Heat equation in a 2D rectangle

This is the solution for the in-class activity regarding the temperature $u(x, y, t)$ in a thin rectangle of dimensions $x \in [0, a], b \in [0, b]$, which is initially all held at temperature T_0, so $u(x, y, t = 0) = T_0$. Then, from $t = 0$ onwards, we keep its $x = 0$ edge at temperature T_L, and all other 3 edges at temperature 0, therefore $u(x = 0, y, t) = T_L, u(x = a, y, t) = u(x, y = 0, t) = u(x, y = b, t) = 0$.

The first part is to calculate the steady-state solution $u_s(x, y) = \lim_{t\to\infty} u(x, y, t)$. It satisfies the heat equation, since u satisfies it as well, however because there is no time-dependence, the time derivative vanishes and we're left with:

$$\frac{\partial^2 u_s}{\partial x^2} + \frac{\partial^2 u_s}{\partial y^2} = 0$$

u_s also satisfies the same boundary conditions like u, so: $u_s(x = 0, y) = T_L, u(x = a, y) = 0, \forall y \in [0, b]$ while $u(x, y = 0) = u(x, y = b), \forall x \in [0, a]$.

We use separation of variables $u_s(x, y) = X(x)Y(y)$. The PDE then becomes:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \to \frac{d^2X}{dx^2} = AX, \frac{d^2Y}{dx^2} = -AY$$

We need to find the eigenvalues A. For this we need additional conditions – let's see what we can get from the boundary conditions. Note that we need to have hom. BC on two opposite sides to be able to solve this: we need to know either 2 BC for X, or two BC for Y, otherwise we cannot find the allowed A. Here we are lucky, because from $u_s(x, y = 0) = u_s(x, y = b) = 0$ for all x, we find that $Y(0) = Y(b) = 0$. So we do have the two needed conditions.

As an aside: for your homework (in conjunction with the pre-reading for today), if the boundary conditions are not such that you have 0 on two opposite sides, then you have to split the problem into a sum of simpler problems, each of which has hom. BC on a pair of opposite edges.

Back to our problem: we need to solve the eigenproblem $\frac{d^2Y}{dx^2} = -AY$ with $Y(0) = Y(b) = 0$. We know that $A = k^2 > 0$ so that the solution is $Y(y) = \alpha\cos(ky) + \beta\sin(ky)$ which has a chance to have two different zeroes (for $A < 0$, the solution would be a sum of exponentials which can never satisfy these BC). Then, as usual, we find an infinite number of eigenvalues $A_n = (n\pi/b)^2$, $n = 1, 2, ...,$ so that $Y_n(y) = \sin\frac{n\pi y}{b}$.

Now we can solve the ODE for X, and for each of these eigenvalues we find a possible solution $X_n(x) = a_n e^{\frac{n\pi}{b}x} + b_n e^{-\frac{n\pi}{b}x}$.

So far, we have found an infinite number of solutions $X_n(x)Y_n(y)$, each of which satisfies the PDE and the 2 BC for y. However, none of these solutions can satisfy the inhom. BC at $x = 0$: $u_s(x = 0, y) = T_L$ since each depends in some nontrivial fashion on y.

So then we try to find a general solution that combines all of these solutions, and hopefully with such coefficients that we can satisfy the remaining BC:

$$u_s(x, y) = \sum_{n=1}^{\infty}(a_n e^{\frac{n\pi}{b}x} + b_n e^{-\frac{n\pi}{b}x})\sin\frac{n\pi y}{b}$$

At $x = 0$, we have:

$$T_L = \sum_{n=1}^{\infty}(a_n + b_n)\sin\frac{n\pi y}{b}, \forall y \in [0, b]$$

which means that we can find $a_n + b_n$ from the odd periodic extension of T_L, which has period $L = 2b$, and which leads to:

$$a_n + b_n = ... = \frac{2}{b}\int_0^b dy T_L \sin\frac{n\pi y}{b} \to$$

$$a_n + b_n = \frac{2T_L}{n\pi}[1 - (-1)^n]$$

Finally, we also need to satisfy the BC at $x = a$:

$$0 = \sum_{n=1}^{\infty}(a_n e^{\frac{n\pi}{b}a} + b_n e^{-\frac{n\pi}{b}a})\sin\frac{n\pi y}{b}, \forall y \in [0, b]$$

from which we find:

$$a_n e^{\frac{n\pi}{b}a} + b_n e^{-\frac{n\pi}{b}a} = 0$$

(it's like replacing $T_L \to 0$ in the above expressions).

