(x-y)$$

and so

$$\delta(x-y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-y)}$$

So this is how one proves this rather strange-looking identity.

Now, back to Fourier transforms. First, in some books people define them more symmetrically, as follows:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{f}(k) e^{ikx}$$

where

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} f(x) e^{-ikx}$$

This is pretty much the same thing, just a rescaling of $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \tilde{f}(k)$.

If x is not a spatial coordinate, but a time coordinate, then it is customary to choose another convention. First, in this case instead of k I will use ω as the continuous variable (it will have units of frequency, and this is the letter we usually use in physics). In this case, it is customary to define the Fourier transform as:

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{f}(\omega) e^{-i\omega t}$$

where

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

In other words, it's as if we start from the Fourier series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{2n\pi x}{L}}$ instead of our usual definition.

This should not put you off. The reason people change conventions is that it is a bit more convenient to use different ones for space vs time Fourier transforms. But mathematically they involve the same sort of operations – just do the proper integrals.

So, let me show you one example of how to use Fourier transforms to solve ODEs. Let's make it a 2nd order ODE, although you should convince yourself that you can do this for higher order ODEs as well, provided that they have constant coefficients. By the way, you may have learned this technique using Laplace transforms, which are quite closely related to the Fourier ones.

Suppose I want to solve:

$$\frac{d^2 u}{dx^2} + \alpha \frac{du}{dx} + \beta u = g(x)$$

for all $x \in (-\infty, \infty)$. Here α, β are some known constants. I can rewrite $g(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{g}(k) e^{ikx}$: once we're given a specific $g(x)$, we can find what $\tilde{g}(k)$ is.

Let's assume that the particular solution also can be Fourier transformed, i.e.

$$u_p(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{u}_p(k) e^{ikx}$$

Then:

$$\begin{aligned} \frac{du_p}{dx} &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{u}_p(k) \frac{d}{dx} e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{u}_p(k) ik e^{ikx} \\ \frac{d^2 u_p}{dx^2} &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{u}_p(k) (ik)^2 e^{ikx} \end{aligned}$$

and putting this guess in the equation, and after grouping terms, we get:

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} [-k^2 \tilde{u}_p(k) + \alpha ik \tilde{u}_p(k) + \beta \tilde{u}_p(k)] e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{g}(k) e^{ikx}$$

Since this must be true for any x , we must have:

$$-k^2 \tilde{u}_p(k) + \alpha ik \tilde{u}_p(k) + \beta \tilde{u}_p(k) = \tilde{g}(k) \rightarrow \tilde{u}_p(k) = \frac{\tilde{g}(k)}{-k^2 + \alpha ik + \beta}$$

and so:

$$u_p(x) = \int_{-\infty}^{\infty} dk \tilde{u}_p(k) e^{ikx} = \int_{-\infty}^{\infty} dk e^{ikx} \frac{\tilde{g}(k)}{-k^2 + \alpha ik + \beta}$$

And we have the solution, once the integral is done.

Note that unlike Green's functions, this requires two integrals. One to find $\tilde{g}(k)$ from $g(x)$, and then the second to find $u_p(x)$. So it is not at all easier than using Green's functions, which only requires one integral. Moreover, Green's functions work for any ODE, not only the simplest ones with constant coefficients.

Also, this only works if we're interested in $x \in (-\infty, \infty)$. In this course, we will (almost) only consider boundary problems defined in a finite range, $x \in [a, b]$. This is why, as you will see, we will not encounter Fourier transforms in the rest of this course. But since you will encounter them a lot in other areas of physics, I thought it's useful to show you how they come about, and how they are linked to the Fourier series. I hope this will be of some use to you, at some point.