Structure of Light Nuclei: Spin, Isospin, Permutation symmetry

- **A=3 structure; isoscalar and isovector μ**

- **Systematic accounting of permutation symmetry: Young diagrams**

Coupling one to four valence-shell nucleons describes many low-lying J^π; T states in light nuclei + dominant L configuration (decays, reactions...)

- **Simple guidelines on configurations with lowest energy then reproduce lowest level order**

Strong interaction, because it's short range and attractive, favors symmetric and lowest L

For a given L, spin-orbit $\vec{L} \cdot \vec{S}$ favors largest J

Demonstrative examples, not proofs. Some naive states will be ruled out. Few explicit (anti)symmeterized $\psi's$: instead arguments for their existence. A=6,5,4,7,8 'Conspiracy' against $(S; T) = (3/2, 3/2)$

Refs. Bohr and Mottleson Appendix 1C; (EGA UW Phys562); PDG;
de-Shalit and Talmi, Nuclear Shell Theory (Dover) Ch. 32 "The Group Theoretical..."; Frank Close "Intro to Quarks and Partons" for more general Young techniques

A=3 $\mu \Leftrightarrow$ 'asymmetric nucleon' calculation

Recall we worked out one 'wrong' constituent quark ψ for the nucleon, antisymmetric in spin and isospin together (i.e., ignoring color).
Assuming ψ_{space} is symmetric (more on that later) and also $L = 0$, we can use that wf for spin-up ^{3}He by replacing $u \to p$ and $d \to n$:

$$\psi_{3\text{He}} = \sqrt{1/6}[ppn(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + pnp(\downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow) - npp(\uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow)]$$

We see that all the like fermions for A=3 are always paired to spin zero in all configurations satisfying permutation symmetry (and L=0).
This, of course, is what we expect if we 'pair' up the identical nucleons, but we can see the explicit physics needed in this simple system.

$\mu(^3\text{He}) = \mu_n$ and $\mu(^3\text{H}) = \mu_p$ in this lowest-order approximation

$\mu(^3\text{He}) = -2.12749772(3)$nm, μ_n =-1.9130427(5)
$\mu(^3\text{H}) \;\; = +2.97896244(4)$ $\qquad \mu_p$=+2.79284734(3)

Note that spin-polarized ^{3}He is used as a polarized neutron target, for experiments at high enough momentum tranfer to be sensitive to the spin dependence of nucleon substructure.

Corrections: Isoscalar and Isovector Magnetic moments

Taking the sum (isoscalar) and difference (isovector) for A=3,
People take the "isoscalar" μ of isobaric analog nuclei and compare to experiment:

				Difference (nm)
$\mu(^3\mathrm{H})$ =	+2.97896244(4)	μ_p=	+2.79284734(3)	0.19
$\mu(^3\mathrm{He})$ =	-2.12749772(3) nm	μ_n =	-1.9130427(5)	-0.20
Isoscalar (sum)	0.852		0.880	-0.028
Isovector (diff)	5.106		4.706	0.401

Isoscalar μ agrees better.

So far, I've been ignoring interactions.

We sketched that the π exchange current is isovector– it doesn't contribute to μ of the $T = 0$ deuteron, nor to this isoscalar sum of μ's.

The isoscalar μ commutes with the central component of the residual interaction, so is sensitive to noncentral components, in particular the tensor interaction (Towner and Khanna Nucl Phys A399 334 (1983); Arima, "A short history of magnetic moments..." Science China 54 188 (2011) doi:10.1007/s11433-010-4224-6)

Young Tableaux: systematic delineation of permutation symmetry.

No formal proofs here: just rules

Example

1	2
3	

Diagram of permutations of *n* objects

In each box is a label for the state: here just the 1st, 2nd, 3rd object; labels are in arbitrary order, but must keep that order

Rows are symmetric under permutation: this example is symmetric for first 2 objects

Columns antisymmetric under permutation

Labels can't decrease going $\rightarrow$ in any row

Labels can't decrease going $\downarrow$ in any column

Can't have same labels in any 2 elements (boxes) in any column (but can duplicate in rows)

Young Tableaux example: A=3 system

$\psi_{\text{space}}\psi_{\text{spin}}\psi_{\text{isospin}}$ **must be antisymmetric**

ψ_{space}

Consider states where all 3 nucleons have L=0

No sublevels: must be symmetric, i.e.