So for each n, we have two equations with wo unknowns:

$$\begin{cases} a_n + b_n = \frac{2T_L}{n\pi}[1-(-1)^n] \\ a_n e^{\frac{n\pi}{b}a} + b_n e^{-\frac{n\pi}{b}a} = 0 \end{cases} \to \ldots \to \begin{cases} a_n = -\frac{T_L}{n\pi \sinh \frac{n\pi a}{b}}[1-(-1)^n]e^{-\frac{n\pi a}{b}} \\ b_n = +\frac{T_L}{n\pi \sinh \frac{n\pi a}{b}}[1-(-1)^n]e^{\frac{n\pi a}{b}} \end{cases}$$

where $\sin(x) = (e^x - e^{-x})/2$ is just a shorthand notation. With this and after rearranging things a little bit, we finally get:

$$u_s(x,y) = \sum_{n=1}^{\infty} \frac{2T_L[1-(-1)^n]}{n\pi \sin \frac{n\pi a}{b}} \sinh \frac{n\pi(a-x)}{b} \sin \frac{n\pi y}{b}$$

which indeed is a linear combination of the allowed solutions, and can be seen to vanish at $y = 0, b$ and $x = a$, while it has the T_L value at $x = 0$. So this part is done.

Now we move on to the transient solution $u_t(x,y,t) = u(x,y,t) - u_s(x,y)$. This also satisfies the full PDE, like u and u_s. However, it has only homogeneous BC:

$$u_t(x=0,y,t) = u(x=0,y,t) - u_s(x=0,y) = T_L - T_L = 0$$

and similarly $u_t(x=a,y,t) = u_t(x,y=0,t) = u_t(x,y=b,t) = 0$ at all times.

So we try again separation of variables. Because we have 3 variables, we try $u_t(x,y,t) = X(x)Y(y)T(t)$. After putting this into the PDE we arrive at:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{\kappa}\frac{1}{T}\frac{dT}{dt}$$

which means that we must solve the eigenequations:

$$\frac{1}{X}\frac{d^2X}{dx^2} = A; \frac{1}{Y}\frac{d^2Y}{dy^2} = B, \frac{1}{\kappa}\frac{1}{T}\frac{dT}{dt} = A + B$$

So we need first to find the eigenvalues A and B so that:

$$\frac{d^2X}{dx^2} = AX; \frac{d^2Y}{dy^2} = BY$$

Note that we can only achieve this is we have 2 BC for X (to find A) and two BC for Y (to find B). This is why here we need ALL BC to be homogeneous, so that we can rewrite them as $X(0) = X(a) = 0; Y(0) = Y(b) = 0$.

It is important to convince yourselves that if we did not do the trick with the steady-state solution first, there would be no way to deal with the inhomog. BC at $x = 0$ directly – we cannot split it in an equation for only X or only Y, so we could not find eigenvalues and there would be no way forward.

But because we made sure that for the transient problem all BC are homog, we can solve for both sets of eigenvalues.

$$\frac{d^2X}{dx^2} = AX, X(0) = X(a) = 0 \to A_n = -\left(\frac{n\pi}{a}\right)^2, X_n(x) = \sin\frac{n\pi x}{a}, n = 1, 2, ...$$

while

$$\frac{d^2Y}{dy^2} = BY, Y(0) = Y(b) = 0 \to B_m = -\left(\frac{m\pi}{b}\right)^2, Y_m(y) = \sin\frac{m\pi y}{b}, m = 1, 2, ...$$

I explicitly use a different integer for Y than for X because there is absolutely nothing that requires these two to be equal!!! For each pair of integers n, m, then I can solve for the time dependence:

$$\frac{dT}{dt} = \kappa(A_n + B_m)T = -\kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right)T \to T_{n,m}(t) = c_{n,m}e^{-\kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right)t}$$

At it should, $T(t\to\infty) \to 0$ (this is part of the transient solution, which dies off at long times).

