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Derivation of β-ν angular correlation

I will admit up front that the next few pages are math.
But
1) I like math
2) I've been saying over and over again that "the S.M. vector/axial vector interaction only makes left-handed leptons and right-handed antileptons" and then using helicity arguments to "derive" angular correlations, given the functional form.
Now you have enough Dirac formalism to derive these things for real, and prove these assumptions.

Consider a nuclear β decay transition. Just take the lowest-order terms for the hadron current, the ones proportional to g_V and g_A. (The induced terms are $< .01$ corrections.)

The transition amplitude is:

$$T_{fi} = -i\frac{4G}{\sqrt{2}}\int J_\mu^{(had)}(x)J^{\mu(lep)}(x)d^4x =$$

$$-i\frac{4G}{\sqrt{2}}\int\left[\bar{\psi}_n(x)\gamma_\mu\frac{1}{2}(g_V - g_A\gamma^5)\psi_p(x)\right]$$

$$\left[\bar{\psi}_\nu(x)\gamma^\mu\frac{1}{2}(1-\gamma^5)\psi_e(x)\right]d^4x$$

• Consider a 0^+ to 0^+ β decay between isobaric analog states (assumed pure).
Then we only have the Fermi part of the hadronic current, and the axial vector part $\bar{\psi}_n\gamma^\mu\gamma^5\psi_p$ will not contribute.
• The energy release in the decay is $<< M_{\text{nucleon}}$, so we can use nonrelativistic spinors for the nucleons. Then only γ^μ with μ=0 contributes. Then:

$$T_{fi} = -i\frac{G}{\sqrt{2}}\int \left[\bar{u_\nu}(p_\nu)\gamma^0(1-\gamma^5)v(p_e)\right]\psi_n^\dagger(x)\psi_p(x)e^{-i(p_\nu+p_e)\cdot x}d^4x$$

where spinor $v(p_e)$ describes an outgoing positron of momentum p_e.
We make the 'long-wavelength'-type expansion and set the lepton plane wave to 1.
We're assuming perfect overlap of the nuclear wavefunctions, so we do the trivial integration over space and get for the invariant amplitude

$$M = \frac{G}{\sqrt{2}}\left[\bar{u}(p_\nu)\gamma^0(1-\gamma^5)v(p_e)\right](2m_N)2\sqrt{2}$$

where $2m_N$ comes from the normalization of the nucleon spinors, and $2\sqrt{2}$ is the isospin factor, though to be honest I just want the angular distribution.

Decay rate in terms of matrix element:

$$d\Gamma = G^2 \sum_{\text{spins}} |\bar{u}(p_\nu)\gamma^0(1-\gamma^5)v(p_e)|^2$$

$$\frac{d^3p_e}{(2\pi)^3 2E_e}\frac{d^3p_\nu}{(2\pi)^3 2E_\nu} 2\pi\delta(E_0 - E_e - E_\nu)$$

The sum over the lepton spins means this product is not as hard as it looks.

$$\sum_{\text{spins}} |\bar{u}(p_\nu)\gamma^0(1-\gamma^5)v(p_e)|^2 =$$

$$\sum \bar{u}(k')\gamma^0(1-\gamma^5)v(k)\bar{v}(k)\gamma^0(1-\gamma^5)u(k') =$$

If we right out some indices, it's OK to rearrange where we put the u spinor; we are summing over repeated indices:

$$\sum \left[\gamma^0(1-\gamma^5)\right]_{\alpha\beta}\left[v(k)\bar{v}(k)\right]_{\beta\gamma}\left[\gamma^0(1-\gamma^5)\right]_{\gamma\delta}\left[u(k')\bar{u}(k')\right]_{\delta\alpha}$$

Summing over the spins of the spinors ends up giving "completeness relations". Halzen and Martin leave this as an exercise. So using

$$\sum_{s=1,2} u^{(s)}(k')\bar{u}^{(s)}(k') = \not{k}' + m$$

$$\sum_{s=1,2} v^{(s)}(k)\bar{v}^{(s)}(k) = \not{k} - m$$

It may seem odd to have rewritten vector outer products as 4x4 matrices, but now we have a complete set of summed-up indices, which reduces the problem to a trace of 4x4 matrices. Using the completeness relations, we have:

$$Tr\left[\gamma^0(1-\gamma^5)(\not{k} - m_\nu)\gamma^0(1-\gamma^5)(\not{k}' + m_e)\right]$$

(You can start the homework problem right there. Replace $\gamma^0\left[1-\gamma^5\right]$ by $\left[C_S+C_S'\gamma^5\right]$)

Now we use some trace theorems. If you don't like trace theorems, it is not very much harder to just keep commuting γ's until you get something recognizable.