$\psi_S(1)\psi_S(2) - \psi_S(2)\psi_S(1) = 0$

1	2	3

ψ_{spin}

We had mixed symmetry

1	2
3	

ψ_{isospin}

We had mixed symmetry

1	3
2	

We've laid these functions out previously– we're just using these diagrams as tools to account for possibilities of number of states with various total quantum numbers and properties

or consider $\psi_{\text{isospin}}\psi_{\text{spin}}$ together for one example (it's ok– we're just changing our labels without explicitly writing it...) and writing the antisymmetric diagram filled with 3 out of 4 states:

1
2
3

Can we have S=3/2 T=1/2?

(Remember Δ(1232) S=3/2 t=3/2 is symmetric)

Evaluate by considering A=3 as a 'hole' in the A=4 system $\rightarrow$

A=3 allowed spin, isospin

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$

S=? T=?

Consider A=4

$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$

4 is the maximum number of particles to be put into 4 states. Closed shell. Totally antisymmetric. S=0, T=0.

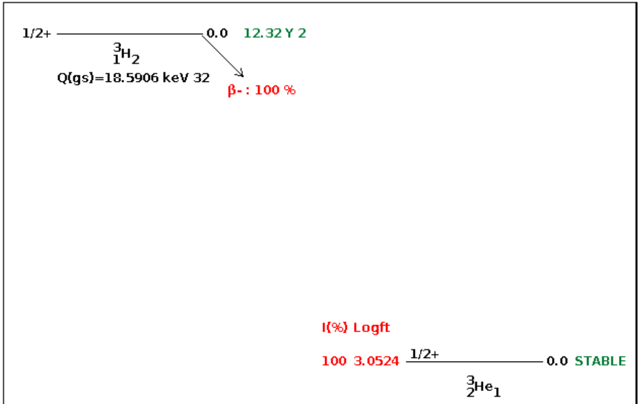
Then

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} - \square$$

S=?, T=?

S=0, T=0

$S = \frac{1}{2}$, $T = \frac{1}{2}$



$(S, T) = (\frac{1}{2}, \frac{1}{2})$ only, no $(\frac{3}{2}, \frac{3}{2})$

These are the only A=3 bound states.
There are no experimentally known unbound resonances.
There are theoretical possibilities for unbound resonances in the 3 proton and 3 neutron systems.

Note tables of Δ that include *t* have no S=3/2, t=1/2, π=+ resonance.

Wilson ‘The Excited States of the Proton’ Comments Nucl. Part. Phys 1 (1967) 128

TABLE OF RESONANCES[a]

Notation	Mass (MeV)	Spin	Parity	Isotopic spin	Width (MeV)
N_α	939	1/2	+	1/2	Stable
	1688	5/2	+	1/2	110
N_γ	1518	3/2	−	1/2	105
	2190	7/2	−	1/2	200
	2650	(11/2)	−	1/2	300
	3030	(15/2)	(−)	1/2	400
	1400	1/2	+	1/2	200
	1570	1/2	−	1/2	130
	1670	5/2	−	1/2	140
	1700	1/2	−	1/2	240
Δ_δ	1236	3/2	+	3/2	120
	1920	7/2	+	3/2	200
	2420	(11/2)	+	3/2	275
	2850	(15/2)	(+)	3/2	300
	3230	(19/2)	(+)	3/2	440
	1670	1/2	−	3/2	180
	2080				40
	2190				40

Young Tableaux example: 'm-scheme' Consider 2 P-shell particles, each with ℓ=1

(Answer is obvious for 2 particles (A=6, 5), but we'll need this for A=7)

- **Label state boxes with $m = \ell_3$**

(Our arbitrary ordinal box labels end up kinda backwards in the simplest way to do it: $m = \ell$ is the 'first' label, $m = \ell - 1$ the 'second', $m = \ell - 2$ the 'third'...)