So, at this moment we have, for any pair of n, m integers, the solutions

$$X_n(x)Y_m(y)T_{n,m}(t) = c_{n,m} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} e^{-\kappa\left((\frac{n\pi}{a})^2+((\frac{m\pi}{b})^2\right)t}$$

which satisfy the PDE and all 4 BC. However, none of these satisfies the initial condition:

$$u_t(x, y, t = 0) = T_0 - u_s(x, y), \forall x, y$$

So again, we put together a combination from all of them, and hope that if we choose the constants $c_{n,m}$ appropriately, we might be able to satisfy this initial condition as well:

$$u_t(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} e^{-\kappa\left((\frac{n\pi}{a})^2+((\frac{m\pi}{b})^2\right)t}$$

where we must have:

$$T_0 - u_s(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}, \forall x, y$$

So this looks like some sort of "double" Fourier series. We can make it look more standard if we introduce the additional functions:

$$g_n(y) = \sum_{m=1}^{\infty} c_{n,m} \sin\frac{m\pi y}{b}$$

in terms of which we find:

$$T_0 - u_s(x, y) = \sum_{n=1}^{\infty} g_n(y) \sin\frac{n\pi x}{a}, \forall x \in [0, a]$$

This last equality looks very familiar (if we treat the g as some coefficients) and leads to:

$$g_n(y) = \frac{2}{a}\int_0^a dx[T_0 - u_s(x, y)] \sin\frac{n\pi x}{a}$$

But once we know these functions, we can use the equality above:

$$g_n(y) = \sum_{m=1}^{\infty} c_{n,m} \sin\frac{m\pi y}{b}, \forall y \in [0, b]$$

to find:

$$c_{n,m} = \frac{2}{b}\int_0^b dy g_n(y) \sin\frac{m\pi y}{b}$$

and so combining the two, we find:

$$c_{n,m} = \frac{2}{a}\int_0^a dx \sin\frac{n\pi x}{a} \frac{2}{b}\int_0^b dy \sin\frac{m\pi y}{b} [T_0 - u_s(x, y)]$$

Note that the final result is not too bad, because u_s is already written as a product of terms depending only on x and only on y, so one can separate easily into a product of two integrals, none of which is too hard. The end result is still quite complicated, but then we didn't really expect a simple solution to this complicated problem, did we?

Let me make one more comment on a different way to extract the $c_{n,m}$ coefficients. This will make sense once we discuss the Sturm-Liouville problem next week.

We are trying to get:

$$T_0 - u_s(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m} X_n(x)Y_m(y), \forall x, y$$

We know that each of the sets $X_n(x)$ and $Y_m(y)$ are orthogonal, because each comes from a Sturm-Liouville equation, and for example $\int_0^a dx X_n(x)X_{n'}(x) = 0$ of $n \neq n'$. And so we have:

$$[T_0 - u_s(x, y)] X_{n'}(x)Y_{m'}(y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m} X_n(x)Y_m(y)X_{n'}(x)Y_{m'}(y)$$

and after integrating on both sides:

$$\int_0^a dx \int_0^b dy \left[T_0 - u_s(x,y)\right] X_{n'}(x) Y_{m'}(y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{n,m} \int_0^a dx X_n(x) X_{n'}(x) \int_0^b dy Y_m(y) Y_{m'}(y)$$

On the rhs, the first integral is zero unless $n = n'$ and the second is zero unless $m = m'$, so only one term survives from the double sum:

$$c_{n',m'} = \frac{\int_0^a dx \int_0^b dy \left[T_0 - u_s(x,y)\right] X_{n'}(x) Y_{m'}(y)}{\int_0^a dx [X_{n'}(x)]^2 \int_0^b dy [Y_{m'}(y)]^2}$$

For the functions $X_n(x)$ and $Y_m(y)$ that we have here (simple sine) its immediate to find:

$$\int_0^a dx [X_{n'}(x)]^2 = \frac{a}{2}$$

and

$$\int_0^b dy [Y_{m'}(y)]^2 = \frac{b}{2}$$

and so we again find:

$$c_{n',m'} = \frac{4}{ab} \int_0^a dx \int_0^b dy \left[T_0 - u_s(x,y)\right] X_{n'}(x) Y_{m'}(y)$$

which is precisely the same formula as before.

This second approach is more general, because it works for any eigenfunctions X_n, Y_m, whereas the Fourier trick only works for simple $\sin \frac{n\pi x}{a}$ etc. However, if the functions are more complicated, the integrals on the bottom will be different from $a/2, b/2$, so one would have to see whatever they are equal to.