The trace of an odd number of γ_μ's vanishes. (γ^5 is an even number of γ^μ's).
So the terms multiplying m_ν and m_e vanish, leaving:

$$Tr\left[\gamma^0(1-\gamma^5)\ \not{k}\gamma^0(1-\gamma^5)\ \not{k}'\right]+ \\ Tr\left[\gamma^0(1-\gamma^5)\gamma^0(1-\gamma^5)m_\nu m_e\right]$$

Commuting one γ^0 matrix through in the 2nd term, we get a product of projection operators which vanishes:

$$Tr\left[\gamma^0(1-\gamma^5)\ \not{k}\gamma^0(1-\gamma^5)\ \not{k}'\right]+ \\ Tr\left[\gamma^0\gamma^0(1+\gamma^5)(1-\gamma^5)m_\nu m_e\right]$$

It looks like we've killed all our lepton masses, but don't worry, they will come back soon when we evaluate some

dot products.
Now we use one trace theorem, which looks fairly capricious:

$$Tr\left[\gamma^\mu(1-\gamma^5)\not{p}_1\gamma^\nu(1-\gamma^5)\not{p}_2\right] =$$
$$2Tr(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2) + 8i\epsilon^{\mu\alpha\nu\beta}p_{1\alpha}p_{2\beta}$$

So we have

$$Tr(\gamma^0\not{k}\gamma^0\not{k}') + 8i\epsilon^{0\alpha0\beta}k_\alpha k'_\beta$$

The 2nd term vanishes because of the completely antisymmetric ϵ. We use a 2nd trace theorem, an important one that lets us evaluate things in terms of 4-vector products:

$$Tr(\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2) = 4(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2)g^{\mu\nu})$$

leaving

$$
\begin{aligned}
8(k^0k'^0 + k'^0k^0 - (k \cdot k')g^{00}) &= \\
8(2E_\nu E_e - (E_eE_\nu - \vec{p_e} \cdot \vec{p_\nu})) &= \\
8(E_eE_\nu + \vec{p_e} \cdot \vec{p_\nu}) &= \\
8E_eE_\nu(1 + \frac{v_e}{c}\frac{v_\nu}{c}\cos\theta_{e\nu})
\end{aligned}
$$

So the β-ν angular distribution for a pure Fermi decay and a V-A interaction is:

$$
W(\theta_{e\nu}) = 1 + \frac{v_e}{c}\frac{v_\nu}{c}\cos\theta_{e\nu})
$$

This is also true for V+A, though the capricious trace theorem hides that a little. It is also true for any combination of V,A (i.e. a small right-handed current won't change this). Since the final observable is even under parity, it is not surprising that it is not sensitive. The first β-ν correlation experiments were done just before parity violation was realized.

The homework problem with a scalar interaction is simpler.
You need the trace theorem $Tr(\not{a}\ \not{b}) = 4a \cdot b$

What if the decaying species is spin-polarized?

I could only find one tractable case, for spin-1/2 μ decay.
(For anything else, I refer you to the Jackson-Treiman-Wyld papers, and to Holstein's RMP, or to recent Severijns RMP for modern notation assuming 'chirality'.)

One difficulty is that you lose the power of the trace theorems when you don't sum over spins.
You can preserve that if you :
1) introduce spin projection operators (they look a lot like the ones we already know),
2) write them in terms of γ matrices
3) then you just have to re-do the traces
Ian Towner's notes on polarized μ decay attached:

Suppose that the muon is polarised

ie power of the trace theorems will be lost, as we are not now summing over the spin components

We can retain the power, if introduce spin projection operators

Define a helicity operator, h, such that

$$h\, u(p, r=1) = +\, u(p, r=1) \qquad \xi_1 = \begin{pmatrix}1\\0\end{pmatrix} \text{ spin 'up' solution}$$

$$h\, u(p, r=2) = -\, u(p, r=2) \qquad \xi_2 = \begin{pmatrix}0\\1\end{pmatrix} \text{ spin 'down' solution}$$

Applying h twice

$$h(hu) = u \qquad \text{ie.} \quad h^2 = I$$

Spin projection operators are therefore

$$\Lambda_+ = \tfrac{1}{2}(I+h) \qquad \Lambda_- = \tfrac{1}{2}(I-h)$$

Projection operators have the property: $\Lambda^2 = \Lambda$

Eg. $\Lambda_+^2 = \frac{1}{4}(I+h)^2 = \frac{1}{4}(I+2h+h^2) = \frac{1}{2}(I+h) = \Lambda_+$

Recall for muon decay: $\mu^-(P) \to e^-(p_1) + \bar{\nu}_e(p_2) + \nu_\mu(p_3)$

$$\frac{1}{2}\sum_{\substack{r\,r_3\\ r_1 r_2}} |T_{fi}|^2 = \frac{g^4}{\cancel{128}\,64\, m_W^4} \sum_{\substack{r\,r_3\\ r_1 r_2}} |\bar{u}(p_3)\gamma_\mu(1+\gamma_5)\, u(P)|^2\, |\bar{u}(p_1)\gamma_\mu(1+\gamma_5)\, v(p_2)|^2$$