- **Consider all configurations possible for each $M = L_3$ (nonnegative for brevity)**
- **Account completely for these configurations with values of L, all M from same permutation symmetry configuration**

$M = 2$: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$ $\begin{array}{|c|}\hline \cancel{1} \\ \hline \cancel{1} \\ \hline\end{array}$

$M = 1$: $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline \cancel{0} & \cancel{1} \\ \hline\end{array}$ $\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array}$

$M = 0$: $\begin{array}{|c|c|}\hline 1 & -1 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline \cancel{-1} & \cancel{1} \\ \hline\end{array}$ $\begin{array}{|c|}\hline 1 \\ \hline -1 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline\end{array}$ $\begin{array}{|c|}\hline \cancel{0} \\ \hline \cancel{0} \\ \hline\end{array}$

L=2 L=1 L=0

Summary:

- **L=2, L=0 with symmetric configurations**

We'll be assuming symmetric configurations have lower energy

- **L=1 has an antisymmetric configuration.**

Note that π given by $(-1)^{\ell_1} \times (-1)^{\ell_2}$ so we don't get $\pi = -$ states with 2 p-shell nucleons

We'll need this 'm-scheme' for 3 particles for A=7

Energy Levels of Neutral Oxygen (O I)

Configuration	Term	J	Level(cm^{-1})	Ref.
$2s^2 2p^4$	^3P	2	0.000	MG93
		1	158.265	MG93
		0	226.977	MG93
$2s^2 2p^4$	^1D	2	15867.862	MG93
$2s^2 2p^4$	^1S	0	33792.583	MG93
$2s^2 2p^3(^4S^\circ)3s$	$^5S^\circ$	2	73768.200	MG93
$2s^2 2p^3(^4S^\circ)3s$	$^3S^\circ$	1	76794.978	MG93
$2s^2 2p^3(^4S^\circ)3p$	^5P	1	86625.757	MG93
		2	86627.778	MG93
		3	86631.454	MG93
$2s^2 2p^3(^4S^\circ)3p$	^3P	2	88631.146	MG93
		1	88630.587	MG93
		0	88631.303	MG93
$2s^2 2p^3(^4S^\circ)4s$	$^5S^\circ$	2	95476.728	MG93
$2s^2 2p^3(^4S^\circ)4s$	$^3S^\circ$	1	96225.049	MG93
$2s^2 2p^3(^4S^\circ)3d$	$^5D^\circ$	4	97420.630	MG93
		3	97420.716	MG93
		2	97420.839	MG93

Parity from orbitals. (Not total L)

$$\pi = \prod_{i=1}^{\text{unpaired}} (-1)^{\ell_i}$$

$\leftarrow$ example from atomic physics

- First ^3P has total orbital angular momentum $L = 1$ (odd), while π is from four p orbitals $(-1)^1(-1)^1(-1)^1(-1)^1 = +1$.
- Similarly, this $^5S^o$ has $L = 0$ but three p orbitals and one s, so π= -1 (thus the o label used in atomic physics)
- while same L=0 ^1S has even number of p orbitals so has π=+1

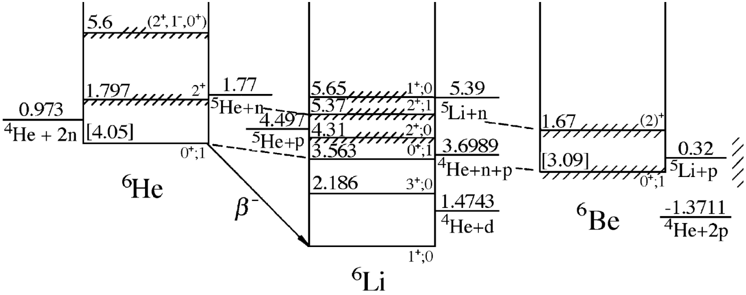
A=6 from two 1p nucleons; assume 1s shell is filled, $S = T = 0$, passive

2 particles in the 1p shell. Identify quantum numbers of partitions, couple them:

$\psi_{\rm space}$ $\boxed{p}\boxed{p}$ L=0, 2, not 1

Yes, $\vec{1} + \vec{1}$ can be 1, but we just showed this was antisymmetric under exchange

So $\psi_{\rm spin}\psi_{\rm isospin}$ must be antisymmetric. For 2 particles exactly as in *d*: (S;T) = (1;0) or (0;1)



There are $\pi = -$ (unbound) states at much higher excitation ? $\rightarrow$

With permutation symmetry under control, couple these possibilities to $\vec{J} = \vec{L} + \vec{S}$:

$J^\pi; T$	$^{2S+1}S_J$
From L=0 "*d*-like"	:
$1^+; 0$	$^3S_1$
$0^+; 1$	$^1S_1$
From L=2	:
$1^+, 2^+, 3^+; 0$	$^3D_{1,2,3}$
$2^+; 1$	$^1D_2$

For given space symmetry, lowest L tends to lie lowest

Highest *J* has lowest energy (spin-orbit $\vec{L}\cdot\vec{S}$)

All states accounted for ☺

$\pi = -$ states?

We found

ψ_{space}

p
p

$\psi_{\text{spin-iso}}$

(S;T)= (0;0)
or (1;1)

$\Rightarrow \pi = +$ **for L=1.**

From next shell?

p	s

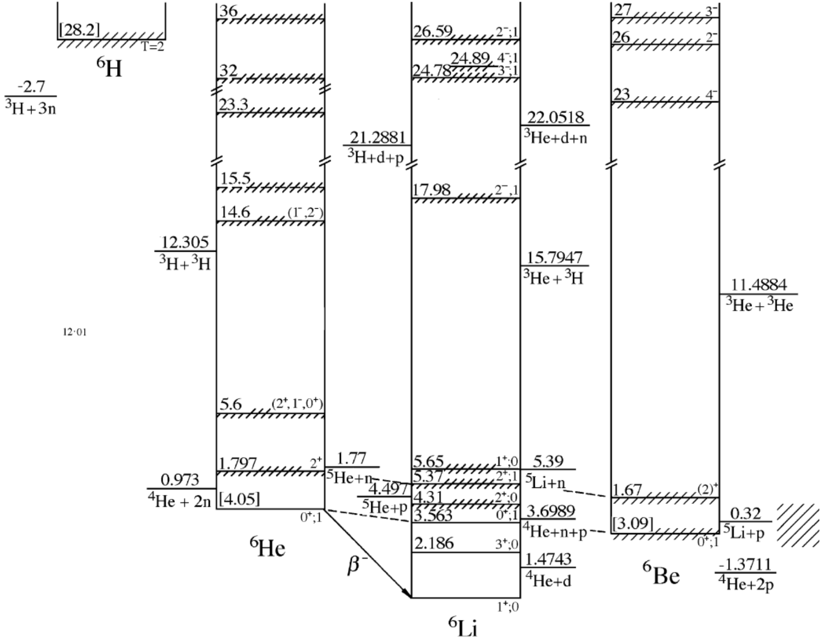
predicts J^π; T

$2^-;0 \quad 1^-;0 \quad 0^-;0 \quad 1^-;1$

none seen

To get $4^-, 3^-$**, maybe?**

s	s	s
p	p	p



Tilley *et al.* **Nuclear Physics A 708 (2002) 3**

A=4: only 1 bound state

ψ_{space}

$$\begin{array}{|c|c|c|c|}\hline s & s & s & s \\ \hline \end{array}$$

L=0

$\psi_{\text{spin}}\psi_{\text{isospin}}$

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

S=0, T=0

^{4}He: $J^\pi; T = 0^+; 0$

Excited states?

ψ_{space}

Since we can symmeterize anything, consider

$$\begin{array}{|c|c|c|c|}\hline s & s & s & p \\ \hline \end{array}$$

The center of mass is moving, and this is spurious.

Abstractly:

in many bases $r\phi_s \propto \phi_p$

i.e. $\vec{r}$ $\begin{array}{|c|c|c|c|}\hline s & s & s & s \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline s & s & s & p \\ \hline \end{array}$

A serious technical issue in many shell-model and other calculations

So instead we consider

ψ_{space}

$$\begin{array}{|c|c|c|}\hline s & s & s \\ \hline p & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$\rightarrow$

A=4 excited states

ψ_{space}

s s s
p

L=1

$\pi = -$

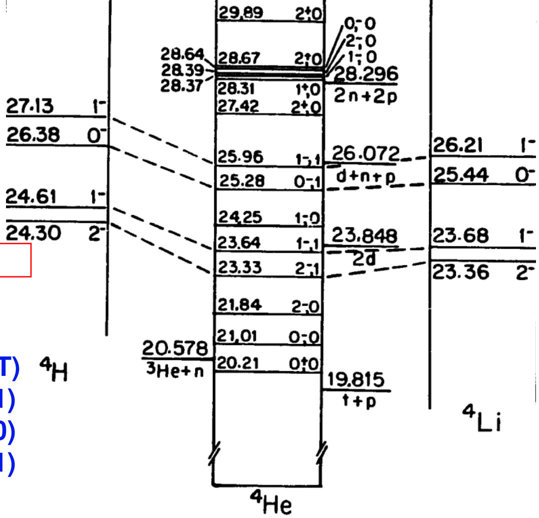
$\psi_{\text{spin-isosp}}$

(S,T)=
(?,?)