↑ remove this averaging factor

↓ replace by

$$|\bar{u}(p_3)\gamma_\mu(1+\gamma_5)\Lambda_+ u(P)|^2$$

then use trace theorems as before

Helicity operator, h

onsider : non-relativistic two-component spin physics

Consider : a free particle in isolation such that we can choose the z-axis along the direction of the particle's momentum

Then: $\underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{p}} \, \underset{\sim}{\xi}_r = \dfrac{\underset{\sim}{\sigma} \cdot \underset{\sim}{p}}{|\underset{\sim}{p}|} \underset{\sim}{\xi}_r = \sigma_3 \underset{\sim}{\xi}_r = \pm \underset{\sim}{\xi}_r$ as required

Generalisation to 4-component spinors :

$$h = \begin{pmatrix} \underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{p}} & 0 \\ 0 & \underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{p}} \end{pmatrix}$$

More convenient for trace theorems to write h in terms of γ-matrices

$$h = -i \gamma_5 \gamma_4 \underset{\sim}{\gamma} \cdot \hat{\underset{\sim}{p}}$$

$$= \frac{-i}{|\underset{\sim}{p}|} \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} 0 & -i\underset{\sim}{\sigma} \cdot \underset{\sim}{p} \\ i\underset{\sim}{\sigma} \cdot \underset{\sim}{p} & 0 \end{pmatrix}$$

$$= \frac{-i}{|\underset{\sim}{p}|} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} 0 & -i\underset{\sim}{\sigma} \cdot \underset{\sim}{p} \\ i\underset{\sim}{\sigma} \cdot \underset{\sim}{p} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{p}} & 0 \\ 0 & \underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{p}} \end{pmatrix} \quad \text{as required}$$

Now it is just a question of redoing the traces

Muon decay rate

$x = E_1 / E_1^{max}$

$$d\Gamma = \frac{g^4}{m_W^4} \frac{1}{192\pi^3} \frac{m_\mu^5}{32} x^2 \left\{ (6-4\rho) - \left(6 - \frac{16}{3}\rho\right) x - \xi \cos\theta \left[(2-4\delta) - \left(2 - \frac{16}{3}\delta\right) x \right] \right\} dx \, d(\cos\theta)$$

$\cos\theta$ = cosine of angle between muon spin polarisation vector and direction of emitted electron.

ρ, ξ and δ are Michel parameters

Coupling	$\gamma_\mu(1+\gamma_5)$ V−A	$\gamma_\mu(1-\gamma_5)$ V+A	$(1+\gamma_5)$ S−P	$\sigma_{\mu\nu}(1+\gamma_5)$ T	Expt
ρ	3/4	3/4	0	1	0·7518 ± 0·0026
ξ	1	−1	−3	7/3	1·0027 ± 0·0080
δ	3/4	3/4	0	3/7	0·7486 ± 0·0040

Result for 100% polarisation

If the muon polarisation is P_μ, then

$$\xi \rightarrow \xi P_\mu$$

3) If a scalar boson is exchanged instead of the W, work out the lepton trace, i.e. derive the β-ν correlation with all the Dirac algebra. The derivation is very similar to the vector case. You will see how a scalar interaction produces leptons with the same helicity.

- Set m_ν=0 (I want only one standard model deviation at a time...)
- Assume C_S and C'_S are real (not necessary)

Starting at the place in the notes:

$$Tr\left[\gamma^0(1-\gamma^5)(\not{k}-m_\nu)\gamma^0(1-\gamma^5)(\not{k}'+m_e)\right]$$

Lepton trace for scalar interaction is then

$$Tr\left[(C_S - C'_S\gamma^5)\ \not{k}(C_S + C'_S\gamma^5)(\not{k}'+m_e)\right]$$

Tr(odd number of γ's)=0, so m_e term is zero

$$Tr\left[(C - C'\gamma^5)\not{k}(C + C'\gamma^5)\not{k}'\right] =$$

$$Tr\left[C^2 \not{k}\not{k}' - C'C\gamma^5\not{k}\not{k}' + CC'\not{k}\gamma^5\not{k}' - C'^2\gamma^5\not{k}\gamma^5\not{k}'\right]$$

$$\gamma^5\gamma^\mu = -\gamma^5\gamma^\mu, \quad \gamma^5\gamma^5 = I$$

$$Tr\left[C^2 \not{k}\not{k}' - 2C'C(\gamma^5\not{k}\not{k}') + C'^2\not{k}\not{k}'\right]$$

$$Tr(\gamma^5\not{a}\not{b}) = 0$$

$$= (C^2 + C'^2)Tr(\not{k}\not{k}')$$

$$\text{use } Tr(\not{a}\not{b}) = 4a \cdot b$$

$$= (C^2 + C'^2)4(E_\beta E_\nu - \vec{p}_\beta \cdot \vec{p}_\nu)$$

$$= 4E_\beta E_\nu (C^2 + C'^2)\left(1 - \frac{\vec{p}_\beta \cdot \vec{p}_\nu}{E_\beta E_\nu}\right)$$

$\Rightarrow$ β-ν correlation a = -1 for scalar interaction