$(\frac{1}{2},\frac{1}{2})$ (one hole) $\otimes$ $(\frac{1}{2},\frac{1}{2})$ = (0, 0) $\oplus$ (S;T) $\Rightarrow$

= (1,1) (1,0) (0,1) (0,0)

(S;T)
(1;1)
(1;0)
(0;1)



$T = 1 \quad {}^3P_{0,1,2}, {}^1P_1 \quad J^\pi = 0^-, 1^-, 1^-, 2^-$ ☺ exactly these are there

$T = 0 \quad {}^3P_{0,1,2} \quad J^\pi = 0^-, 1^-, 2^-$ ☺ exactly these are there

The first 0^+, $T = 0$ state does not fit ☹: similar states exist in ^{16}O, ^{40}Ca, considered 'intruder' states from a higher shell. Higher-lying $\pi = +$ states are ... two p particles?

A=5: no bound states

Can't have for space:

1	2	3	4	5

because would need for spin $\times$ isospin:

1
2
3
4
5

yet we only have 2 spin $\times$ 2 isospin = 4 possibilities

Instead we have: space:

s	s	s	s
p			

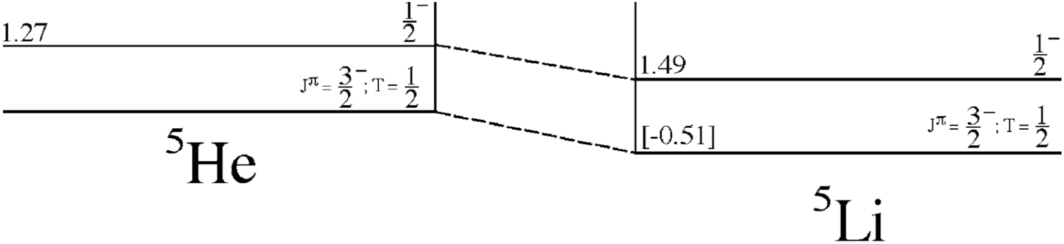
L=1 (finally, a $\pi = -$ state)

spin-isospin:

S,T= 1/2,1/2 = S,T= 0,0 + S,T= 1/2,1/2

With antisymmeterization known ok, we can couple these possibilities: $\vec{J} = \vec{L} + \vec{S}$ so $J^\pi; T = 1/2^-; 1/2$ and $3/2^-; 1/2$ with $^{2S+1}L_J = {}^2P_{3/2}, {}^2P_{1/2}$, and higher J lower E

Consistent with shell model, with fewer assumptions. Excited states $\rightarrow$



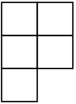
A=5 excited states

ψ_{space}

s	s	s
p	p	

L=0, 2 $\quad \pi = +$

$\psi_{\text{spin-isospin}}$ to ‘balance’



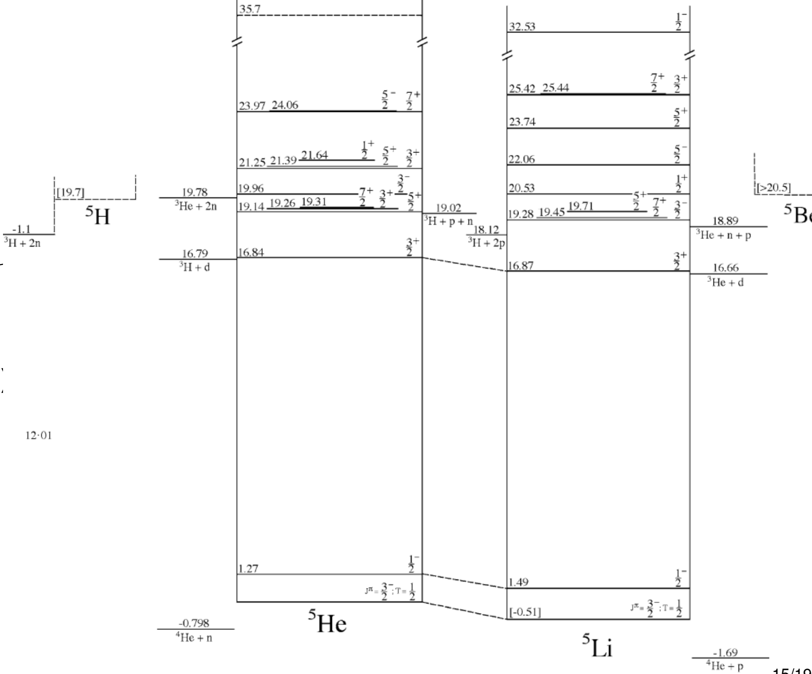
$\cancel{(S;T) = (3/2;3/2)}$ is fully stretched and symmetric

(S;T)=(3/2;1/2),(1/2;3/2),(1/2;1/2)

$J^{\pi} = 3/2^{+}, 1/2^{+}, 5/2^{+}, 7/2^{+}$ ☺

The $5/2^{-}$ ☹ is maybe 3 p-shell particles?

The 16.8 MeV $^4S_{3/2}$ $J^{\pi} = 3/2^{+}$ decays mostly to d not α which needs final state L=2



low-lying A=7 **will have**

ψ_{space} **symmetric**

$\boxed{p}\boxed{p}\boxed{p}$

$\Rightarrow$

$\psi_{\text{spin-isosp}}$

$(S;T) = (\frac{1}{2};\frac{1}{2})$

What L's allowed? 'm-scheme' labels by $m = L_3$

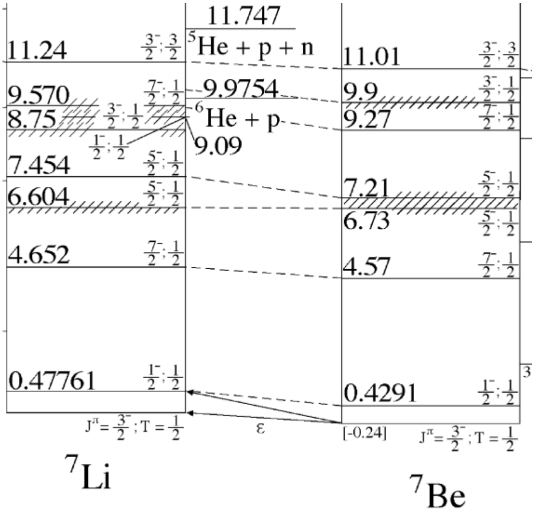
$\leftarrow$ rows don't decrease m

	L=3		L=1
M=3	1 1 1		
M=2	1 1 0	+	1 1 -1
M=1	1 0 0		1 0 -1
M=0	0 0 0		

Lowest L lies lowest

For given L, highest J is lowest $^2P_{3/2}, {}^2P_{1/2}, {}^2F_{7/2}\ {}^2F_{5/2}$

L=1,3 fully account for m's: No $L = 0, 2$, No $T = 3/2$. Need mixed symmetry partitions for $T = 3/2$, or for $\pi = +$ of which none are known.



A=8 low-lying states

Assume inert 1s core of 4 nucleons

ψ_{space} **symmetric**

p	p	p	p

$\Rightarrow$

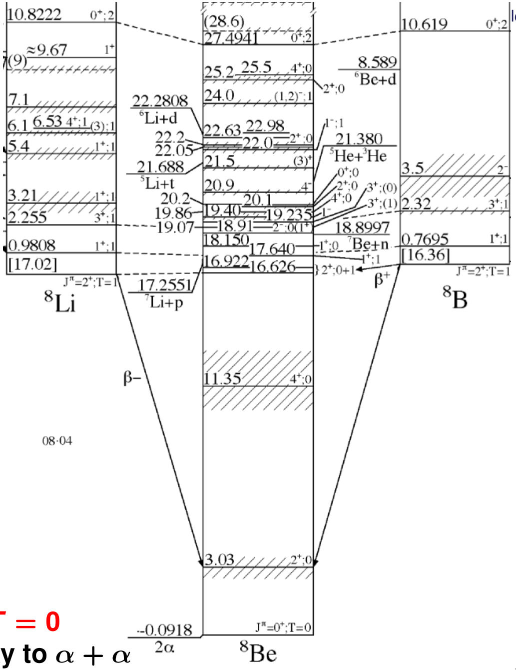
$\psi_{\text{spin-isosp}}$ **antisymmetric**

$M=4$	1 1 1 1		
$M=3$	1 1 1 0		
$M=2$	1 1 0 0	1 1 1 -1	
$M=1$	1 0 0 0	1 1 0 -1	
$M=0$	0 0 0 0	1 0 0 -1	1 1 -1 -1
	L=4	L=2	L=0

S=0,T=0

$S=0 \Rightarrow J=L$: ${}^1S_0$, ${}^1D_2$, ${}^1G_4$: 0^+, 2^+, 4^+, all $T=0$

All ^{8}Be unbound to decay to $\alpha+\alpha$



A=8 higher-lying states:

Assume inert core of four 1s nucleons

ψ_{space}

p	p	p
p		

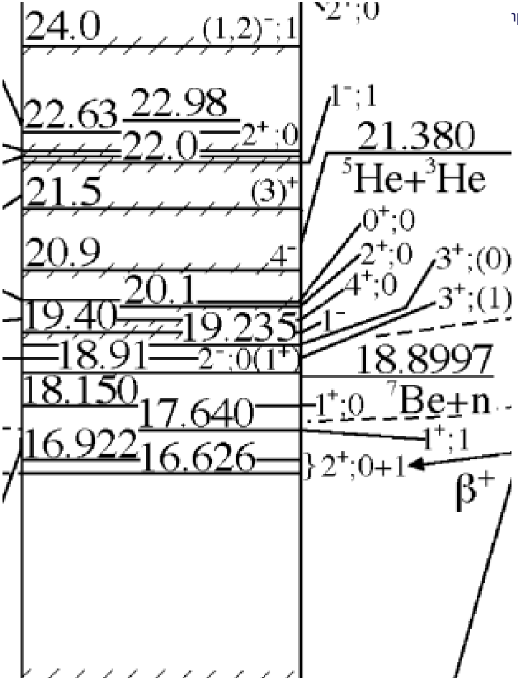
$\Rightarrow$

Should be able to make $L = 1, 2, 3$

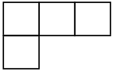
$\psi_{\text{spin-isosp}}$

$(S, T) =$ $(1, 1), (1, 0), (0, 1)$

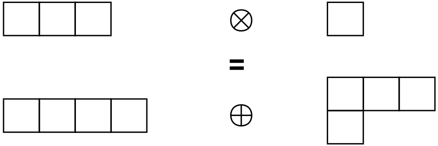
(All $\pi = +$, 4 p-shell particles)
Lowest-energy expected states:
$^3P_{0,1,2}$ $T = 1$
$^3P_{0,1,2}$ $T = 0$
$^1P_0$ $T = 1$
with highest J lowest E
Then:
$^3D_{1,2,3}$ $T = 1$
$^3D_{1,2,3}$ $T = 0$
$^1D_2$ $T = 1$



Observations ^{8}Be 2^+ T=0,1



$\leftarrow$ **These configurations look like ^{7}Be+n and ^{7}Li+p $\rightarrow$**



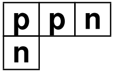
In terms of isospin, we can decompose

$$|^7\mathrm{Be}+n\rangle = \frac{|10\rangle+|00\rangle}{\sqrt{2}}$$

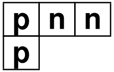
$$|^7\mathrm{Li}+p\rangle = \frac{|10\rangle-|00\rangle}{\sqrt{2}}$$

They would decay into these channels, except it's (slightly) energy forbidden

Can investigate by $^7\mathrm{Li}(d,n)^8\mathrm{Be}$ and (nowadays) $^7\mathrm{Be}(d,p)^8\mathrm{Be}$



$\leftarrow$ **$p's$ are symmetric so closer together, Coulomb energy higher than $\rightarrow$**



So the 16.92 MeV state looks more like $^7\mathrm{Be}+n$